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We record with deep sadness the passing of Anton Evseev on February 21, 2017.

ABSTRACT. This paper initiates a systematic study of the cyclotomic KLR algebras of affine types A and C. We start by introducing a graded deformation of these algebras and then constructing all of the irreducible representations of the deformed cyclotomic KLR algebras using *content systems*, which are recursively defined using Rouquier's Q-polynomials. This leads to a generalisation of the Young's seminormal forms for the symmetric groups in the KLR setting. Quite amazingly, the same theory captures the representation theory of the cyclotomic KLR algebras of affine types A and C, with the main difference being that the definition of the residue sequence of a tableau depends on the Cartan type. We use our semisimple deformations to construct two "dual" cellular bases for the non-semisimple KLR algebras of affine types A and C. As applications we recover many of the main features from the representation theory in type A, simultaneously proving them for the cyclotomic KLR algebras of types A and C. These results are completely new in type C and we, usually, give more direct proofs in type A. In particular, we show that these algebras categorify the irreducible integrable highest weight modules of the corresponding Kac–Moody algebras, we construct and classify their simple modules, we investigate links with canonical bases and we generalise Kleshchev's modular branching rules to these algebras.

1. INTRODUCTION

The *KLR algebras* are a remarkable family of graded algebras that were independently introduced by Khovanov–Lauda [36] and Rouquier [62, 63]. These algebras are now central to many of the recent developments in representation theory, not least because these algebras categorify the positive part of quantised Kac–Moody algebras [68].

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The *cyclotomic KLR algebras* are natural finite dimensional quotients of the KLR algebras that categorify the irreducible highest weight representations of the corresponding quantum groups [10, 14, 31, 69]. These algebras are only well understood for quivers of type $A_{e-1}^{(1)}$ and A_∞, where it has been possible to bootstrap results using the Brundan–Kleshchev isomorphism theorem [10], which shows that the cyclotomic KLR algebras of type A are isomorphic to the (ungraded) Ariki–Koike algebras. Using the Brundan–Kleshchev isomorphism, there is now an extensive literature in type A including a categorification theorem [11], cellular bases [9, 24], and results on Specht modules [13, 25, 40].

Very little explicit information is known about the cyclotomic KLR algebras for other Cartan types and even in type A our understanding is imperfect because it is seen through the lens of the Brundan–Kleshchev isomorphism theorem, which does not keep track of the grading. Hu and Shi have proved an amazing general formula that gives the graded dimensions of the weight spaces of the cyclotomic KLR algebras of symmetrisable Cartan type [28]. Recent work of the second author and Tubbenhauer [56, 57] shows that the cyclotomic KLR algebras of types $A_{2e}^{(2)}$, B_∞, $C_{e-1}^{(1)}$ and $D_{e-1}^{(1)}$ are graded cellular algebras, in the sense of [21, 24], using the weighted KLRW algebras pioneered by Webster [69, 70, 71] and Bowman [9], who mainly consider type A. The combinatorics in this paper is influenced by a beautiful series of papers by Ariki and Park [5, 6, 7], which determine the representation type of the cyclotomic KLR algebras in certain types, and by the attempts of Ariki, Park and Speyer [8] to construct Specht modules for the cyclotomic KLR algebras of affine type C. The semisimplicity of the cyclotomic KLR algebras of types A and C is determined in the papers [52, 65].

The cyclotomic KLR algebras are defined by generators and relations with the most important relations being encoded in Rouquier's Q-polynomials. Modulo a choice of signs, which do not affect the algebras up to isomorphism, the "standard" Q-polynomials in literature take the form

$$Q_{i,j}(u,v) = \begin{cases} u - v & \text{if } i \to j, \\ (u-v)(v-u) & \text{if } i \leftrightarrows j, \\ u - v^2 & \text{if } i \Rightarrow j, \end{cases}$$

where i and j are vertices of the underlying quiver and u and v are indeterminates of degree 2 (see subsection 2B for more detailed definitions.) Our starting point is to consider "deformations" of these polynomials, such as

$$Q_{i,j}^x(u,v) = \begin{cases} u - v - x^2 & \text{if } i \to j, \\ \left(u-v+x^2\right)\left(v-u+x^2\right) & \text{if } i \leftrightarrows j, \\ u - \left(v-x^2\right)^2 & \text{if } i \Rightarrow j, \end{cases}$$

where x is an indeterminate over $\mathbb{K}$ of degree 1. (We allow more general deformations.) Using the standard Q-polynomials $Q_{i,j}(u,v)$, and a dominant weight Λ, we define the "standard" (cyclotomic) KLR algebras $\mathscr{R}_n^\Lambda$ via Definition 2C.2. Using the deformed Q-polynomials $Q_{i,j}^x(u,v)$, the same definition gives us the deformed (cyclotomic) KLR algebras R_n^Λ, for $n \geq 0$. For quivers of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$ we show that the deformed cyclotomic KLR algebras R_n^Λ are split semisimple graded algebras over $\mathbb{K}[\underline{x}^\pm] = \mathbb{K}[x, x^{-1}]$. We prove this by introducing *content systems*, which are a generalisation of the classical content functions from the semisimple representation theory of the symmetric groups. Unlike the classical situation, a content system consists of two functions that determine

"contents" and "residues", where the content function is determined by the Q-polynomials. We use content systems to construct irreducible representations of the deformed cyclotomic KLR algebras of types A and C over $\mathbb{K}[\underline{x}^{\pm}]$, giving a generalisation of Young's seminormal forms in the KLR setting. The appearance of seminormal forms in the representation theory of the KLR algebras of type A is not surprising but, at least for us, this was unexpected for the algebras of type C.

The graded semisimple deformations of the cyclotomic KLR algebras gives a new way of approaching the non-semisimple representation theory of the cyclotomic KLR algebras, even though these algebras are rarely semisimple. The deformed cyclotomic KLR algebras are semisimple over $\mathbb{K}[\underline{x}^{\pm}]$ but they stop being semisimple when x is not invertible, which allows us to recover the standard cyclotomic KLR algebras from the deformed algebras by specialising $x = 0$. In this way, we can use the semisimple representation theory of R_n^Λ over $\mathbb{K}[x, x^{-1}]$ to understand the non-semisimple representation theory of $\mathscr{R}_n^\Lambda$ over $\mathbb{K}$. In fact, throughout the paper we work mainly with the deformed KLR algebra R_n^Λ, both because R_n^Λ is easier to work with and because it has a richer representation theory that encodes everything about $\mathscr{R}_n^\Lambda$.

The first main result of this paper, Theorem 4F.1, is the following.

Theorem A. *Let $\mathscr{R}_n^\Lambda$ be a cyclotomic KLR algebra of type $A_{e-1}^{(1)}$ or $C_{e-1}^{(1)}$. Then $\mathscr{R}_n^\Lambda$ is a graded cellular algebra.*

Knowing that an algebra is cellular gives a framework for understanding its representation theory, including a construction of the irreducible representations of the algebra. We actually prove several enhanced versions of Theorem **A**. First, over a positively graded ring K, such as $\mathbb{K}[x]$, we show that the deformed KLR algebra R_n^Λ over K is a graded K-cellular algebra, where K-cellularity further generalises cellular algebras to the category of finite dimensional graded algebras that are defined over graded rings. Secondly, we construct four different cellular bases of R_n^Λ, two of which specialise to give cellular bases of $\mathscr{R}_n^\Lambda$, and two of which give bases for the split semisimple algebra R_n^Λ when we extend scalars to $\mathbb{K}[x^{\pm}]$.

The proof of Theorem **A** starts by using our generalisation of Young's seminormal forms to show that R_n^Λ has two seminormal cellular bases, $\{f_{\mathsf{st}}^\lhd\}$ and $\{f_{\mathsf{st}}^\rhd\}$, over $\mathbb{K}[x^{\pm}]$. The seminormal bases are characterised as bases of simultaneous eigenvectors for the generators $y_1, \ldots, y_n$ of R_n^Λ, where the eigenvalues are given by our content systems. The seminormal bases are then used to show that R_n^Λ has two "integral" cellular bases, $\{\psi_{\mathsf{st}}^\lhd\}$ and $\{\psi_{\mathsf{st}}^\rhd\}$ (Definition 4A.5), that are defined over $\mathbb{K}[x]$ and which specialise to give cellular bases of $\mathscr{R}_n^\Lambda$. In type A, the ψ-bases of R_n^Λ generalise the ψ-bases constructed in [24]. The transition matrix from the $f^\lhd$-basis to the $\psi^\lhd$-basis is unitriangular, as is the transition matrix from the $f^\rhd$-basis to the $\psi^\rhd$-basis, so it is very easy to deduce properties of ψ-bases from the seminormal bases.

The key difference between the $\psi^\lhd$-basis and the $\psi^\rhd$-basis, and between the $f^\lhd$-basis and the $f^\rhd$-basis, is that one is defined using the *reverse dominance order* on the set of ℓ-partitions and the other is defined using the *dominance order*. (Here ℓ is the level of the dominant weight Λ; see subsection 3B.) That is, by reversing the choice of partial order in our definitions we can switch between these two families of cellular bases. In turn, this leads to the construction of two closely related families of *cell modules*, or *Specht modules*, $\{S_{\boldsymbol{\mu}}^\lhd\}$ and $\{S_{\boldsymbol{\nu}}^\rhd\}$, and two families of simple $\mathsf{R}_n^\Lambda(F[x])$-modules $\{D_{\boldsymbol{\mu}}^\lhd\}$ and $\{D_{\boldsymbol{\nu}}^\rhd\}$. Throughout the paper we keep track of these two families of modules because, aside from the notation, doing this requires almost no extra work, with the only real difference being

whether we work with the dominance or reverse dominance order. In fact, we need to work with these two "dual" families of modules because some of our main results are proved by exploiting the close connections between these two families of modules.

Once we have proved that R_n^Λ and $\mathscr{R}_n^\Lambda$ are cellular algebras, we next turn to understanding their representation theory. We first use the semisimple representation theory to show that R_n^Λ (and $\mathscr{R}_n^\Lambda$), is a graded symmetric algebra. There is a natural symmetrising form that is defined using *defect polynomials* (Definition 4D.2), which are graded analogues of the *generic degrees* from the representation theory of cyclotomic Hecke algebras [50]. In particular, this allows us to show that $S_\lambda^\triangleleft$ is isomorphic to the dual of $S_\lambda^\triangleright$, up to shift. The *defect* of a Specht module is equal to the degree of its defect polynomial. Defect is a key invariant of the blocks of the cyclotomic KLR algebras, which generalises the p-weight of a partition in the modular representation theory of the symmetric groups.

As a second application of the semisimple representation theory, we give explicit Specht filtrations of the modules obtained by inducing and restricting the Specht modules of R_n^Λ over an arbitrary ring. Together with the combinatorics based on the defect polynomials, the graded branching rules for the Specht modules translate into our next main result, which is a categorification theorem. To state this we need to introduce some notation.

Let $\mathbb{K}$ be a field and x an indeterminate over $\mathbb{K}$. We consider $\mathbb{K}[x]$ as a positively graded ring, with x in degree 1, and set $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Let $\mathrm{Rep}_\mathbb{K} R_n^\Lambda(\mathbb{K}[x])$ be the category of graded $R_n^\Lambda(\mathbb{K}[x])$-modules that are finite dimensional as $\mathbb{K}$-vector spaces and let $\mathrm{Proj}_\mathbb{K} R_n^\Lambda(\mathbb{K}[x])$ be the full subcategory of projective $R_n^\Lambda(\mathbb{K}[x])$-modules. Let

$$\left[\mathrm{Rep}_\mathbb{K} R_\bullet^\Lambda(\mathbb{K}[x]) \right] \quad \text{and} \quad \left[\mathrm{Proj}_\mathbb{K} R_\bullet^\Lambda(\mathbb{K}[x]) \right]$$

be the direct sum of the Grothendieck groups of these categories for $n \geq 0$, which we consider as free $\mathcal{A}$-modules by letting q act as the grading shift functor.

Suppose that Γ is a quiver of type $A_{e-1}^{(1)}$ or type $C_{e-1}^{(1)}$. Let $U_q(\mathfrak{g}_\Gamma)$ be the corresponding quantised Kac–Moody algebra and let $U_\mathcal{A}(\mathfrak{g}_\Gamma)$ be Lusztig's $\mathcal{A}$-form of $U_q(\mathfrak{g}_\Gamma)$. For a dominant weight Λ, let $L(\Lambda)_\mathcal{A}$ be the $\mathcal{A}$-form of the corresponding irreducible integrable highest weight module for $U_\mathcal{A}(\mathfrak{g}_\Gamma)$ and let $L(\Lambda)^*$ be is dual, with respect to the Cartan pairing.

Theorem B (Cyclotomic categorification theorem). *Suppose that Γ is a quiver of type $A_{e-1}^{(1)}$ or $C_{e-1}^{(1)}$ and let Λ be a dominant weight. Then, as $U_\mathcal{A}(\mathfrak{g}_\Gamma)$-modules,*

$$L(\Lambda)_\mathcal{A} \cong \left[\mathrm{Proj}_\mathbb{K} R_\bullet^\Lambda(\mathbb{K}[x]) \right] \qquad \text{and} \qquad L(\Lambda)_\mathcal{A}^* \cong \left[\mathrm{Rep}_\mathbb{K}(R_\bullet^\Lambda(\mathbb{K}[x])) \right].$$

This result, which is Theorem 6D.20, is not new. In type $A_{e-1}^{(1)}$ it is one of the main results of [11]. More generally, [31] establishes this result whenever Γ is a quiver of symmetrisable Cartan type. What is new about this result is that it is deduced almost directly from the graded branching rules for the Specht modules of $R_n^\Lambda(\mathbb{K}[x])$, which directly encode the action of $U_q(\mathfrak{g}_\Gamma)$ on the Grothendieck groups. This explicit link with the representation theory of $R_n^\Lambda(\mathbb{K}[x])$ makes it much easier to apply this result to the representation theory of $R_n^\Lambda(\mathbb{K}[x])$. In fact, the information flow is stronger in both directions, so we also use the representation theory of $R_n^\Lambda(\mathbb{K}[x])$ to better understand $L(\Lambda)$. In particular, we are able to give detailed information about the canonical bases of $L(\Lambda)_\mathcal{A}$ and $L(\Lambda)_\mathcal{A}^*$ and their role in this categorification theorem.

Our approach to Theorem **B** is partly based on [11], although our perspective is fundamentally different because we work almost exclusively inside the Grothendieck groups

of the cyclotomic KLR algebras whereas [11] works mainly inside a combinatorial Fock space, which we also use. In particular, we use Theorem **A**, and the triangularity of the decomposition matrices of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$, to show that Lusztig's bar involution is triangular on the basis of Specht modules. Our arguments work simultaneously for the algebras of type $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$ and, via Theorem **A**, we obtain two versions of Theorem **B** corresponding to the $\psi^\lhd$ and $\psi^\rhd$ cellular bases. This gives two explicit realisations of the irreducible integrable highest weight $U_\mathcal{A}(\mathfrak{g}_\Gamma)$-modules $L(\Lambda)_\mathcal{A}$ and $L(\Lambda)_\mathcal{A}^*$.

Our next main goal is to classify the irreducible graded $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules. Our parallel theories, using the $\psi^\lhd$ and $\psi^\rhd$ cellular bases, leads to two combinatorial descriptions of the crystal graph of $L(\Lambda)$, which we call the $\lhd$-crystal graph and the $\rhd$-crystal graphs in this introduction. To describe these, let I be the vertex set of the quiver Γ. The paths in the crystal graphs of $L(\Lambda)$ are labelled by n-tuples $\mathbf{i} \in I^n$, corresponding to generalisations of Kleshchev's *good node sequences* (Definition 6F.5). Each good node sequence $\mathbf{i}$ determines two paths: one path $\underline{\mathbf{0}}_\ell \overset{\mathbf{i}\lhd}{\rightsquigarrow} \boldsymbol{\mu}$ in the $\lhd$-crystal graph and a second path $\underline{\mathbf{0}}_\ell \overset{\mathbf{i}\rhd}{\rightsquigarrow} \boldsymbol{\nu}$ path in the $\rhd$-crystal graph. Here, $\underline{\mathbf{0}}_\ell$ is the empty ℓ-partition and $\boldsymbol{\mu}, \boldsymbol{\nu}$ are ℓ-partitions of n. Let

$$\mathcal{K}_n^\lhd = \left\{ \boldsymbol{\mu} \in \mathcal{P}_n^\ell \,\middle|\, 0 \overset{\mathbf{i}\lhd}{\rightsquigarrow} \boldsymbol{\mu} \text{ for some } \mathbf{i} \in I^n \right\}$$

and

$$\mathcal{K}_n^\rhd = \left\{ \boldsymbol{\nu} \in \mathcal{P}_n^\ell \,\middle|\, 0 \overset{\mathbf{j}\rhd}{\rightsquigarrow} \boldsymbol{\mu} \text{ for some } \mathbf{j} \in I^n \right\}$$

be the vertex sets of the two crystal graphs. Calculations with the canonical bases in the Grothendieck groups $\mathrm{Rep}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])$ and $\mathrm{Proj}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])$ allows us to classify the irreducible $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules over a field, for $n \geq 0$. As Theorem 6F.14, we prove.

Theorem C. *Let $\mathbb{K}$ be a field. Up to shift, $\{D_{\boldsymbol{\mu}}^\lhd | \boldsymbol{\mu} \in \mathcal{K}_n^\lhd\}$ and $\{D_{\boldsymbol{\nu}}^\rhd | \boldsymbol{\nu} \in \mathcal{K}_n^\rhd\}$ are both complete sets of pairwise non-isomorphic irreducible $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules.*

In particular, over any field, this result classifies the irreducible modules of the cyclotomic KLR algebras of type $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$.

Theorem **C** implies that there is a bijection $\mathsf{m}\colon \mathcal{K}_n^\lhd \longrightarrow \mathcal{K}_n^\rhd$ such that $D_{\boldsymbol{\mu}}^\lhd \cong D_{\mathsf{m}(\boldsymbol{\mu})}^\rhd$. In Corollary 5E.8 we show that if

$$\boldsymbol{\mu} \in \mathcal{K}_n^\lhd \quad \text{and} \quad \underline{\mathbf{0}}_\ell \overset{\mathbf{i}\lhd}{\rightsquigarrow} \boldsymbol{\mu}$$

is a path in the $\lhd$-crystal graph of $L(\Lambda)$ then there is a unique ℓ-partition

$$\boldsymbol{\nu} = \mathsf{m}(\boldsymbol{\mu}) \quad \text{such that} \quad \underline{\mathbf{0}}_\ell \overset{\mathbf{i}\rhd}{\rightsquigarrow} \boldsymbol{\nu}$$

is a path in the $\rhd$-crystal graph. This gives a way to compute the ℓ-partition $\mathsf{m}(\boldsymbol{\mu})$. In the special case of the symmetric groups, this gives another description of the *Mullineux map*, which describes what happens to the simple modules of the symmetric group when they are tensored with the sign representation. We introduce a sign representation for the algebras $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ and show in our setting, which generalises that of the symmetric groups, the Mullineux map is the function $\boldsymbol{\mu} \mapsto \mathsf{m}(\boldsymbol{\mu})'$, where $\boldsymbol{\mu}'$ is the ℓ-partition conjugate to $\boldsymbol{\mu}$; see subsection 4A.

Finally, we show that Kleshchev's modular branching rules [38] extend to give branching rules for the simple $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules. For $i \in I$, let E_i^Λ and F_i^Λ be the corresponding i-restriction and i-induction functors and let e_i and f_i be Kashiwara's operators on the

crystal graph of $L(\Lambda)$. We refer the reader to subsection 6G for the precise definitions and statements, but the main results take the form:

Theorem D. *Suppose that* $\boldsymbol{\mu} \in \mathcal{K}_n^\lhd$, $\boldsymbol{\nu} \in \mathcal{K}_n^\rhd$ *and* $i, j \in I$. *Then, up to grading shift,*

$$D_{e_i\boldsymbol{\mu}}^\lhd = \operatorname{soc}\left(E_i^\Lambda D_{\boldsymbol{\mu}}^\lhd\right), \qquad D_{f_i\boldsymbol{\mu}}^\lhd = \operatorname{head}\left(F_i^\Lambda D_{\boldsymbol{\mu}}^\lhd\right),$$
$$D_{e_j\boldsymbol{\nu}}^\rhd = \operatorname{soc}\left(E_j^\Lambda D_{\boldsymbol{\nu}}^\rhd\right) \quad and \quad D_{f_j\boldsymbol{\nu}}^\rhd = \operatorname{head}\left(F_j^\Lambda D_{\boldsymbol{\nu}}^\rhd\right).$$

In type $A_{e-1}^{(1)}$, Brundan and Kleshchev [11, Theorem] prove one version of this result by lifting Ariki's [1, 4] and Grojnowski's work [22], from the ungraded representation theory, into the KLR world. More generally, for any symmetrisable Cartan type, Lauda and Vazirani [44] show that analogues of these modular branching rules categorify the crystal graph of $L(\Lambda)$ by lifting parts of Grojnowski's approach to the KLR setting. Lauda and Vazirani's result does not imply Theorem **D** because it is not clear how their crystal graph is related to the labelling of the simple modules given in Theorem **C**. Our proof of Theorem **D** is almost axiomatic in that it uses Theorem **B** to lift the result from Theorem **B** and properties of the canonical basis.

Throughout the paper we work almost exclusively with a deformed cyclotomic KLR algebra R_n^Λ that has a content system to prove our results, after which the results for $\mathscr{R}_n^\Lambda$ are obtained by specialising the deformation parameters to 0. We show by example that every cyclotomic KLR algebra of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$ has a graded content system over $\mathbb{Z}[x]$, so our results apply to all cyclotomic KLR algebras of affine types A and C over any ring. In type $A_{e-1}^{(1)}$, the results we obtain for $\mathscr{R}_n^\Lambda$ were known but those for R_n^Λ are new. In type $C_{e-1}^{(1)}$, all of these results are completely new. As we note in subsection 2B, the results in this paper also extend to quivers of type A_∞ and C_∞. It likely that the general framework that we develop can be modified to work in other types.

It is quite striking that we are able to prove all of these results using a common framework for the cyclotomic KLR algebras of type $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$. Ultimately, the reason why this works is that our deformation arguments show that the algebra R_n^Λ over $\mathbb{K}[x^\pm]$ is isomorphic to a direct sum of matrix algebras that depend only on n and ℓ, and not on the choice of dominant weight Λ or even on the quiver Γ. In fact, Theorem 3F.8 shows that if n and ℓ are fixed then, for any choice of content system, the deformed cyclotomic KLR algebras over $\mathbb{K}[x^\pm]$ are canonically isomorphic as ungraded algebras.

An index of notation is included at the end of the document, before the list of references.

2. KLR ALGEBRAS

2A. **Graded rings, algebras and modules.** Throughout this paper we work with $\mathbb{Z}$-graded rings, algebras and modules. For convenience, we refer to each of these structures as being *graded*. This section recalls the basic definitions that we need for modules over graded rings.

All rings in the paper will be commutative integral domains with 1. A *graded ring* is a ring K that has a decomposition $K = \bigoplus_{d\in\mathbb{Z}} K_d$ as an additive abelian group such that $K_d K_e \subseteq K_{e+d}$. In particular, note that K_0 is a subring of K.[1]

Let K be a graded commutative domain. Then:

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[1]We apologise to the readers who instantly think that K is a field. In the body of the paper we mostly work with a field $\mathbb{K}$, which is a $\Bbbk$-algebra (often the field of fractions of the ring $\Bbbk$), and we consider modules over the graded rings $\Bbbk[\underline{x}]$, $\mathbb{K}[\underline{x}]$ and $\mathbb{K}[\underline{x}^\pm]$.

- A *graded K-module* is a K-module M that admits a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ as a K_0-module such that $K_d M_e \subseteq M_{d+e}$.
- A *graded K-algebra* is a K-algebra A that admits an decomposition $A = \bigoplus_{d \in \mathbb{Z}} A_d$ as a graded K-module such that $K_d A_e \subseteq A_{d+e}$.
- A *graded A-module* is an A-module M that admits a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ as a graded K-module such that $A_d M_e \subseteq M_{d+e}$.

If $R = \bigoplus_d R_d$ is a graded ring, algebra or module let $\underline{R}$ be the corresponding structure obtained by forgetting the grading. An element $m \in R$ is *homogeneous* of degree d if $0 \neq m \in R_d$, in which case we set $\deg(m) = d$. By definition, 0 is not homogeneous. In particular, note that if $r \in R$ and $m \in M$ are homogeneous then $\deg(rm) = \deg(r) + \deg(m)$. Further, R is *positively graded* if there are no elements of negative degree (that is, these are non-negatively graded structures) and R is *concentrated in degree d* if $R = R_d$.

In this paper the three types of graded rings K that we consider are:

- commutative domains $\Bbbk$ with 1,
- polynomial rings $\Bbbk[\underline{x}] = \Bbbk[\underline{x}]$, where $\underline{x}$ is a (possibly empty) family of indeterminates over $\Bbbk$ with each indeterminate having degree 1,
- Laurent polynomial rings $\mathbb{K}[\underline{x}^{\pm}] = \mathbb{K}[\underline{x}, \underline{x}^{-1}]$, where $\mathbb{K}$ is a field that is a $\Bbbk$-algebra, such as the field of fractions of $\Bbbk$.

In these rings, the elements of $\Bbbk$ and $\mathbb{K}$ are in degree 0.

A *graded field* is a graded ring in which every nonzero homogeneous element has a multiplicative inverse. In particular, $\mathbb{K}$ and $\mathbb{K}[\underline{x}^{\pm}]$ are graded fields. By [67, Theorem 4.1] all graded fields are of this form.

If A is a graded K-algebra and M is a graded A-module then graded submodules, quotient modules, projective modules, ... are defined in the obvious ways. If K is a graded field and A is a graded K-algebra then an *irreducible graded A-module* is a graded A-module that has no non-trivial proper graded A-submodules. We emphasise that irreducible graded modules make sense when the ground ring is a graded field that is not a field.

Remark 2A.1. Let K be a field and A a graded K-algebra. Then a graded A-module D is an irreducible graded A-module if and only if $\underline{D}$ is an irreducible $\underline{A}$-module by [60, Theorem 4.4.4 and Theorem 9.6.8]. In contrast, if A is a graded $\mathbb{K}[\underline{x}^{\pm}]$-algebra then an irreducible graded A-module is not necessarily irreducible when we forget the grading. For example, if $A = \mathbb{K}[x^{\pm}]$ and $D = \mathbb{K}[x^{\pm}]$ then D is an irreducible graded A-module but D is not irreducible as an $\underline{A}$-module because, for example, it contains the (non-homogeneous) ideal $(x+1)\mathbb{K}[x^{\pm}]$.

If M and N are graded A-modules then a *homogeneous A-module homomorphism* of degree d is an A-module homomorphism $f: M \longrightarrow N$ such that $\deg f(m) = \deg(m) + d$ whenever $m \in M$ is homogeneous. In this case we write $\deg f = d$. The map f is an *A-module isomorphism* if it is bijective and it is homogeneous of degree 0.

Let q be an indeterminate and set $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and $\mathbb{A} = \mathbb{Q}(q)$. If $M = \bigoplus_d M_d$ is a graded A-module and $s \in \mathbb{Z}$ let $q^s M = \bigoplus_d (q^s M)_d$ be the graded A-module that is equal to $\underline{M}$ as an ungraded module, has $(q^s M)_d = M_{d-s}$ and with A-action inherited from the action on M.

If M and N are graded A-modules let $\text{Hom}_A(M, N)$ be the homogeneous A-module homomorphisms of degree 0. Then $\text{Hom}_A(q^d M, N) \cong \text{Hom}_A(M, q^{-d} N)$ is naturally identified with the set of homogeneous maps $M \to N$ of degree d, for $d \in \mathbb{Z}$. Set $\text{HOM}_A(M, N) = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_A(q^d M, N)$. Define $\text{End}_A(M)$ and $\text{END}_A(M)$ similarly.

Remark 2A.2. For geometric reasons, indeterminates are usually put in degree 2. It is more convenient for us to put the indeterminates in $\underline{x}$ in degree 1 because then the graded field $\mathbb{K}[x^{\pm 1}]$ has a unique irreducible graded representation, namely itself; see Remark 2A.1. (In contrast, if we set $\deg(x) = 2$ then $\mathbb{K}[x^{\pm 1}]$ and $q\mathbb{K}[x^{\pm 1}]$ are non-isomorphic irreducible graded $\mathbb{K}[x^{\pm 1}]$-modules.) On the other hand, $\{q^d\mathbb{K} \,|\, d \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic unique irreducible graded $\mathbb{K}[x]$-modules, where the $\mathbb{K}[x]$-module $q^d\mathbb{K}$ is concentrated in degree d and x acts as multiplication by 0.

If A is a graded K-algebra then we will usually work in the category $\operatorname{Rep} A$ of finitely generated A-modules with homogeneous maps of degree 0. If $K = \bigoplus_d K_d$ and $\mathbb{K} = K_0$ is a field let $\operatorname{Rep}_\mathbb{K} A$ be the full subcategory of $\operatorname{Rep} A$ consisting of A-modules that are *finite dimensional* as $\mathbb{K}$-vector spaces. Similarly, let $\operatorname{Proj} A$ be the additive subcategory of $\operatorname{Rep} A$ consisting of *projective graded A-modules* and let $\operatorname{Proj}_\mathbb{K} A$ be the corresponding subcategory of $\operatorname{Rep}_\mathbb{K} A$. The proofs of our Main Theorem **B**-Theorem **D** take place in the categories $\operatorname{Rep}_\mathbb{K} \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$ and $\operatorname{Proj}_\mathbb{K} \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$.

Let $[\operatorname{Rep}_\mathbb{K} A]$ and $[\operatorname{Proj}_\mathbb{K} A]$ be the *Grothendieck groups* of the categories $\operatorname{Rep}_\mathbb{K} A$ and $\operatorname{Proj}_\mathbb{K} A$, respectively. Given a module M in $\operatorname{Rep}_\mathbb{K} A$, or in $\operatorname{Proj}_\mathbb{K} A$, let $[M]$ be its image in $[\operatorname{Rep}_\mathbb{K} A]$ or $[\operatorname{Proj}_\mathbb{K} A]$, respectively. Both $[\operatorname{Rep}_\mathbb{K} A]$ and $[\operatorname{Proj}_\mathbb{K} A]$ are free $\mathcal{A}$-modules where q acts by grading shift. That is, $[qM] = q[M]$.

2B. Quivers and Q-polynomials.

In this section we fix the Lie theoretic data that will be used throughout the paper. Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ be the set of non-negative integers.

Let Γ be a symmetrisable *quiver* Γ with vertex set I. Let $(C, P, P^\vee, \Pi, \Pi^\vee)$ be the *Cartan data* attached to Γ, consisting of:

- A *symmetrisable Cartan matrix*, $C = (\mathsf{c}_{ij})_{i,j \in I}$ satisfies $\mathsf{c}_{ii} = 2$, $\mathsf{c}_{ij} \leq 0$ for $i \neq j$, $\mathsf{c}_{ij} = 0$ whenever $\mathsf{c}_{ji} = 0$. Since C is symmetrisable, there exists a diagonal matrix $D = \operatorname{diag}(\mathsf{d}_i | i \in I)$ such that DC is symmetric
- The *weight lattice P* is a free abelian group with basis the *simple roots* $\Pi = \{\alpha_i \,|\, i \in I\}$.
- The *dual weight lattice* is $P^\vee = \operatorname{Hom}(P, \mathbb{Z})$ has basis the *simple coroots* $\Pi^\vee = \{\alpha_i^\vee \,|\, i \in I\}$.

The *Cartan pairing* $\langle\ ,\ \rangle \colon P^\vee \times P \longrightarrow \mathbb{Z}$ and *fundamental weights* $\{\Lambda_i \,|\, i \in I\} \subset P$ are given by

$$\langle \alpha_i^\vee, \alpha_j \rangle = \mathsf{c}_{ij} \quad \text{and} \quad \langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}, \qquad \text{for } i, j \in I.$$

The *positive root lattice* is $Q^+ = \sum_{i \in I} \mathbb{N}\alpha_i$, and $P^+ = \bigoplus_{i \in I} \mathbb{N}\Lambda_i$ is the set of *dominant weights* of Γ. The *height* of $\alpha = \sum_{i \in I} h_i \alpha_i \in Q^+$ is $\operatorname{ht}(\alpha) = \sum_{i \in I} h_i$. Let Q_n^+ be the set of all elements of Q^+ of height n. Set $\mathfrak{h} = \mathbb{Q} \otimes_\mathbb{Z} P^\vee$. As C is symmetrisable, there exists a symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ such that

$$(\alpha_i | \alpha_j) = \mathsf{d}_i \mathsf{c}_{ij} = \mathsf{c}_{ij} \mathsf{d}_j \qquad \text{and} \qquad \langle \alpha_i^\vee, \lambda \rangle = \frac{2(\alpha_i | \lambda)}{(\alpha_i | \alpha_i)}, \qquad \text{for } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

Fix $n \in \mathbb{N}$ and let $\mathfrak{S}_n$ be the *symmetric group* on n letters. As a Coxeter group, $\mathfrak{S}_n$ is generated by the simple transpositions $\sigma_1, \ldots, \sigma_{n-1}$, where $\sigma_k = (k, k+1)$ for $1 \leq k < n$. Let $L \colon \mathfrak{S} \to \mathbb{N}$ be the *length function* on $\mathfrak{S}_n$, so if $w \in \mathfrak{S}_n$ then $L(w) = l$ if l is minimal such that $w = \sigma_{a_1} \ldots \sigma_{a_l}$, for some $1 \leq a_j < n$. A *reduced expression* for $w \in \mathfrak{S}_n$ is any expression $w = \sigma_{a_1} \ldots \sigma_{a_l}$ with $l = L(w)$.

The group $\mathfrak{S}_n$ acts from the left on the set $I^n = I \times \cdots \times I$ by place permutations: if $w \in \mathfrak{S}^n$ and $\mathbf{i} = (i_1, \ldots, i_n) \in I^n$ write $w\mathbf{i} = (i_{w(1)}, \ldots, i_{w(n)})$.

In this paper we will mainly consider the quivers of type $A_{e-1}^{(1)}$ $(e \geq 2)$ and $C_{e-1}^{(1)}$ $(e \geq 3)$, for which we use the following quivers:

Type	Dynkin diagram	δ	$(\mathsf{d}_0, \ldots, \mathsf{d}_{e-1})$
$A_{e-1}^{(1)}$		$\alpha_0 + \alpha_1 + \cdots + \alpha_{e-2} + \alpha_{e-1}$	$(1,1,\ldots,1,1)$
$C_{e-1}^{(1)}$		$\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{e-2} + \alpha_{e-1}$	$(2,1,\ldots,1,2)$

Here, δ is the *null root*, which satisfies $\langle \delta, \alpha_i^\vee \rangle = 0$, for $i \in I$. Notice that for both of these quivers we have $I = \{0, 1, \ldots, e-1\}$. Our arguments apply equally well to the infinite quivers A_∞ and C_∞ but there is no real gain in considering these because the cyclotomic KLR algebras for these quivers are isomorphic to cyclotomic KLR algebras for a suitably large finite quiver.

Fix a (graded) commutative domain $K = \bigoplus_{d \in \mathbb{Z}} K_d$ and let u, v be indeterminates over K. Following Rouquier [63, Definition 3.2.2] and Kashiwara–Kang [31], a family of *Q-polynomials* for Γ is a collection of polynomials $Q_{ij}(u,v) \in K[u,v]$, for $i, j \in I$, such that $Q_{i,j}(u,v) = Q_{j,i}(v,u)$, $Q_{i,i}(u,v) = 0$ and if $i \neq j$ then

$$Q_{i,j}(u,v) = \sum_{p,q \geq 0} t_{i,j;p,q} u^p v^q, \qquad \text{where} \quad t_{i,j;-\mathsf{c}_{ij},0} \in K_0^\times \quad \text{and} \quad t_{i,j;p,q} \in K_d, \qquad (2\text{B}.1)$$

where $d = -2(\alpha_i|\alpha_j) - p(\alpha_i|\alpha_i) - q(\alpha_j|\alpha_j)$. That is, $Q_{i,j}(u,v)$ is homogeneous of degree d. By assumption, $Q_{i,j}(u,v) = Q_{j,i}(v,u)$, so $t_{i,j;p,q} = t_{j,i;q,p}$. One standard choice for these polynomials is

$$Q_{i,j}(u,v) = \begin{cases} u - v & \text{if } i \to j, \\ (u-v)(v-u) & \text{if } i \leftrightarrows j, \\ u - v^2 & \text{if } i \Rightarrow j. \end{cases} \qquad (2\text{B}.2)$$

As discussed in the introduction, this paper uses "deformed analogues" of these standard Q-polynomials. More examples can be found in Example 3A.2 below.

For $i, j, k \in I$ and indeterminates u, v and w over K, define the three variable Q-polynomials

$$Q_{i,j,k}(u,v,w) = \delta_{ik} \frac{Q_{ij}(u,v) - Q_{jk}(v,w)}{u - w}, \qquad (2\text{B}.3)$$

where δ_{ik} is the Kronecker delta. Then $Q_{i,j,k}(u,v,w) \in K[u,v,w]$.

2C. KLR algebras. This section defines the (cyclotomic) KLR algebras, which are one of the main objects of interest in this paper. Unless otherwise stated, all of our algebras are K-algebras, where K is a (graded) commutative integral domain with one.

As in the last section, let $K = \bigoplus_d K_d$ be a graded commutative ring with one and fix algebraically independent indeterminates $u_1, \ldots, u_n$ over K. The symmetric group $\mathfrak{S}_n$ acts on $K[u_1, \ldots, u_n]$ by permuting indeterminates $f \mapsto {}^w f = f(u_{w(1)}, \ldots, u_{w(n)})$, for $w \in \mathfrak{S}_n$ and $f \in K[u_1, \ldots, u_n]$.

Recall from subsection 2B, that $I = \{0, 1, \ldots, e-1\}$ is the (finite) vertex set of the quiver Γ and that we have fixed a family $\mathbf{Q}_I = (Q_{ij}(u, v))_{i,j \in I}$ of Rouquier's Q-polynomials. In addition, fix a family of homogeneous *weight polynomials*

$$\mathbf{W}_I = (W_i(u))_{i \in I}$$

such that

$$W_i(u) = \sum_{k=0}^{(\alpha_i^\vee \mid \Lambda)} a_{i;k} u^{(\alpha_i^\vee \mid \Lambda)-k}, \qquad \text{for } a_{i;k} \in K_{\mathsf{d}_i k} \text{ and } a_{i;0} = 1. \qquad (2\text{C}.1)$$

The weight polynomials $\mathbf{W}_I$ determine a dominant weight $\Lambda = \Lambda_{\mathbf{W}_I} = \sum_{i \in I} l_i \Lambda_i \in P^+$, where $l_i = \deg W_i(u)$ for $i \in I$. The *level* of Λ is $\ell = \sum_i l_i$. We assume $\ell \geq 1$.

A *cyclotomic KLR datum* is a triple $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$, where Γ is a quiver and $\mathbf{Q}_I$ and $\mathbf{W}_I$ are families of Q-polynomials and weight polynomials for Γ, respectively. The quiver Γ has vertex set I and comes equipped with a Cartan datum as in subsection 2B.

If $\alpha \in Q_n^+$ let $I^\alpha = \{\mathbf{i} \in I^n \mid \alpha = \alpha_{i_1} + \cdots + \alpha_{i_n}\}$.

Definition 2C.2. Let $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$ be a cyclotomic KLR datum and suppose that $\alpha \in Q_n^+$. The *KLR algebra* $\mathscr{R}_\alpha = \mathscr{R}_\alpha(\mathbf{Q}_I)$ is the unital associative K-algebra generated by

$$\{1_{\mathbf{i}} \mid \mathbf{i} \in I^\alpha\} \cup \{\psi_k \mid 1 \leq k < n\} \cup \{y_m \mid 1 \leq m \leq n\}$$

subject to the relations:

(KLR1) $1_{\mathbf{i}} 1_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}} 1_{\mathbf{i}}$ and $\sum_{\mathbf{i} \in I^\alpha} 1_{\mathbf{i}} = 1$
(KLR2) $y_k 1_{\mathbf{i}} = 1_{\mathbf{i}} y_k$ and $y_k y_m = y_m y_k$
(KLR3) $\psi_k y_m = y_m \psi_k$ if $m \neq k, k+1$
(KLR4) $\psi_k \psi_m = \psi_m \psi_k$ if $|m - k| > 1$
(KLR5) $\psi_k 1_{\mathbf{i}} = 1_{\sigma_k \mathbf{i}} \psi_k$,
(KLR6) $(\psi_k y_{k+1} - y_k \psi_k) 1_{\mathbf{i}} = \delta_{i_k, i_{k+1}} 1_{\mathbf{i}} = (y_{k+1} \psi_k - \psi_k y_k) 1_{\mathbf{i}}$
(KLR7) $\psi_k^2 1_{\mathbf{i}} = Q_{i_k, i_{k+1}}(y_k, y_{k+1}) 1_{\mathbf{i}}$
(KLR8) $(\psi_{k+1} \psi_k \psi_{k+1} - \psi_k \psi_{k+1} \psi_k) 1_{\mathbf{i}} = Q_{i_k, i_{k+1}, i_{k+2}}(y_k, y_{k+1}, y_{k+2}) 1_{\mathbf{i}}$

for all $\mathbf{i} \in I^\alpha$ and all admissible k and m. The *cyclotomic KLR algebra* is the quotient algebra

$$\mathscr{R}_\alpha^\Lambda = \mathscr{R}_\alpha^\Lambda(\mathbf{Q}_I, \mathbf{W}_I) = \mathscr{R}_\alpha / \mathscr{W}_\alpha^\Lambda(\mathbf{W}_I), \qquad (2\text{C}.3)$$

where $\mathscr{W}_\alpha^\Lambda(\mathbf{W}_I)$ is the two-sided ideal of $\mathscr{R}_\alpha$ generated by $\{W_{i_1}(y_1) 1_{\mathbf{i}} \mid \mathbf{i} \in I^\alpha\}$.

Set $\mathscr{R}_n = \bigoplus_{\alpha \in Q_n^+} \mathscr{R}_\alpha$ and $\mathscr{R}_n^\Lambda = \bigoplus_{\alpha \in Q_n^+} \mathscr{R}_\alpha^\Lambda$.

We abuse notation and use $1_{\mathbf{i}}$, y_r and ψ_r for both the generators of $\mathscr{R}_\alpha$ and $\mathscr{R}_n$ and for their images in $\mathscr{R}_\alpha^\Lambda$ and $\mathscr{R}_n^\Lambda$. When we want to emphasise the base ring K we write $\mathscr{R}_n(K) = \mathscr{R}_n(\mathbf{Q}_I, \mathbf{W}_I, K)$ and $\mathscr{R}_n^\Lambda(K) = \mathscr{R}_n(\mathbf{Q}_I, \mathbf{W}_I, K)$.

Importantly, the algebras $\mathscr{R}_n$ and $\mathscr{R}_n^\Lambda$ are graded K-algebras with degree function

$$\deg 1_{\mathbf{i}} = 0, \quad \deg y_m 1_{\mathbf{i}} = (\alpha_{i_m} \mid \alpha_{i_m}) = 2\mathsf{d}_i, \quad \text{and} \quad \deg \psi_k 1_{\mathbf{i}} = -(\alpha_{i_k} \mid \alpha_{i_{k+1}}),$$

for $\mathbf{i} \in I^n$, $1 \leq k < n$ and $1 \leq m \leq n$.

Inspecting the relations, there is a unique anti-isomorphism $*$ of $\mathscr{R}_n$, and of $\mathscr{R}_n^\Lambda$, that fixes each of the generators. If M is a graded $\mathscr{R}_n^\Lambda$-module then the *graded dual* of M is

$$M^\circledast = \mathrm{HOM}_{\mathscr{R}_n^\Lambda}(M, K), \qquad (2\text{C}.4)$$

where the $\mathscr{R}_n^\Lambda$-action on $M^\circledast$ is given by $(af)(m) = f(a^* m)$, for $a \in \mathscr{R}_n^\Lambda$, $f \in M^\circledast$ and $m \in M$.

We reserve the notation $\mathscr{R}_n^\Lambda$ for the cyclotomic KLR algebras that are defined using Q-polynomials such that $Q_{i,j}(u,v) \in K_0[u,v]$, such as the standard Q-polynomials given in (2B.2). For most of this paper we work with cyclotomic KLR algebras R_n^Λ that are defined using "deformations" of the standard Q-polynomials, such as those in Example 3A.2 below.

Remark 2C.5. There is an extensive literature for the cyclotomic KLR algebras of affine type A. Almost all of these papers work with the quiver $A_{e-1}^{(1)}$. In particular, in characteristic $p > 0$ the group algebra of the symmetric group is isomorphic to a cyclotomic KLR algebra of type $A_{p-1}^{(1)}$. As this paper simultaneously treats affine types A and C, we have chosen our notation to be consistent with the literature in affine type A and so that both quivers have the same vertex set $\{0, 1, \ldots, e-1\}$. This is why we work with quivers of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$ even though a more natural notation would be to work with quivers of types $A_e^{(1)}$ and $C_e^{(1)}$.

When K is positively graded the algebras in this paper fit into the general framework developed by Kang and Kashiwara in [31]. In particular, [31] proves the following result using an intricate induction on n.

Proposition 2C.6 (Kang–Kashiwara [31, Theorem 4.5]). *Suppose that K is a positively graded ring. Then $\mathsf{R}_n^\Lambda(K)$ is free as a K-module.*

Proof. By [31, Theorem 4.5], $\mathsf{R}_n^\Lambda(K)$ is projective as an $\mathsf{R}_{n-1}^\Lambda(K)$-module, which implies that $\mathsf{R}_n^\Lambda(K)$ is projective as an $\mathsf{R}_0^\Lambda(K)$-module. This gives the result since $\mathsf{R}_0^\Lambda(K) \cong K$. $\qquad\square$

A cyclotomic KLR datum $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$ is *standard* if $Q_{i,j}(u,v), W_i(u) \in K_0[u,v]$, for all $i, j \in I$. A (cyclotomic) KLR algebra is *standard* if its cyclotomic KLR datum is standard. Many papers in the literature define KLR algebras over positively graded rings $K = \bigoplus_{d \geq 0} K_d$ but in almost all cases they only consider standard Q-polynomials, like those in (2B.2). Non-standard Q-polynomials, such as those in Example 3A.2 below, play an important role in this paper.

Let $\Bbbk$ be a commutative integral domain with 1. Let $\mathbb{K}$ be a field that is a $\Bbbk$-algebra. (Often, $\mathbb{K}$ will be the field of fractions of $\Bbbk$.) Let $\underline{x}$ be a (possibly empty) tuple of indeterminates over $\Bbbk$. In this and later sections, we work over the polynomial ring $\Bbbk[\underline{x}] = \Bbbk[\underline{x}]$ and the Laurent polynomial ring $\mathbb{K}[\underline{x}^\pm] = \mathbb{K}[\underline{x}, \underline{x}^{-1}]$ with indeterminates $\underline{x}$. We consider $\Bbbk[\underline{x}]$ as a positively graded ring, and $\mathbb{K}[\underline{x}^\pm]$ as a $\mathbb{Z}$-graded ring, with the indeterminates in $\underline{x}$ all having degree 1; compare Remark 2A.1.

Fix a standard family of standard Q-polynomials $\mathbf{Q}_I$ together with a family of standard weight polynomials $\mathbf{W}_I$, both with coefficients in $\Bbbk$. Let $\mathscr{R}_n^\Lambda(\Bbbk) = \mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{W}_I, \Bbbk)$ be the corresponding cyclotomic KLR algebra over $\Bbbk$. A $\Bbbk[\underline{x}]$-*deformation* of $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$ is a cyclotomic KLR datum $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ such that $\mathbf{Q}_I^x = (Q_{i,j}^x(u,v))_{i,j \in I}$ is a family of Q-polynomials with coefficients in $\Bbbk[\underline{x}]$ and $\mathbf{W}_I^x = (W_i^x(u))_{i \in I}$ is a family of weight polynomials such that the polynomials in $\mathbf{Q}_I$ and $\mathbf{W}_I$ are the degree zero terms of the polynomials in $\mathbf{Q}_I^x$ and $\mathbf{W}_I^x$, respectively. That is, $\mathbf{Q}_I = \mathbf{Q}_{I|\underline{x}=0}^x$ and $\mathbf{W}_I = \mathbf{W}_{I|\underline{x}=0}^x$. (Here, and below, if $f(\underline{x}) \in \Bbbk[\underline{x}]$ then $f(\underline{x})_{|\underline{x}=0}$ is the constant term of $f(\underline{x})$.)

Notation 2C.7. Suppose that $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ is a $\Bbbk[\underline{x}]$-deformation of $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$. Let

$$\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}]) = \mathscr{R}_n^\Lambda(\mathbf{Q}_I^x, \mathbf{W}_I^x, \Bbbk[\underline{x}]) \qquad \text{and} \qquad \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm]) = \mathscr{R}_n^\Lambda(\mathbf{Q}_I^x, \mathbf{W}_I^x, \mathbb{K}[\underline{x}^\pm])$$

be the corresponding cyclotomic KLR algebras over $\Bbbk[\underline{x}]$ and $\mathbb{K}[\underline{x}^\pm]$, respectively.

The $\Bbbk[\underline{x}]$-deformations $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ used in this paper are part of the data of a *content system*, which is the subject of the next section. Non-trivial examples of the polynomials $\mathbf{Q}_I^x$ and $\mathbf{W}_I^x$ are given in Example 3A.2 below. We will sometimes use the deformed KLR algebras $\mathsf{R}_n(\Bbbk[\underline{x}]) = \mathsf{R}_n(\mathbf{Q}_I^x, \Bbbk[\underline{x}])$ and $\mathsf{R}_n(\mathbb{K}[\underline{x}^\pm]) = \mathsf{R}_n(\mathbf{W}_I^x, \mathbb{K}[\underline{x}^\pm])$ determined by the polynomials $\mathbf{Q}_I^x$. Let $Q_{ijk}^x(u, v, w)$ be the analogue of the three variable Q-polynomials in (2B.3) determined by $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$.

As before, let $\mathscr{R}_n^\Lambda(\Bbbk) = \mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{W}_I, \Bbbk)$ be the standard cyclotomic KLR algebra determined by $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$. By specialising the indeterminates in $\underline{x}$ to zero, the relations of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])|_{\underline{x}=0}$ coincide with those of the algebra $\mathscr{R}_n^\Lambda(\Bbbk)$, so we have the following trivial but useful observation

Proposition 2C.8. Suppose that $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ is a $\Bbbk[\underline{x}]$-deformation of $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$. Consider $\Bbbk$ as a graded $\Bbbk[\underline{x}]$-module by letting $\underline{x}$ act as zero. Then $\mathscr{R}_n^\Lambda(\Bbbk) \cong \mathsf{R}_n^\Lambda(\Bbbk) = \Bbbk \otimes \mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ as graded algebras.

That is, the standard cyclotomic KLR algebra $\mathscr{R}_n^\Lambda(\Bbbk)$ is isomorphic, as a graded algebra, to the specialisation of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ at $\underline{x} = 0$. Equivalently, $\mathscr{R}_n^\Lambda(\Bbbk)$ is the degree zero component, with respect to the $\underline{x}$-grading, of the algebra $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$. Note also that $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ is free as a $\Bbbk[\underline{x}]$-module by Proposition 2C.6.

It turns out that the representation theories of the algebras $\mathscr{R}_n^\Lambda(\Bbbk)$ and $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ are very similar, with the theory for $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ being slightly richer. In contrast, under the assumptions introduced below, the algebra $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ is semisimple, which makes it a useful tool for studying the algebras $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ and $\mathsf{R}_n^\Lambda(\Bbbk) \cong \mathscr{R}_n^\Lambda(\Bbbk)$. Note that $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ embeds into $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ by Proposition 2C.6.

2D. **Bases of KLR algebras.** For each $w \in \mathfrak{S}_n$, fix a *preferred reduced expression* $w = \sigma_{a_1} \ldots \sigma_{a_l}$ and define $\psi_w = \psi_{a_1} \ldots \psi_{a_l}$. In general, ψ_w depends on the choice of the preferred reduced expression for w.

Theorem 2D.1 (Khovanov–Lauda [36, Theorem 2.5], Rouquier [62, Theorem 3.7])**.** The algebra R_n is free as a K-algebra with basis $\{\psi_w y_1^{m_1} \ldots y_n^{m_n} 1_{\mathbf{i}} | w \in \mathfrak{S}_n, m_1, \ldots, m_n \in \mathbb{N}, \mathbf{i} \in I^n\}$.

Given $1 \le k < n$, define the *divided difference operator*

$$\partial_k \colon K[u_1, \ldots, u_n] \longrightarrow K[u_1, \ldots, u_n]; f \mapsto \frac{f - {}^{\sigma_k}f}{u_k - u_{k+1}}.$$

The next result follows easily from the relations in Definition 2C.2.

Lemma 2D.2 (Kang–Kashiwara [31, Lemma 4.2])**.** Let V be an R_n-module and $f \in K[u_1, \ldots, u_n]$ such that $f(y_1, \ldots, y_n) 1_{\mathbf{i}} V = 0$, for $\mathbf{i} \in I^n$. Suppose that $i_k = i_{k+1}$, for some $1 \le k < n$. Then

$$(\sigma_k f)(y_1, \ldots, y_n) 1_{\mathbf{i}} V = 0 \quad \text{and} \quad (\partial_k f)(y_1, \ldots, y_n) 1_{\mathbf{i}} V = 0.$$

Lemma 2D.3. Let $f = (u_1 - a_1) \ldots (u_1 - a_t) \in K[u_1, u_2]$, for $a_1, \ldots, a_t \in K$. Then

$$(\partial_1 f)(a_1, u) = (u - a_2) \ldots (u - a_t).$$

Proof. This follows easily by induction on t using the general identity $\partial_k(fg) = (^{\sigma_k}f)\partial_k g + (\partial_k f)g$. $\qquad\square$

Following [32, (1.6)], if $1 \le r < n$, define $\varphi_r = \sum_{\mathbf{i} \in I^n} \varphi_r \mathbf{1_i} \in \mathscr{R}_n$ by

$$\varphi_r \mathbf{1_i} = \begin{cases} (\psi_r(y_r - y_{r+1}) + 1)\mathbf{1_i} & \text{if } i_r = i_{r+1}, \\ \psi_r \mathbf{1_i} & \text{if } i_r \ne i_{r+1}. \end{cases} \tag{2D.4}$$

By definition, $\varphi_r \mathbf{1_i}$ is homogeneous and $\deg \varphi_r \mathbf{1_i} \ge 0$. If $w = \sigma_{a_1} \ldots \sigma_{a_m}$ is a reduced expression for $w \in \mathfrak{S}_d$ define $\varphi_w = \varphi_{a_1} \cdots \varphi_{a_m}$. Parts (b) and (c) of the next lemma show that φ_w does not depend on the choice of the reduced expression.

Lemma 2D.5 (Kang, Kashiwara and Kim [32, Lemma 1.5]). *The following identities hold:*

(a) *If $1 \le r < n$, then $\varphi_r^2 \mathbf{1_i} = (Q_{i_r, i_{r+1}}(y_r, y_{r+1}) + \delta_{i_r, i_{r+1}})\mathbf{1_i}$.*
(b) *If $1 \le r < n-1$, then $\varphi_r \varphi_{r+1} \varphi_r = \varphi_{r+1} \varphi_r \varphi_{r+1}$.*
(c) *If $|r - s| > 1$, then $\varphi_r \varphi_s = \varphi_s \varphi_r$.*
(d) *If $w \in \mathfrak{S}_n$ and $1 \le t \le n$, then $\varphi_w y_t = y_{w(t)} \varphi_w$.*
(e) *If $1 \le k < n$ and $w(k+1) = w(k) + 1$, then $\varphi_w \psi_k = \psi_{w(k)} \varphi_w$.*
(f) *If $w \in \mathfrak{S}_n$, then $\varphi_{w^{-1}} \varphi_w \mathbf{1_i} = \displaystyle\prod_{\substack{1 \le a < b \le n \\ w(a) > w(b)}} (Q_{i_a, i_b}(y_a, y_b) + \delta_{i_a, i_b})\mathbf{1_i}$.*

3. Content systems for KLR algebras

This section introduces *content systems*, which are the basic combinatorial tool underpinning this paper. Using content systems, we will give analogues of Young's seminormal forms for cyclotomic KLR algebras of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$, which are then used to prove the main results of this paper.

3A. **Content systems.** As in subsection 2C, in this subsection we let $\Bbbk$ be a commutative ring with 1 and fix a family of indeterminates $\underline{x}$ and work over the rings $\Bbbk[\underline{x}]$. In this section, $\mathbb{K}$ is the field of fractions of $\Bbbk$ and we will mainly work over $\mathbb{K}[\underline{x}^\pm]$. Let $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ be a $\Bbbk[\underline{x}]$-deformation of the standard cyclotomic KLR datum $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$. This section studies the algebras $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ and $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ under the additional assumption that they come equipped with a *content system*, which is the subject of this subsection.

As in subsection 2C, the cyclotomic KLR datum $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ determines a dominant weight $\Lambda = \Lambda_{\mathbf{W}_I^x} \in P^+$ of level ℓ. Fix an ℓ-tuple $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_\ell) \in I^\ell$, the ℓ-*charge*, such that $\Lambda = \sum_{l=1}^\ell \Lambda_{\rho_l}$.

Let Γ_ℓ be the quiver of type $A_\infty^{\times \ell} = A_\infty \times \cdots \times A_\infty$, with ℓ factors. More explicitly, Γ_ℓ has vertex set $J_\ell = \{1, 2, \ldots, \ell\} \times \mathbb{Z}$ and edges $(l, a) \longrightarrow (l, a+1)$, for all $(l, a) \in J_\ell$. Given $(k, a), (l, b) \in J_\ell$, write $(k, a) \!-\! (l, b)$ if $(k, a) \ne (l, b)$ and there is an arrow between (k, a) and (l, b), in either direction. Similarly, write $(k, a) \not\!-\! (l, b)$ if $(k, a) \ne (l, b)$ and there are no arrows between (k, a) and (l, b). By definition, if $k \ne l$ then $(k, a) \not\!-\! (l, b)$.

Definition 3A.1. A *content system* for $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ with values in $\Bbbk[\underline{x}]$ is a pair of maps (c, r), with

$$\mathsf{c} \colon J_\ell \longrightarrow \Bbbk[\underline{x}] \quad \text{and} \quad \mathsf{r} \colon J_\ell \longrightarrow I,$$

such that:

(a) If $1 \le l \le \ell$ then $\mathsf{r}(l, 0) = \rho_l$. Moreover, if $i \in I$ then
$$W_i^x(u) = \prod_{l \in [1,\ell], \rho_l = i} (u - \mathsf{c}(l, 0)).$$

(b) If $(k, a) \in J_\ell$ and $j \in \{\mathsf{r}(k, a-1), \mathsf{r}(k, a+1)\}$ then there exists a unit $\epsilon = \epsilon_{k,a,j} \in \Bbbk^\times$ such that
$$Q_{\mathsf{r}(k,a),j}^x(\mathsf{c}(k,a), v) = \epsilon \prod_{\substack{b \in \{a-1,a+1\} \\ \mathsf{r}(k,b)=j}} (\mathsf{c}(k,b) - v).$$

(c) If $(k, a), (l, b) \in J_\ell$ with $-n < a, b < n$ then $\mathsf{r}(k, a) = \mathsf{r}(l, b)$ and $\mathsf{c}(k, a) = \mathsf{c}(l, b)$ if and only if $(k, a) = (l, b)$.

The function c is the *content function* of the content system and r is the *residue function*. A content system (c, r) is *graded* if $\mathsf{c}(k, a)$ is homogeneous of degree $(\alpha_i | \alpha_i) = 2\mathsf{d}_i$, where $i = \mathsf{r}(k, a) \in I$ for $(k, a) \in J_\ell$.

Almost all of the content systems that we consider will be graded. Even though content systems are defined using a quiver of type Γ_ℓ, the quiver Γ is not assumed to be of this type. Notice that the roots of the polynomials $W_i^x(u)$ are pairwise distinct by conditions (a) and (c) of Definition 3A.1.

By definition, a content system (c, r) depends on the choices of $K = \Bbbk[\underline{x}]$, Γ, $\mathbf{Q}_I^x$, $\mathbf{W}_I^x$, $\boldsymbol{\rho}$ and n. To define a content system we need to specify all of this data. As we will see, content systems are closely related to semisimple representations. In particular, the theory below implies that content systems do not exist for most choices of (standard) Q-polynomials or over fields of positive characteristic. As we explain in Theorem 3F.8 below, if a content system exists then the algebra $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}^\pm])$ is uniquely determined up to non-homogeneous isomorphism. On the other hand, the examples below show that by deforming the standard Q-polynomials we can always find content systems for any standard cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$ of type $A_{e-1}^{(1)}$ or type $C_{e-1}^{(1)}$.

In the examples below, we give the minimum information necessary to specify the Q-polynomials. Recall from (2B.1) that $Q_{i,j}^x(u, v) = Q_{j,i}^x(v, u)$, $Q_{i,i}^x(u, v) = 0$ and that $Q_{i,j}^x(u, v) = 1$ if i and j are not connected in Γ, so we only need to specify one of the polynomials $Q_{i,j}^x(u, v)$ and $Q_{j,i}^x(v, u)$ whenever i and j are connected in Γ.

Example 3A.2. The content systems below are completely new, so the use of the adjectives *classical* and *reduced* is purely descriptive. For parts (a)–(e), we allow $n \ge 0$ to be arbitrary and we take $K = \mathbb{Z}[\underline{x}] = \mathbb{Z}[x]$, where $\underline{x} = (x)$ and x is an indeterminate of degree 1 over $\mathbb{Z}$. For the examples of level $\ell = 1$ we identify J_ℓ with $\mathbb{Z}$ via the obvious map $(1, a) \mapsto a$ and set $\boldsymbol{\rho} = (0)$. Throughout we use the weight polynomials $\mathbf{W}_I^x = (W_i(u))$, where $W_i^x(u) = \prod_{l \in [1,\ell], \rho_l = i}(u - \mathsf{c}(l, 0))$ in accordance with Definition 3A.1 (a). If $a, b \in \mathbb{Z}$ with $b \ne 0$ let $\lfloor \frac{a}{b} \rfloor$ be the integer part of $\frac{a}{b}$ and set $\bar{a} = a \pmod{e} \in I$.

(a) (The quiver Γ_ℓ) Let $\Gamma = \Gamma_\ell$, the quiver of type $A_\infty^{\times \ell}$, and let $\rho = ((1, 0), \ldots, (\ell, 0))$. Let $\mathbf{Q}_I^x = \mathbf{Q}_I$ be the standard Q-polynomials for Γ_ℓ given by (2B.2). Let r^{J_ℓ} be the identity map on J_ℓ and define c^{J_ℓ} to be identically zero. Then $(\mathsf{r}^{J_\ell}, \mathsf{c}^{J_\ell})$ is a content system for $\mathscr{R}_n^\Lambda = \mathsf{R}_n^\Lambda$, where $\Lambda = \Lambda_{(1,0)} + \cdots + \Lambda_{(\ell,0)}$.

(b) (Classical contents) Let Γ be a quiver a type $A_{e-1}^{(1)}$. Define
$$Q_{i,j}^x(u, v) = \begin{cases} (v - u + x^2)(u + x^2 - v) & \text{if } i \leftrightarrows j, \\ (u + x^2 - v) & \text{if } i \to j, \end{cases}$$

for $i, j \in I = \{0, 1, \ldots, e-1\}$. Then $\Lambda = \Lambda_0$ and $\ell = 1$. Then a content system for R_n^Λ is given by the functions $\mathsf{c}(a) = ax^2$ and $\mathsf{r}(a) = \overline{a}$, for $a \in \mathbb{Z}$. More explicitly, (c, r) is given by the table:

a	-1	0	1	$\ldots$	$e{-}1$	e	$\ldots$	$2e{-}1$	$2e$	$\ldots$	$3e{-}1$	$\ldots$
$\mathsf{r}(a)$	$e{-}1$	0	1	$\ldots$	$e{-}1$	0	$\ldots$	$e{-}1$	0	$\ldots$	$e{-}1$	$\ldots$
$\mathsf{c}(a)$	$-x^2$	0	x^2	$\ldots$	$(e{-}1)x^2$	ex^2	$\ldots$	$(2e{-}1)x^2$	$2ex^2$	$\ldots$	$(3e{-}1)x^2$	$\ldots$

Here, and below, the shading in the table highlights how the content function depends on $e = |I|$. The residue function r is the standard residue function for type $A_{e-1}^{(1)}$. We call this a *classical* content system because we recover the content function used in the classical semisimple representation theory of the symmetric groups by setting $x = 1$. For more details, see Example 3B.3.

To verify this example, and the examples that follow, observe that if $e > 2$ and $\mathsf{r}(a) = i$ and $\mathsf{c}(a) = cx$ then $(c+1)x - v = Q_{i,i+1}^x(\mathsf{c}(a), v) = \epsilon(\mathsf{c}(a+1) - v)$ by Definition 3A.1(c), so we require $\mathsf{c}(a+1) = (c+1)x$ (and $\epsilon = +1$). The calculation when $e = 2$ is similar except that we also need to inductively assume that $\mathsf{c}(a-1) = (c-1)x$. In this way, the content function c is completely determined by the Q^x-polynomials and the "initial condition" given by the weight polynomial $W_0^x(u) = u - \mathsf{c}(0) = u$.

There is a related content system $(\mathsf{c}', \mathsf{r}')$ that is, in a certain sense, dual to (c, r), which is given by $\mathsf{c}'(a) = \mathsf{c}(-a)$ and $\mathsf{r}'(a) = \mathsf{r}(-a)$, for $a \in \mathbb{Z}$. This is a special case of a general construction given in subsection 5E, so similar remarks apply to every example below.

(c) (Reduced contents) Let Γ be a quiver a type $A_{e-1}^{(1)}$. Define

$$Q_{i,j}^x(u, v) = \begin{cases} (u-v)(v+x^2-u) & \text{if } e = 2 \text{ and } (i, j) = (0, 1), \\ (u-v-x^2) & \text{if } e > 2 \text{ and } (i, j) = (0, e), \\ (u-v) & \text{if } i \to j \neq e, \end{cases}$$

for $i, j \in I$. As in the last example, $\Lambda = \Lambda_0$ and $\ell = 1$. Then a content system (c, r) for R_n^Λ is given by the functions $\mathsf{r}(a) = \overline{a}$ and $\mathsf{c}(a) = \lfloor \frac{a}{e} \rfloor x^2$, for all $a \in \mathbb{Z}$. More explicitly, (c, r) is given by the table:

a	-1	0	1	$\ldots$	$e-1$	e	$e+1$	$\ldots$	$2e-1$	$2e$	$2e+1$	$\ldots$	$3e-1$	$3e$	$\ldots$
$\mathsf{r}(a)$	$e-1$	0	1	$\ldots$	$e-1$	0	1	$\ldots$	$e-1$	0	1	$\ldots$	$e-1$	0	$\ldots$
$\mathsf{c}(a)$	$-x^2$	0	0	$\ldots$	0	x^2	x^2	$\ldots$	x^2	$2x^2$	$2x^2$	$\ldots$	$2x^2$	$3x^2$	$\ldots$

(d) (Classical contents) Let Γ be a quiver a type $C_{e-1}^{(1)}$. Define

$$Q_{i,j}^x(u, v) = \begin{cases} u - (v-x^2)^2 & \text{if } i = 0 \Rightarrow 1 = j, \\ (u+x^2)^2 - v & \text{if } i = e-1 \Leftarrow e = j, \\ (u-v+x^2) & \text{if } i \to j, \end{cases}$$

for $i, j \in I$. As in the last example, $\Lambda = \Lambda_0$ and $\ell = 1$. For an integer a set $a' = \lfloor \frac{a}{e-1} \rfloor$ and let $\overline{\overline{a}}$ be the unique integer such that $a \equiv \overline{\overline{a}} \pmod{2(e-1)}$ and $0 \leq \overline{\overline{a}} < 2e - 1$. A content system (c, r) for R_n^Λ is given by the functions

$$\mathsf{c}(a) = \begin{cases} (a+1)^2 x^4 & \text{if } \overline{\overline{a}} = 0, \\ (-1)^{a'}(a+1)x^2 & \text{if } \overline{\overline{a}} > 0 \end{cases} \quad \text{and} \quad \mathsf{r}(a) = \begin{cases} \overline{\overline{a}} & \text{if } \overline{\overline{a}} < e, \\ \overline{-\overline{\overline{a}} - 2} & \text{otherwise,} \end{cases}$$

for $a \in \mathbb{Z}$. More explicitly, (c, r) is given by the table:

a	-1	0	1	$\ldots$	$e-2$	$e-1$	e	$\ldots$	$2e-3$	$2e-2$	$2e-1$	$\ldots$
$\mathsf{r}(a)$	1	0	1	$\ldots$	$e-2$	$e-1$	$e-2$	$\ldots$	1	0	1	$\ldots$
$\mathsf{c}(a)$	$0x^2$	$1^2 x^4$	$2x^2$	$\ldots$	$(e-1)x^2$	$e^2 x^4$	$-(e+1)x^2$	$\ldots$	$-(2e-2)x^2$	$(2e-1)^2 x^4$	$2ex^2$	$\ldots$

Notice that we cannot set $\mathsf{c}(0) = 0$ because this would force $c(-1) = x^2 = \mathsf{c}(1)$, which would violate Definition 3A.1 (c). As we will see, the residue function r is the type $C_{e-1}^{(1)}$ residue function used by Ariki, Park and Speyer [8]. (Again, compare with Example 3B.3.)

(e) (Reduced contents) Let Γ be a quiver a type $C_{e-1}^{(1)}$. Define

$$Q_{i,j}^x(u,v) = \begin{cases} u - (v - x^2)^2 & \text{if } i = 0 \Rightarrow 1 = j, \\ (u + x^2)^2 - v & \text{if } i = e - 2 \Leftarrow e - 1 = j, \\ (u - v) & \text{if } i \to j, \end{cases}$$

for $i, j \in I$. As in the last example, $\Lambda = \Lambda_0$ and $\ell = 1$. A content system (c, r) for R_n^Λ is given by the functions

$$\mathsf{c}(a) = \begin{cases} (2a' + 1)^2 x^4 & \text{if } \overline{a} = 0, \\ (-1)^{a'} (2a' + 2) x^2 & \text{if } \overline{a} > 0 \end{cases} \quad \text{and} \quad \mathsf{r}(a) = \begin{cases} \overline{a} & \text{if } \overline{a} < e, \\ \overline{\overline{a}} - 2 & \text{otherwise,} \end{cases}$$

for $a \in \mathbb{Z}$. More explicitly, (c, r) is given by the table:

a	-1	0	1	$\ldots$	$e-2$	$e-1$	e	$\ldots$	$2e-3$	$2e-2$	$2e-1$	$\ldots$
$\mathsf{r}(a)$	1	0	1	$\ldots$	$e-2$	$e-1$	$e-2$	$\ldots$	1	0	1	$\ldots$
$\mathsf{c}(a)$	$0x^2$	$1^2 x^4$	$2x^2$	$\ldots$	$2x^2$	$3^2 x^4$	$-4x^2$	$\ldots$	$-4x^2$	$5^2 x^4$	$6x^2$	$\ldots$

(f) (Higher levels, many parameters) We extend the examples of content systems for level one algebras given in Examples (b)–(e) to algebras of level $\ell > 1$. Let Γ be a quiver of type $A_{e-1}^{(1)}$ or $C_{e-1}^{(1)}$, as above, and let $\Lambda \in P^+$ be a dominant weight with ℓ-charge $\boldsymbol{\rho} \in I^\ell$. Fix a family of indeterminates $\underline{x} = (x, x_1, \ldots, x_\ell)$ over $\mathbb{Z}$ and set $K = \mathbb{Z}[\underline{x}]$. Let $\mathbf{Q}_I^x$ be one of the families of Q-polynomials given in Examples (b)–(e) and let $(\mathsf{r}_0, \mathsf{c}_0)$ be the corresponding level one content system for $\Lambda = \Lambda_0$. A content system for the algebra R_n^Λ is then given by setting $\mathsf{r}(k, a) = i = \mathsf{r}_0(\rho_k + a) \in I$ and $\mathsf{c}(k, a) = \mathsf{c}_0(\rho_k + a) + x_k^{2\mathsf{d}_i}$, for $(k, a) \in J_\ell$.

(g) (Higher levels, one parameter) We can tweak the last example to give a content system that is defined over $\mathbb{Z}[x]$ for any $\ell \geq 1$. For example, in type $A_{e-1}^{(1)}$ to satisfy Definition 3A.1 (c) we can fix integers $c_1 > c_2 + 2n > \cdots > c_\ell + 2n$, and then specialise x_k to $c_k x^2$ in example (f), for $1 \leq k \leq \ell$. For type $C_{e-1}^{(1)}$, we need $c_1 > c_2 + 2n^2 > \cdots > c_\ell + 2n^2$. More generally, if $\Bbbk$ is a "large enough" ring such that $2n \cdot 1_\Bbbk \neq 0$ then a higher level content system with values in $\Bbbk[x]$ is given by defining $\mathsf{c}(k, a) = (c_k + a)x$, for suitable choices $c_1, \ldots, c_\ell \in \Bbbk$ such that $c_k + a = c_l + b$ only if $(k, a) = (l, b)$ for $-n < a, b < n$ and $1 \leq k, l \leq \ell$. The content system in Example 3A.2 (d)–(f) extend to higher levels in essentially the same way except that extra care is required in choosing the "initial contents" $\mathsf{c}(k, 0)$, for $1 \leq k \leq \ell$, to ensure that Definition 3A.1 (c) is satisfied. We leave the details to the reader.

(h) (Non-graded content systems) In characteristic zero, the content systems given in Examples (a)–(f) are all graded content systems for any $n \geq 0$. By Proposition 2C.8, the standard cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$ is isomorphic to the algebra $\mathsf{R}_n^\Lambda / \underline{x}\mathsf{R}_n^\Lambda$ obtained by specialising all of the indeterminates at 0. We can obtain ungraded content systems for R_n^Λ over $\mathbb{Z}$ by specialising the indeterminates to a fixed prime p. Reducing modulo p, it follows that the algebra $\mathsf{R}_n^\Lambda / p\mathsf{R}_n^\Lambda$ is isomorphic to the corresponding standard cyclotomic KLR algebra $\mathscr{R}_n^\Lambda(\mathbb{Z}/p\mathbb{Z})$, defined over the finite field $\mathbb{Z}/p\mathbb{Z}$.

(i) (Finite type) It is possible to construct content systems for some quivers of finite type, such as type A_e, but we do not consider these here. The main difference is that in finite type the irreducible modules defined in Proposition 3C.2 below exist only for certain ℓ-partitions.

In particular, (b)–(e) and (g) of Example 3A.2 show the following:

Lemma 3A.3. Let Γ be a quiver of type $A_{e-1}^{(1)}$ or $C_{e-1}^{(1)}$ and suppose that $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$ is a standard cyclotomic KLR datum for $\mathscr{R}_n^\Lambda(\mathbb{Z})$. Then there exists a $\mathbb{Z}[x]$-deformation $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ of $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$ such that the algebra $\mathsf{R}_n^\Lambda = \mathsf{R}_n^\Lambda(\mathbf{Q}_I^x, \mathbf{W}_I^x, \mathbb{Z}[x])$ has a content system (c, r) with values in $\mathbb{Z}[x]$.

If $\Bbbk$ is a field of characteristic $p > 0$ then the functions (c, r) from Example 3A.2 (b)–(h) define content systems only for "small" values of n because the uniqueness requirement of Definition 3A.1 (c) fails whenever n is too large. For example, in characteristic 2 examples (c) and (d) define contents systems in type $C_{e-1}^{(1)}$ only when $n = 1$. However, since content systems for cyclotomic KLR algebras of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$ always exist over $\mathbb{Z}[x]$ we can use content systems to construct cellular bases for these algebras by base change from $\mathbb{Z}[x]$.

Lemma 3A.4. Suppose that (c, r) is a content system and $i = \mathsf{r}(l, a)$ and $j = \mathsf{r}(l, a+1)$, for $(l, a) \in J_\ell$. Then $j \text{---} i$ and, in particular, $i \neq j$. Moreover, $j = \mathsf{r}(l, a-1)$ if and only if $i \Longrightarrow j$ or $j \leftrightarrows i$.

Proof. By Definition 3A.1 (b), $Q_{i,j}^x(\mathsf{c}(k, a), v)$ is a nonzero polynomial in v, so $i \neq j$ and $(\alpha_i | \alpha_j) \neq 0$ by (2B.1). Hence, $j \text{---} i$. If, in addition, $\mathsf{r}(l, a-1) = j$ then $Q_{i,j}^x(\mathsf{c}(k, a), v)$ is a polynomial of degree 2 in v. $\square$

Lemma 3A.4 implies that if (c, r) is a content system for R_n^Λ and Γ is a quiver of type $A_{e-1}^{(1)}$ and $1 \leq l \leq \ell$ then either $\mathsf{r}(l, a) = \overline{\rho_l + a}$ or $\mathsf{r}(l, a) = \overline{\rho_l - a}$, for all $a \in \mathbb{Z}$. Similarly, if Γ is of type $C_{e-1}^{(1)}$ then $\mathsf{r}(l, a) = \mathsf{r}(\rho_l + a)$ or $\mathsf{r}(l, a) = \mathsf{r}(\rho_l - a)$, where r is the level one residue function used in (c) and (d) of Example 3A.2. As sketched in example (b) above, the content function is almost uniquely determined by the cyclotomic KLR datum $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ because $\mathsf{c}(l, 0)$ is a root of the polynomial $W_{\mathsf{r}(l,0)}^x(u)$ and $\mathsf{c}(l, a+1)$ is a root of the polynomial $Q_{i,j}^x(\mathsf{c}(l, a), v)$, where $i = \mathsf{r}(l, a)$ and $j = \mathsf{r}(l, a+1)$. So, defining a content system (c, r) amounts to finding a $\Bbbk[\underline{x}]$-deformation $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$ of the cyclotomic KLR datum.

3B. **Tableau combinatorics.** By Definition 3A.1, a content system (c, r) with values in $\Bbbk[\underline{x}]$, is just a pair of functions. This subsection extends these functions to maps on ℓ-partitions and standard tableaux, and the next subsection uses this combinatorics

to construct irreducible graded representations of the deformed KLR algebra R_n^Λ over $\mathbb{K}[\underline{x}^\pm]$. These representations, which are modelled on Young's seminormal forms, are the foundations that this paper are built on. We start by setting up the required combinatorics.

A *partition* is a weakly decreasing sequence of positive integers. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition, then the *size* of λ is $|\lambda| = \sum_{t=1}^r \lambda_t$, and we set $\lambda_t = 0$ for $t > r$. An *ℓ-partition* is an ordered tuple $\boldsymbol{\lambda} = (\lambda^{(1)}|\ldots|\lambda^{(\ell)})$ of partitions. The *size* of $\boldsymbol{\lambda}$ is $|\boldsymbol{\lambda}| = \sum_{c=1}^\ell |\lambda^{(c)}|$. Let $\mathcal{P}_n^\ell$ be the set of ℓ-partitions of size n. We identify partitions and 1-partitions in the obvious way.

If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$ then $\boldsymbol{\lambda}$ *dominates* $\boldsymbol{\mu}$, written $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$, if

$$\sum_{c=1}^{k-1} \left|\lambda^{(c)}\right| + \sum_{r=1}^s \lambda_r^{(k)} \geq \sum_{c=1}^{k-1} \left|\mu^{(c)}\right| + \sum_{r=1}^s \mu_r^{(k)}, \qquad \text{for } 1 \leq k \leq \ell \text{ and } s \geq 1.$$

Similarly, the *reverse dominance order* $\unlhd$ is defined by $\boldsymbol{\lambda} \unlhd \boldsymbol{\mu}$ if $\boldsymbol{\mu} \unrhd \boldsymbol{\lambda}$. Write $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ and $\boldsymbol{\mu} \lhd \boldsymbol{\lambda}$ if $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$.

In this paper, we consider the set of ℓ-partitions $\mathcal{P}_n^\ell$ both as the poset $(\mathcal{P}_n^\ell, \unrhd)$, under dominance, and as the poset $(\mathcal{P}_n^\ell, \unlhd)$, under reverse dominance. As we will see, the interplay between the dominance and reverse dominance partial orders corresponds to a duality in the representation theory.

Let $\mathcal{N}_n^\ell = \{(k, r, c) \mid 1 \leq k \leq \ell \text{ and } r, c \in \mathbb{Z}_{>0}\}$ be the set of *nodes*, which we consider as a totally ordered set under the *lexicographic order* $\geq$. We also use the reverse lexicographic order $\leq$. (We emphasize that our use of, and notation for, the lexicographic and reverse lexicographic orders coincides with how we use the notation the dominance and reverse dominance orders.) Identify an ℓ-partition $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ with its *Young diagram*, which is the set of nodes:

$$\boldsymbol{\lambda} = \left\{ (k, r, c) \,\middle|\, 1 \leq k \leq \ell \text{ and } 1 \leq c \leq \lambda_r^{(k)} \right\}.$$

Remark 3B.1. In this paper the node $(k, r, c) \in \mathcal{N}_n^\ell$ sits in component k, row r and column c of an ℓ-partition. This is different to the conventions of [19], where the components of the nodes are indexed in order (r, c, k). The convention used in this paper is preferable because many places in this paper order the nodes lexicographically, or reverse lexicographically, looking first at the component index and then at the row and column indices.

A $\boldsymbol{\lambda}$-*tableau* is a bijection $\mathsf{t} \colon \boldsymbol{\lambda} \longrightarrow \{1, 2, \ldots, n\}$. The group $\mathfrak{S}_n$ naturally acts from the left on the set of all $\boldsymbol{\lambda}$-tableaux. A $\boldsymbol{\lambda}$-tableau t is *standard* if $\mathsf{t}(k, r, c) < \mathsf{t}(k, r + 1, c)$, and $\mathsf{t}(k, r, c) < \mathsf{t}(k, r, c + 1)$, whenever these nodes are in $\boldsymbol{\lambda}$. That is, the entries in each component of a standard tableau increase along rows and down columns. Let $\mathrm{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux. For $\mathcal{P} \subseteq \bigcup_{n \geq 0} \mathcal{P}_n^\ell$, set

$$\mathrm{Std}(\mathcal{P}) = \{\mathsf{s} \mid \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}\}$$

and

$$\mathrm{Std}^2(\mathcal{P}) = \{(\mathsf{s}, \mathsf{t}) \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}\}.$$

Write $\mathrm{Shape}(\mathsf{t}) = \boldsymbol{\lambda}$ if $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Given $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and $1 \leq m \leq n$ let $\mathsf{t}_{\downarrow m}$ be the subtableau of t containing the numbers in $\{1, \ldots, m\}$. That is, $\mathsf{t}_{\downarrow m}$ is the restriction of t to $\mathsf{t}^{-1}(\{1, \ldots, m\})$.

Armed with this notation, we can now extend (c, r) to functions on ℓ-partitions and tableaux.

Definition 3B.2. Let $A = (k, r, c) \in \mathcal{N}_n^\ell$ be a node. The *content* of A is $c(A) = \mathsf{c}(k, c - r) \in \Bbbk[\underline{x}]$ and the *residue* of A is $\mathsf{r}(A) = \mathsf{r}(k, c - r) \in I$. If $i \in I$, then A is an *i-node* if $\mathsf{r}(A) = i$.

Let $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ a standard $\boldsymbol{\lambda}$-tableau, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Fix $1 \leq m \leq n$. Define

$$\mathsf{c}_m(\mathsf{t}) = \mathsf{c}\left(\mathsf{t}^{-1}(m)\right) \qquad \text{and} \qquad \mathsf{r}_m(\mathsf{t}) = \mathsf{r}\left(\mathsf{t}^{-1}(m)\right),$$

which are the *content* and *residue* of m in t, respectively. Similarly, the *content sequence* and the *residue sequence* of t are

$$\mathsf{c}(\mathsf{t}) = (\mathsf{c}_1(\mathsf{t}), \ldots, \mathsf{c}_n(\mathsf{t})) \in \Bbbk[\underline{x}]^n \quad \text{and} \quad \mathsf{r}(\mathsf{t}) = (\mathsf{r}_1(\mathsf{t}), \ldots, \mathsf{r}_n(\mathsf{t})) \in I^n,$$

respectively. Let $\mathrm{Std}(\mathbf{i}) = \{\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell) \mid \mathsf{r}(\mathsf{t}) = \mathbf{i}\}$ be the set of standard tableaux with residue sequence $\mathbf{i}$.

Example 3B.3. Suppose that $\ell = 1$ and let $\boldsymbol{\lambda} = (5, 3, 2)$. Using the content systems from parts (b)–(e) of Example 3A.2 for the quivers $A_2^{(1)}$ and $C_2^{(1)}$, the different residues and contents in $\boldsymbol{\lambda}$ are:

Quiver	Example 3A.2	Contents					Residues				
$A_2^{(1)}$	(b)	0	x	$2x$	$3x$	$4x$	0	1	2	0	1
		$-x$	0	x			2	0	1		
		$-2x$	$-x$				1	2			
$A_2^{(1)}$	(c)	0	0	0	x	x	0	1	2	0	1
		$-x$	0	0			2	0	1		
		$-x$	$-x$				1	2			
$C_2^{(1)}$	(d) and (e)	x^2	$2x$	$3^2 x^2$	$4x$	$5^2 x^2$	0	1	2	1	0
		0	x^2	$2x$			1	0	1		
		$-2x^2$	0				2	1			

The symmetric group $\mathfrak{S}_n$ acts on I^n and $\Bbbk[\underline{x}]^n$ by place permutations. Write $w\mathsf{c}(\mathsf{t})$ and $w\mathsf{r}(\mathsf{t})$ for the content and residue sequences obtained by acting with w, for $w \in \mathfrak{S}_n$.

From subsection 2B, recall that $\sigma_j = (j, j+1) \in \mathfrak{S}_n$, for $1 \leq j < n$.

Lemma 3B.4. Suppose that $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\mu})$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$.
(a) We have $\mathsf{s} = \mathsf{t}$ if and only if $\mathsf{c}(\mathsf{s}) = \mathsf{c}(\mathsf{t})$ and $\mathsf{r}(\mathsf{s}) = \mathsf{r}(\mathsf{t})$.
(b) Suppose $\boldsymbol{\lambda} = \boldsymbol{\mu}$, $\mathsf{c}(\mathsf{s}) = \sigma_m \mathsf{c}(\mathsf{t})$ and $\mathsf{r}(\mathsf{s}) = \sigma_m \mathsf{r}(\mathsf{t})$, for some $1 \leq m < n$. Then $\mathsf{s} = \sigma_m \mathsf{t}$.

Proof. (a) If $\mathsf{s} \neq \mathsf{t}$ then let m be minimal such that $\mathsf{s}_{\downarrow m} \neq \mathsf{t}_{\downarrow m}$. Set $\boldsymbol{\mu} = \mathrm{Shape}(\mathsf{s}_{\downarrow(m-1)})$ and let $A = (k, r, c) = \mathsf{s}^{-1}(m)$ and $B = (l, s, d) = \mathsf{t}^{-1}(m)$. Then A and B are addable nodes of $\boldsymbol{\mu}$. If $k = l$ then it is well-known and easy to check that $c - r \neq d - s$. Consequently, $(k, c - r) \neq (l, d - s)$ and, hence, $(\mathsf{c}_m(\mathsf{s}), \mathsf{r}_m(\mathsf{s})) \neq (\mathsf{c}_m(\mathsf{t}), \mathsf{r}_m(\mathsf{t}))$ by Definition 3A.1(c). Therefore, $(\mathsf{c}(\mathsf{s}), \mathsf{r}(s)) \neq (\mathsf{c}(\mathsf{t}), \mathsf{r}(t))$, giving (a).

Now consider (b). By assumption, $c(\sigma_m s) = c(t)$ and $r(\sigma_m s) = r(t)$, so $\sigma_m s = t$ by (a). Hence, $s = \sigma_m t$ as claimed. $\square$

Part (b) implies that if $\sigma_m t \notin \mathrm{Std}(\mathcal{P}_n^\ell)$ then no standard tableau has content sequence $\sigma_m c(t)$ and residue sequence $\sigma_m r(t)$.

Given $1 \le m < n$ and $t \in \mathrm{Std}(\mathbf{i})$, for $\mathbf{i} \in I^n$, define scalars in $\mathbb{K}[\underline{x}^\pm]$ by

$$Q_m(t) = Q_{r_m(t),r_{m+1}(t)}^{\underline{x}}(c_m(t), c_{m+1}(t)) - \frac{\delta_{r_m(t),r_{m+1}(t)}}{\left(c_{m+1}(t) - c_m(t)\right)^2}. \tag{3B.5}$$

Note that $Q_{r_m(t),r_{m+1}(t)}^{\underline{x}}(c_m(t), c_{m+1}(t)) \in \mathbb{k}[\underline{x}]$, so $Q_m(t) \in \mathbb{k}[\underline{x}]$ unless $r_m(t) = r_{m+1}(t)$. Further, if $r_m(t) = r_{m+1}(t)$ then $Q_m(t)$ is well-defined because $c_m(t) \ne c_{m+1}(t)$ by Definition 3A.1 (c) and Definition 3B.2.

The following result looks innocuous but it is the key to constructing the seminormal representations of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$.

Lemma 3B.6. Suppose that $t \in \mathrm{Std}(\boldsymbol{\lambda})$ and let $s = \sigma_m t$, where $1 \le m < n$. Then $Q_m(t) \ne 0$ if and only if $s \in \mathrm{Std}(\boldsymbol{\lambda})$. Consequently, if (c, r) is a graded content system and $s \in \mathrm{Std}(\boldsymbol{\lambda})$ then $Q_m(t)$ is a nonzero homogeneous element of $\mathbb{K}[\underline{x}^\pm]$.

Proof. For the duration of the proof set $(k, a, b) = t^{-1}(m)$ and $(l, c, d) = t^{-1}(m + 1)$, so that $c_m(t) = c(k, b - a)$, $r_m(t) = r(k, b - a)$, $c_{m+1}(t) = c(l, d - c)$ and $r_{m+1}(t) = r(l, d - c)$

Suppose first that $s = \sigma_m t \in \mathrm{Std}(\boldsymbol{\lambda})$. If $r_m(t) = r_{m+1}(t)$ then $c_m(t) \ne c_{m+1}(t)$ by Lemma 3B.4, so that $Q_m(t) = -1/(c_{m+1}(t) - c_m(t))^2 \ne 0$. Now suppose that $r_m(t) \ne r_{m+1}(t)$. By (3B.5), $Q_m(t) = 0$ only if $c(l, d - c)$ is a root of $Q_{r(k,a),r(l,d-c)}^{\underline{x}}(c(k, b-a), v)$. By axioms (b) and (c) of Definition 3A.1, $c(l, d-c)$ is not a root of $Q_{r(k,b-a),r(l,d-c)}^{\underline{x}}(c(k, b - a), v)$ if $(k, a) \not\to (l, c)$, so we can assume that $k = l$ and $d - c = b - a \pm 1$ since otherwise $(k, a) \not\to (l, c)$. However, if $d - c = b - a \pm 1$ then m and $m + 1$ are on adjacent diagonals in $\boldsymbol{\lambda}$, which is not possible since t and $s = \sigma_m t$ are both standard. Hence, $Q_m(t) \ne 0$ when s is standard.

Now, suppose that $s \notin \mathrm{Std}(\boldsymbol{\lambda})$. This happens if and only if m and $m + 1$ are in the same row or same column of the same component of t. That is, $k = l$ and either $a = c$ and $d = b + 1$, or $b = d$ and $c = a + 1$. That is, either $r_{m+1}(t) = r(k, b - a + 1)$ and $c_{m+1}(t) = c(k, b - a + 1)$, or $r_{m+1}(t) = r(k, b - a - 1)$ and $c_{m+1}(t) = c(k, b - a - 1)$. Hence, in both cases, $Q_m(t) = Q_{r_m(t),r_{m+1}(t)}^{\underline{x}}(c_m(t), c_{m+1}(t)) = 0$ by Definition 3A.1 (b).

Finally, if (c, r) is a graded content system and $s \in \mathrm{Std}(\boldsymbol{\lambda})$ then $Q_m(t) \ne 0$, so it is homogeneous and nonzero in view of the remarks before the lemma. Moreover, $Q_m(t)$ has the expected degree by (2B.1) since $c(k, a)$ is homogeneous of degree $(\alpha_i | \alpha_i)$ by Definition 3A.1, where $i = r(k, a)$. $\square$

3C. Seminormal forms.

We continue to assume that (c, r) is a (graded) content system that takes values in $\mathbb{k}[\underline{x}]$. Even though (c, r) takes values in $\mathbb{k}[\underline{x}]$ the representations that we construct are modules for the $\mathbb{K}[\underline{x}^\pm]$-algebra $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ because the action of the KLR algebra on these modules involves the scalars $Q_m(t)$ from (3B.5), and these scalars typically belong to $\mathbb{K}[\underline{x}^\pm]$, not $\mathbb{k}[\underline{x}]$. To prove irreducibility we also use the following elements, which are not defined over $\mathbb{k}[\underline{x}]$.

Definition 3C.1. Let $\mathbf{i} \in I^n$. If $\mathsf{t} \in \mathrm{Std}(\mathbf{i})$, define

$$F_\mathsf{t} = \prod_{k=1}^n \prod_{\substack{\mathsf{s} \in \mathrm{Std}(\mathbf{i}) \\ \mathsf{c}_k(\mathsf{s}) \neq \mathsf{c}_k(\mathsf{t})}} \frac{y_k - \mathsf{c}_k(\mathsf{s})}{\mathsf{c}_k(\mathsf{t}) - \mathsf{c}_k(\mathsf{s})} \cdot 1_\mathbf{i} \in \mathsf{R}_n^\Lambda \left(\mathbb{K}[\underline{x}^\pm]\right).$$

If (c, r) is a graded content system then F_t is homogeneous element of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ of degree 0 since $\mathsf{c}_k(\mathsf{s})$ appears in the product only if $\mathsf{r}_k(\mathsf{t}) = \mathsf{r}_k(\mathsf{s})$. Note that $1_\mathbf{i} = 1_{\mathsf{r}(\mathsf{t})}$, for $\mathsf{t} \in \mathrm{Std}(\mathbf{i})$.

The next result gives a generalisation of Young's classical seminormal forms to KLR algebras with content systems. As noted in subsection 2A, $\mathbb{K}[\underline{x}^\pm]$ is a graded field, which explains the claim that the module $V_{\boldsymbol\lambda}$ is an irreducible graded $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module. Recall that $\mathbb{K}$ is the field of fractions of $\Bbbk$.

Proposition 3C.2. Let $\boldsymbol\lambda \in \mathcal{P}_n^\ell$. Suppose that there exist scalars

$$\{\beta_k(\mathsf{t}) \in \mathbb{K}[\underline{x}^\pm] \mid 1 \leq k < n \text{ and } \mathsf{t}, \sigma_k \mathsf{t} \in \mathrm{Std}(\boldsymbol\lambda)\}$$

satisfying the following conditions:
   (a) $\beta_k(\sigma_k\mathsf{t})\beta_k(\mathsf{t}) = Q_k(\mathsf{t})$ if $1 \leq k < n$ and $\sigma_k\mathsf{t} \in \mathrm{Std}(\boldsymbol\lambda)$;
   (b) $\beta_k(\mathsf{t})\beta_t(\sigma_k\mathsf{t}) = \beta_t(\mathsf{t})\beta_k(\sigma_l\mathsf{t})$ if $1 \leq k, l < n$, $|k - l| \neq 1$ and $\sigma_k\mathsf{t}, \sigma_l\mathsf{t} \in \mathrm{Std}(\boldsymbol\lambda)$;
   (c) $\beta_k(\sigma_{k+1}\sigma_k\mathsf{t})\beta_{k+1}(\sigma_k\mathsf{t})\beta_k(\mathsf{t}) = \beta_{k+1}(\sigma_k\sigma_{k+1}\mathsf{t})\beta_k(\sigma_{k+1}\mathsf{t})\beta_{k+1}(\mathsf{t})$ if $1 \leq k < n - 1$ and all the tableaux appearing in this equation are standard.

Then there exists a graded $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module $V_{\boldsymbol\lambda}$ that is free as an $\mathbb{K}[\underline{x}^\pm]$-module with homogeneous basis $\{v_\mathsf{t} \mid \mathsf{t} \in \mathrm{Std}(\boldsymbol\lambda)\}$ and where $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-action is determined by

$$1_\mathbf{i} v_\mathsf{t} = \delta_{\mathbf{i}\,\mathsf{r}(\mathsf{t})} v_\mathsf{t}, \quad y_k v_\mathsf{t} = \mathsf{c}_k(\mathsf{t}) v_\mathsf{t}, \quad \psi_k v_\mathsf{t} = \beta_k(\mathsf{t}) v_{\sigma_k \mathsf{t}} + \frac{\delta_{\mathsf{r}_k(\mathsf{t}), \mathsf{r}_{k+1}(\mathsf{t})}}{\mathsf{c}_{k+1}(\mathsf{t}) - \mathsf{c}_k(\mathsf{t})} v_\mathsf{t}$$

for all admissible k, $\mathbf{i} \in I^n$ and $\mathsf{t} \in \mathrm{Std}(\boldsymbol\lambda)$ and where $v_\mathsf{s} = 0$ if $\mathsf{s} \notin \mathrm{Std}(\boldsymbol\lambda)$. Moreover, if $\mathbb{K}[\underline{x}^\pm]$ is a graded field then $V_{\boldsymbol\lambda}$ is irreducible.

Proof. To prove that $V_{\boldsymbol\lambda}$ is an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module it is enough to check that the action of the generators of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ on $V_{\boldsymbol\lambda}$ respects the relations of Definition 2C.2. The action respects the cyclotomic relation

$$W_{i_1}(y_1)1_\mathbf{i} = 0, \qquad \text{for all } \mathbf{i} \in I^n,$$

by Definition 3A.1(a). The relations (KLR1)–(KLR4) and (KLR6) are easily checked by direct calculation, with condition (b) of the proposition used for (KLR4) and relation (KLR5) following by Lemma 3B.4(b).

To check relation (KLR7), for each $\mathsf{t} \in \mathrm{Std}(\boldsymbol\lambda)$ it is enough to prove that

$$\psi_k^2 1_\mathbf{i} v_\mathsf{t} = Q_{i_k, i_{k+1}}^x(y_k, y_{k+1}) 1_\mathbf{i} v_\mathsf{t}, \qquad 1 \leq k < n \text{ and } \mathbf{i} \in I^n. \tag{3C.3}$$

If $\sigma_k\mathsf{t}$ is not standard, then $\mathsf{r}_k(\mathsf{t}) \neq \mathsf{r}_{k+1}(\mathsf{t})$ by Lemma 3B.4(b) and $Q_k(\mathsf{t}) = 0$ by Lemma 3B.6. So,

$$\psi_k^2 1_\mathbf{i} v_\mathsf{t} = 0 = \delta_{\mathbf{i}\,\mathsf{r}(\mathsf{t})} Q_{\mathsf{r}_k(\mathsf{t}), \mathsf{r}_{k+1}(\mathsf{t})}^x(\mathsf{c}_k(\mathsf{t}), \mathsf{c}_{k+1}(\mathsf{t})) v_\mathsf{t} = Q_{i_k, i_{k+1}}^x(y_k, y_{k+1}) 1_\mathbf{i} v_\mathsf{t}.$$

On the other hand, if $\sigma_k\mathsf{t}$ is standard then

$$\psi_k^2 1_\mathbf{i} v_\mathsf{t} = \left(\beta_k(\sigma_k\mathsf{t})\beta_k(\mathsf{t}) + \frac{\delta_{\mathsf{r}_k(\mathsf{t}), \mathsf{r}_{k+1}(\mathsf{t})}}{(\mathsf{c}_{k+1}(\mathsf{t}) - \mathsf{c}_k(\mathsf{t}))^2}\right) v_\mathsf{t} = Q_{\mathsf{r}_k(\mathsf{t}), \mathsf{r}_{k+1}(\mathsf{t})}^x(y_k, y_{k+1}) 1_\mathbf{i} v_\mathsf{t}, \tag{3C.4}$$

where the second equality follows using condition (a) of the proposition and the definition of $Q_k(\mathsf{t})$. Hence, (3C.3) holds in all cases.

We now verify relation (KLR8). Let $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, $1 \leq k < n-1$ and $\mathbf{i} \in I^n$. To simplify notation, set $i = i_k$, $i' = i_{k+1}$ and $i'' = i_{k+2}$ and define $\mathsf{t}_1 = \sigma_k \mathsf{t}$, $\mathsf{t}_2 = \sigma_{k+1} \mathsf{t}$, $\mathsf{t}_{21} = \sigma_{k+1} \mathsf{t}_1$, $\mathsf{t}_{12} = \sigma_k \mathsf{t}_2$ and $\mathsf{t}_{121} = \sigma_k \mathsf{t}_{21} = \sigma_{k+1} \mathsf{t}_{12}$. Note that if $\mathsf{t}_1 \notin \mathrm{Std}(\boldsymbol{\lambda})$, then $\mathsf{t}_{21} \notin \mathrm{Std}(\boldsymbol{\lambda})$. Similarly, $\mathsf{t}_{12} \notin \mathrm{Std}(\boldsymbol{\lambda})$ if $\mathsf{t}_2 \notin \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathsf{t}_{121} \notin \mathrm{Std}(\boldsymbol{\lambda})$ if either $\mathsf{t}_{12} \notin \mathrm{Std}(\boldsymbol{\lambda})$ or $\mathsf{t}_{21} \notin \mathrm{Std}(\boldsymbol{\lambda})$. Using these facts and some routine, although slightly lengthy calculations for the first equality (cf. [26, Lemma 3.8]), shows that

$$
(\psi_k \psi_{k+1} \psi_k - \psi_{k+1} \psi_k \psi_{k+1}) 1_{\mathbf{i}} v_{\mathsf{t}}
$$

$$
= \left(\delta_{ii'} \delta_{i'i''} \frac{\mathsf{c}_k(\mathsf{t}) + \mathsf{c}_{k+2}(\mathsf{t}) - 2\mathsf{c}_{k+1}(\mathsf{t})}{(\mathsf{c}_{k+1}(\mathsf{t}) - \mathsf{c}_k(\mathsf{t}))^2 (\mathsf{c}_{k+2}(\mathsf{t}) - \mathsf{c}_{k+1}(\mathsf{t}))^2} \right.
$$
$$
\left. + \delta_{ii''} \frac{\beta_k(\mathsf{t})\beta_k(\mathsf{t}_1) - \beta_{k+1}(\mathsf{t})\beta_{k+1}(\mathsf{t}_2)}{\mathsf{c}_{k+2}(\mathsf{t}) - \mathsf{c}_k(\mathsf{t})} \right) v_{\mathsf{t}}
$$
$$
+ \left(\beta_k(\mathsf{t}_{21})\beta_{k+1}(\mathsf{t}_1)\beta_k(\mathsf{t}) - \beta_{k+1}(\mathsf{t}_{12})\beta_k(\mathsf{t}_2)\beta_{k+1}(\mathsf{t}) \right) v_{\mathsf{t}_{121}}
$$
$$
= \delta_{ii''} \frac{Q_k(\mathsf{t}) - Q_{k+1}(\mathsf{t})}{\mathsf{c}_{k+2}(\mathsf{t}) - \mathsf{c}_k(\mathsf{t})} v_{\mathsf{t}} = \delta_{i'i''} \frac{Q_{ij}^x(y_{k+2}, y_{k+1}) - Q_{ij}^x(y_k, y_{k+1})}{y_k - y_{k+2}} 1_{\mathbf{i}} v_{\mathsf{t}}
$$
$$
= Q_{ii'i''}(y_k, y_{k+1}, y_{k+2}) 1_{\mathbf{i}} v_{\mathsf{t}}
$$

where we have used conditions (a) and (c) of the proposition, and (3B.5), for the second equality. Hence, relation (KLR8) is satisfied. We have now shown that all of the relations in Definition 2C.2 are satisfied, so $V_{\boldsymbol{\lambda}}$ is an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module.

We next prove that $V_{\boldsymbol{\lambda}}$ is an irreducible graded $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module when $\mathbb{K}[\underline{x}^\pm] = \mathbb{K}[x^\pm]$ is a graded field. First note that

$$
F_{\mathsf{t}} v_{\mathsf{s}} = \delta_{\mathsf{ts}} v_{\mathsf{s}}, \qquad \text{for all } \mathsf{t}, \mathsf{s} \in \mathrm{Std}(\mathcal{P}_n^\ell), \tag{3C.5}
$$

by Definition 3C.1 and Lemma 3B.4 since v_{s} is a eigenvector for the y_k's. Now suppose that $v \in V_{\boldsymbol{\lambda}}$ belongs to a graded $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-submodule M of $V_{\boldsymbol{\lambda}}$ and write $v = \sum_{\mathsf{s}} r_{\mathsf{s}} v_{\mathsf{s}}$, for $r_{\mathsf{s}} \in \mathbb{K}[\underline{x}^\pm]$. If $r_{\mathsf{t}} \neq 0$ then $r_{\mathsf{t}} v_{\mathsf{t}} = F_{\mathsf{t}} v \in M$. Hence, $v_{\mathsf{t}} \in M$ since M is a graded submodule and $\mathbb{K}[\underline{x}^\pm]$ is a graded field. To show that $M = V_{\boldsymbol{\lambda}}$ it is enough to show that $v_{\sigma_k \mathsf{t}} \in \mathsf{R}_n^\Lambda v_{\mathsf{t}}$ whenever $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\sigma_k \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, for $1 \leq k < n$. Under these assumptions, $F_{\sigma_k \mathsf{t}} \psi_k v_{\mathsf{t}} = \beta_k(\mathsf{t}) v_{\sigma_k \mathsf{t}}$. So it is enough to prove that $\beta_k(\mathsf{t}) \neq 0$, which follows from assumption (a) since $\beta_k(\mathsf{t})\beta_k(\sigma_k \mathsf{t}) = Q_k(\mathsf{t})$ and $Q_k(\mathsf{t}) \neq 0$ by Lemma 3B.6.

Finally, it remains to determine the grading on $V_{\boldsymbol{\lambda}}$. Since we have already shown that the action of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ on $V_{\boldsymbol{\lambda}}$ respects the relations and that $V_{\boldsymbol{\lambda}}$ is irreducible, and $\{v_{\mathsf{s}}\}$ is a homogeneous basis, we can fix a grading on $V_{\boldsymbol{\lambda}}$ by fixing the degree of one of these basis elements. The degrees of the other basis elements are now uniquely determined by the $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-action since $V_{\boldsymbol{\lambda}}$ is cyclic. $\square$

Remark 3C.6. Suppose that the content system (c, r) is not graded and takes values in $\mathbb{k}$. Then the argument of Proposition 3C.2 shows that $V_{\boldsymbol{\lambda}}$ is an irreducible $\mathsf{R}_n^\Lambda(\mathbb{K})$-module.

Proposition 3C.2 constructs the module $V_{\boldsymbol{\lambda}}$ subject to the existence of suitable scalars $\beta_k(\mathsf{t})$, for $1 \leq k < n$ and $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. There are two natural choices (see (4A.8)), but for now we define:

$$
\beta_k(\mathsf{t}) = \begin{cases} 1 & \text{if } \sigma_k \mathsf{t} \rhd \mathsf{t}, \\ Q_k(\sigma_k \mathsf{t}) & \text{if } \mathsf{t} \rhd \sigma_k \mathsf{t}. \end{cases} \tag{3C.7}
$$

Lemma 3C.8. *The coefficients $\beta_k(\mathsf{t})$ defined by (3C.7) satisfy the conditions of Proposition 3C.2.*

Proof. The only condition that is not obvious is that the β-coefficients satisfy the "β-braid relation"

$$\beta_k(\sigma_{k+1}\sigma_k\mathsf{t})\beta_{k+1}(\sigma_k\mathsf{t})\beta_k(\mathsf{t}) = \beta_{k+1}(\sigma_k\sigma_{k+1}\mathsf{t})\beta_k(\sigma_{k+1}\mathsf{t})\beta_{k+1}(\mathsf{t}),$$

for $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and $1 \le r < d$ such that all the tableaux in this identity are standard. In fact, since $\beta_k(\mathsf{t})$ depends only on the nodes $\mathsf{t}^{-1}(k)$ and $\mathsf{t}^{-1}(k+1)$, we have $\beta_k(\mathsf{t}) = \beta_{k+1}(\sigma_k\sigma_{k+1}\mathsf{t})$, $\beta_{k+1}(\sigma_k\mathsf{t}) = \beta_k(\sigma_{k+1}\mathsf{t})$ and $\beta_k(\sigma_{k+1}\sigma_k\mathsf{t}) = \beta_{k+1}(\mathsf{t})$. These equalities imply the β-braid relation above. □

For each $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ Proposition 3C.2 constructs an irreducible $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module $V_{\boldsymbol{\lambda}}$. We now fix the choice of β-coefficients given by (3C.7) and define $V_{\boldsymbol{\lambda}}$ to be the $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module defined by Proposition 3C.2.

If t is a standard tableau then it is not clear from Definition 3C.1 that the element F_t is nonzero. This now follows by virtue of (3C.5) and Lemma 3C.8.

Corollary 3C.9. *Let* $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, *for* $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. *Then* $F_\mathsf{t} \ne 0$ *in* $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$.

The next result shows that the representations constructed in Proposition 3C.2 are pairwise non-isomorphic and, up to isomorphism, independent of the choice of β-coefficients in Proposition 3C.2.

Corollary 3C.10. *Suppose that* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$. *Then* $V_{\boldsymbol{\lambda}} \cong V_{\boldsymbol{\mu}}$ *as* $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-*modules if and only if* $\boldsymbol{\lambda} = \boldsymbol{\mu}$. *Moreover, up to isomorphism,* $V_{\boldsymbol{\lambda}}$ *is independent of the choice of homogeneous scalars* $\{\beta_k(\mathsf{t}) \,|\, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ *satisfying conditions* (a)–(b) *of Proposition 3C.2.*

Proof. Suppose first that $\boldsymbol{\lambda} \ne \boldsymbol{\mu}$. By Lemma 3B.4 and (3C.5), if $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $F_\mathsf{t}V_{\boldsymbol{\lambda}} \ne 0$ and $F_\mathsf{t}V_{\boldsymbol{\mu}} = 0$. Hence, $V_{\boldsymbol{\lambda}} \not\cong V_{\boldsymbol{\mu}}$.

To prove the second statement suppose that $V_{\boldsymbol{\lambda}} \cong V_{\boldsymbol{\mu}}$ and that $V_{\boldsymbol{\lambda}} = \langle v_\mathsf{t} | \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})\rangle$ and $V'_{\boldsymbol{\lambda}} = \langle v'_\mathsf{t} | \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})\rangle$ are two $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-modules with homogeneous structure constants $\{\beta_r(\mathsf{t})\}$ and $\{\beta'_r(\mathsf{t})\}$, respectively, satisfying the conditions of Proposition 3C.2. In particular, note that if $\sigma_r\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $\beta_r(\mathsf{t})$ and $\beta'_r(\mathsf{t})$ are both nonzero by Proposition 3C.2 (a) and Lemma 3B.6. Define a $\mathbb{K}[\underline{x}^\pm]$-linear map $\theta \colon V_{\boldsymbol{\lambda}} \to V'_{\boldsymbol{\lambda}}$ inductively as follows. First, fix any tableau $\mathsf{t}_1 \in \mathrm{Std}(\boldsymbol{\lambda})$ and set $\theta(v_{\mathsf{t}_1}) = v'_{\mathsf{t}_1}$. By way of induction, suppose that $\theta(v_{\mathsf{t}_1}), \dots, \theta(v_{\mathsf{t}_{m-1}})$ have been defined and that $\mathsf{t}_m \in \mathrm{Std}(\boldsymbol{\lambda}) \setminus \{\mathsf{t}_1, \dots, \mathsf{t}_{m-1}\}$ is a standard tableau such that $\mathsf{t}_m = \sigma_k\mathsf{t}_l$, where $1 \le k < n$ and $1 \le l < m$. Set

$$\theta(v_{\mathsf{t}_m}) = \frac{1}{\beta_k(\mathsf{t}_l)}\left(\psi_k - \frac{\delta_{r_k(\mathsf{t}_m), r_k(\mathsf{t}_l)}}{c_k(\mathsf{t}_m) - c_k(\mathsf{t}_l)}\right)\theta(v_{\mathsf{t}_l}).$$

By Proposition 3C.2, if $\theta(v_{\mathsf{t}_l}) \ne 0$ then $\theta(v_{\mathsf{t}_m}) \ne 0$. By induction, $\theta(v_\mathsf{t})$ is defined and nonzero for all $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. In particular, θ is a $\mathbb{K}[\underline{x}^\pm]$-module isomorphism. Moreover, $\theta(v_\mathsf{t}) \in F_\mathsf{t}V'_{\boldsymbol{\lambda}} = \mathbb{K}[\underline{x}^\pm]v'_\mathsf{t}$ by (3C.5), so $\theta(v_\mathsf{t}) = \xi_\mathsf{t}v'_\mathsf{t}$, for some scalar $\xi_\mathsf{t} \in \mathbb{K}[\underline{x}^\pm]$. Since $V_{\boldsymbol{\lambda}}$ and $V'_{\boldsymbol{\lambda}}$ are both $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-modules, the construction of Proposition 3C.2 guarantees that θ is an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module homomorphism and that $V_{\boldsymbol{\lambda}} \cong V'_{\boldsymbol{\lambda}}$, as claimed. □

Motivated by the seminormal forms of Proposition 3C.2, we now use (graded) content systems to study the algebras $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$. Our next goal is to prove a semisimplicity result for $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$, which we will use to study the algebras $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}])$ and $\mathscr{R}_n^\Lambda(\mathbb{k})$.

3D. **Weight modules.** This subsection looks at $\mathsf{R}_n(\mathbb{K}[\underline{x}^\pm])$=modules that are spanned by simultaneous eigenvectors of $y_1, \ldots, y_n$. This is a first step towards finding a basis for $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$.

Suppose that V is an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm]$-module. Let $\mathsf{c} = (c_1, \ldots, c_n) \in \mathbb{K}[\underline{x}^\pm]^n$ and $\mathbf{i} \in I^n$, where c_k is homogeneous of degree $(\alpha_{i_k}|\alpha_{i_k})$, for $1 \leq k \leq n$. The $(\mathsf{c}, \mathbf{i})$-*weight space* of V is the $\mathbb{K}[\underline{x}^\pm]$-module

$$V_{\mathsf{c},\mathbf{i}} = \{v \in V \mid y_k 1_{\mathbf{i}} v = c_k v \text{ for } 1 \leq k \leq n\}.$$

A *weight module* is an $\mathsf{R}_n(\mathbb{K}[\underline{x}^\pm])$-module that is a direct sum of $(\mathsf{c}, \mathbf{i})$-weight spaces and is of finite rank as a $\mathbb{K}[\underline{x}^\pm]$-module. For example, the module $V_{\boldsymbol{\lambda}}$ of Proposition 3C.2 is an $\mathsf{R}_n(\mathbb{K}[\underline{x}^\pm])$-weight module.

The next result is similar to the classification of the irreducible representations of the affine Hecke algebras of rank 2. The connection with the seminormal forms of Proposition 3C.2 is evident in part (b).

Proposition 3D.1. Let V be a weight module for $\mathsf{R}_2(\mathbb{K}[\underline{x}^\pm])$ and suppose that $0 \neq v \in V$ is a homogeneous vector such that $y_1 v = c_1 v$, $y_2 v = c_2 v$ and $1_{ij} v = v$, where $c_1, c_2 \in \mathbb{K}[\underline{x}^\pm]$ and $i, j \in I$ with c_1 and c_2 homogeneous of the appropriate degree. Then one of the following of the following mutually exclusive cases occurs:

(a) If $Q_{ij}^x(c_1, c_2) \neq 0$ then $\langle v, w \rangle$ is an $\mathsf{R}_2(\mathbb{K}[\underline{x}^\pm])$-weight module of rank 2 such that $w = \psi_1 v$, $y_1 w = c_2 w$, $y_2 w = c_1 w$ and $1_{ji} w = w$.

(b) If $i = j$ then $c_1 \neq c_2$ and $V = \langle v, w \rangle$ is an $\mathsf{R}_2(\mathbb{K}[\underline{x}^\pm])$-weight module of rank 2 such that $w = (\psi_1 - \frac{1}{c_2 - c_1})v$, $y_1 w = c_2 w$, $y_2 w = c_1 w$ and $1_{ii} w = w$.

(c) If $i \neq j$ and $Q_{ij}^x(c_1, c_2) = 0$ then either $V = \langle v \rangle$ is an $\mathsf{R}_2(\mathbb{K}[\underline{x}^\pm])$-weight module of rank 1 with $\psi_1 v = 0$, or $\langle v, w \rangle$ is an $\mathsf{R}_2(\mathbb{K}[\underline{x}^\pm])$-weight module of rank 2 with $w = \psi_1 v$ and $\psi_1 w = 0$.

Proof. As in the statement of the proposition, suppose that $v \in V$ and $1_{\mathbf{i}} v = v$, $y_1 v = c_1 v$ and $y_2 v = c_2 v$. As in part (a), we first assume that $Q_{ij}^x(c_1, c_1) \neq 0$. Then $i \neq j$ since $Q_{ii}^x(u, v) = 0$. Let $w = \psi_1 v$. Then $\psi_1 w = Q_{ij}^x(c_1, c_2) v \neq 0$, so $w \neq 0$. The remaining claims in (a) now follow easily from the relations.

Next, suppose that (b) holds, so that $i = j$. If $\psi_1 v = 0$ then $0 = y_2 \psi_1 v = (\psi_1 y_1 + 1)v = v$, which is a contradiction so $\psi_1 v \neq 0$. By assumption, $V = \langle v, \psi_1 v \rangle$ and v is a weight vector, so $\psi_1 v + av$ must be a weight vector for some $0 \neq a \in \mathbb{K}[\underline{x}^\pm]$. Applying the relations, $y_2(\psi_1 v + av) = c_1 \psi_1 v + (ac_2 + 1)v$. Since this is a weight vector, comparing coefficients, $ac_1 = ac_2 + 1$. Hence, $c_1 \neq c_2$ and $w = \psi_1 v - \frac{1}{c_2 - c_1} v$ is a weight vector. The remaining claims in part (b) now follow easily.

Finally, it remains to consider (c), when $i \neq j$ and $Q_{ij}^x(c_1, c_2) = 0$. If $w = \psi_1 v \neq 0$ then $\psi_1 w = \psi_1^2 v = 0$ since $Q_{ij}^x(c_1, c_2) = 0$. In this case $1_{ij} v = v$ and $1_{ji} w = w$, so $\langle v, w \rangle$ is $\mathbb{K}[\underline{x}^\pm]$-free of rank 2. On the other hand, if $w = 0$ then $\mathbb{K}[\underline{x}^\pm]v$ is an irreducible $\mathsf{R}_2(\mathbb{K}[\underline{x}^\pm])$-module that is free of rank 1 as claimed. $\square$

The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{K}[\underline{x}^\pm]^n$ and I^n by place permutations. Recall the definition of the elements $\varphi_r \in \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ from (2D.4).

Corollary 3D.2. Let V be a weight module for $\mathsf{R}_n(\mathbb{K}[\underline{x}^\pm])$ and let $0 \neq v \in V_{\mathsf{c},\mathbf{i}}$ be homogeneous, for $\mathbf{i} \in I^n$ and $\mathsf{c} \in \mathbb{K}[\underline{x}^\pm]^n$. Suppose that $1 \leq r < n$ and that $(c_r, i_r) \neq (c_{r+1}, i_{r+1})$. Then $0 \neq \varphi_r v \in V_{s_r \mathsf{c}, s_r \mathbf{i}}$.

Proof. By (KLR6), $\psi_r v \in V_{s_r\mathsf{c}, s_r\mathbf{i}} + \delta_{i_r i_{r+1}} V_{\mathsf{c},\mathbf{i}}$. In particular, $\psi_r v \in V_{s_r\mathsf{c}, s_r\mathbf{i}}$ if $i_r \neq i_{r+1}$. If $i_r = i_{r+1}$ then $\varphi_r v \in V_{s_r\mathsf{c}, s_{r+1}\mathbf{i}}$ in view of Proposition 3D.1 (b) since

$$\psi_r 1_\mathbf{i} = (\psi_r(y_r - y_{r+1}) + 1)1_\mathbf{i}$$

in this case. Finally, φ_r is invertible in $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ by Lemma 2D.5 (a), so $\varphi_r v \neq 0$. $\qquad\square$

3E. **Content reduction.** One of the main results of this subsection is Corollary 3E.9, which shows that $\{F_\mathsf{t} \,|\, \mathsf{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)\}$ is a family of pairwise orthogonal idempotents in $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$. To prove this we argue by induction on n to classify all weight modules for $\mathscr{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ by showing that the eigenvalues of $y_1, \ldots, y_n$ are given by the content functions on the standard tableaux.

If $\mathbf{i} \in I^n$ and $1 \leq m \leq n$ define $\mathbf{i}_{\downarrow m} = (i_1, \ldots, i_m) \in I^m$. If $\mathbf{i} \in I^m$ and $j \in I$ let $\mathbf{i}j = (i_1, \ldots, i_m, j) \in I^{m+1}$. Let $I_{\operatorname{Std}}^m = \{\mathsf{r}(\mathsf{s}) \,|\, \mathsf{s} \in \operatorname{Std}(\mathcal{P}_m^\ell)\}$ be the set of residue sequences of the standard tableaux of size m. If $\mathbf{j} \in I^m$ set

$$1_{\mathbf{j},n} = \sum_{\substack{\mathbf{i} \in I^n \\ \mathbf{i}_{\downarrow m} = \mathbf{j}}} 1_\mathbf{i} \quad \in \mathsf{R}_n^\Lambda\left(\mathbb{K}[\underline{x}^\pm]\right).$$

By (KLR1), if $\mathbf{i}, \mathbf{j} \in I^m$ then $1_{\mathbf{i},n} 1_{\mathbf{j},n} = \delta_{\mathbf{ij}} 1_{\mathbf{i},n}$ and, moreover, $1_{\mathsf{R}_n^\Lambda} = \sum_{\mathbf{j} \in I^m} 1_{\mathbf{j},n}$.

Let V be an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module and suppose that $1 \leq m \leq n$. For $\mathsf{s} \in \operatorname{Std}(\mathcal{P}_m^\ell)$ define V_s to be the simultaneous $\mathsf{c}_k(\mathsf{s})$-eigenspace of y_k acting on $1_{\mathsf{r}(\mathsf{s})} V$, for $1 \leq k \leq m$. That is, V_s is the $\mathbb{K}[\underline{x}^\pm]$-module

$$V_\mathsf{s} = \left\{ v \in 1_{\mathsf{r}(\mathsf{s}),n} V \,\middle|\, y_k v = \mathsf{c}_k(\mathsf{s}) v \text{ for } 1 \leq k \leq m \right\}.$$

An $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module V is *m-content reduced* if V is free as a $\mathbb{K}[\underline{x}^\pm]$-module and $V = \sum_{\mathsf{s} \in \operatorname{Std}(\mathcal{P}_m^\ell)} V_\mathsf{s}$ as a $\mathbb{K}[\underline{x}^\pm]$-module. The module V is *content reduced* if it is n-content reduced. If V is m-content reduced then the sum $V = \sum_{\mathsf{s} \in \operatorname{Std}(\mathcal{P}_m^\ell)} V_\mathsf{s}$ is necessarily direct because $V_\mathsf{s} \cap V_\mathsf{t} = 0$, for $\mathsf{s} \neq \mathsf{t} \in \operatorname{Std}(\mathcal{P}_m^\ell)$. In particular, every content reduced module is a weight module for $\mathsf{R}_n(\mathbb{K}[\underline{x}^\pm])$.

Suppose that V is an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module. We can consider V as an $\mathsf{R}_n(\mathbb{K}[\underline{x}^\pm])$-module using the canonical surjection $\mathsf{R}_n(\mathbb{K}[\underline{x}^\pm]) \to \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$. By Theorem 2D.1 and Definition 2C.2, over any ring there is an algebra embedding of R_m into R_n that sends $1_\mathbf{j}$ to $1_{\mathbf{j},n}$, for $\mathbf{j} \in I^m$. Therefore, V is an $\mathsf{R}_m(\mathbb{K}[\underline{x}^\pm])$-module by restriction. Since V is an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module, it is killed by the weight polynomials $\mathbf{W}_I^x$, so the $\mathsf{R}_m(\mathbb{K}[\underline{x}^\pm])$-action on V makes V into an $\mathsf{R}_m^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module. Let $\operatorname{Res}_{\mathsf{R}_m^\Lambda}(V)$ and $\operatorname{Res}_{\mathsf{R}_m}(V)$ be the restrictions of V to an $\mathsf{R}_m^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module and $\mathsf{R}_m(\mathbb{K}[\underline{x}^\pm])$-module, respectively.

The modules $V_{\boldsymbol{\lambda}}$ of Proposition 3C.2 are content reduced. Conversely, we have:

Lemma 3E.1. Let V be an m-content reduced $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module, where $1 \leq m \leq n$. Then

$$\operatorname{Res}_{\mathsf{R}_m^\Lambda}(V) \cong \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}_m^\ell} V_{\boldsymbol{\lambda}}^{\oplus a_{\boldsymbol{\lambda}}}, \qquad \text{for some } a_{\boldsymbol{\lambda}} \geq 0,$$

as an $\mathsf{R}_m^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module.

Proof. Since V is m-content reduced, by definition, it is free as a $\mathbb{K}[\underline{x}^\pm]$-module and has a homogeneous basis of weight vectors. Let $v_\mathsf{s}' \in V_\mathsf{s}$ be such a basis vector, where $\mathsf{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \in \mathcal{P}_m^\ell$. To prove the lemma it is enough to show that $\mathsf{R}_m^\Lambda(\mathbb{K}[\underline{x}^\pm]) v_\mathsf{s} \cong V_{\boldsymbol{\lambda}}$. Let $d_\mathsf{s} = d_\mathsf{s}^{\triangleleft} \in \mathfrak{S}_n$ be the permutation such that $\mathsf{s} = d_\mathsf{s} \mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}$ and set $v_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}} = \varphi_{w^{-1}} v_\mathsf{s}'$ and $v_\mathsf{t} = \psi_{d_\mathsf{t}} v_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}}$, where $\mathsf{t} = d_\mathsf{t} \mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}$ for $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Then v_t is a nonzero element of V_t by Corollary 3D.2.

Moreover, $\{v_{\mathsf{t}} \,|\, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ is linearly independent since these weight spaces are disjoint. Let W be the submodule of V spanned by the $\{v_{\mathsf{t}} \,|\, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$. By Proposition 3D.1 and Lemma 3B.4(b), if $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $1 \le k < n$ then there exist scalars $\beta_k(\mathsf{t})$ such that

$$\psi_k v_{\mathsf{t}} = \beta_k(\mathsf{t}) v_{\sigma_k \mathsf{t}} + \tfrac{\delta_{r_k(\mathsf{t}), r_{k+1}(\mathsf{t})}}{\mathsf{c}_{k+1}(\mathsf{t}) - \mathsf{c}_k(\mathsf{t})} v_{\mathsf{t}}.$$

In particular, W is an $\mathsf{R}_m^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$-submodule of V. Further, since W is an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$-module, relations (KLR7), (KLR4) and (KLR8) imply that these coefficients satisfy conditions (a)–(c), respectively, of Proposition 3C.2. (In fact, the reader can check that $\beta_k(\mathsf{t}) \in \mathbb{k}[\underline{x}]$ is given by (3C.7).) Therefore, $W \cong V_{\boldsymbol{\lambda}}$ by Corollary 3C.10, completing the proof. $\qquad\square$

Remark 3E.2. Using Definition 2C.2, it is easy to see that if $1 \le m \le n$ then there is a surjective algebra map from $\mathsf{R}_m^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$ onto the subalgebra of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$ generated by $\psi_1, \ldots, \psi_{m-1}, y_1, \ldots, y_m$ and $1_{\mathbf{j},n}$, for $\mathbf{j} \in I^m$. It follows from Corollary 4A.12 below that this map is an isomorphism, but we cannot prove this yet. For now it is enough to work with m-content reduced modules, which are combinatorial shadows of these isomorphisms.

The next lemma can be viewed as the module theoretic origin of Definition 3C.1. In the lemma we assume that $c_1, \ldots, c_N \in \mathbb{k}[\underline{x}]$ only because (c, r) takes values in $\mathbb{k}[\underline{x}]$.

Lemma 3E.3. Let V be an $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$-module. Suppose that $\prod_{k=1}^N (y_r - c_k) 1_{\mathbf{i}} V = 0$, where $1 \le r \le n$ and $c_1, \ldots, c_N \in \mathbb{k}[\underline{x}]$ are pairwise distinct and $\mathbf{i} \in I^n$. Then

$$1_{\mathbf{i}} V = \bigoplus_{k=1}^N V_{\mathbf{i},k}, \qquad \text{where} \qquad V_{\mathbf{i},k} = \{v \in 1_{\mathbf{i}} V \,|\, y_r v = c_k v\}, \text{ for } 1 \le k \le N.$$

Proof. This follows by applying the easy (polynomial) identity

$$\sum_{k=1}^N \prod_{l \ne k} \frac{(y_r - c_l)}{(c_k - c_l)} = 1.$$

$\qquad\square$

We now show that every $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$-module is content reduced, which is the linchpin of this subsection.

Theorem 3E.4. Let V be a $\mathbb{K}[\underline{x}^{\pm}]$-free $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$-module. Then V is content reduced.

Proof. We argue induction on m to show that V is m-content reduced, for $1 \le m \le n$.

Suppose $m = 1$. Fix $\mathbf{i} = (i) \in I$. By Definition 3A.1(a),

$$\prod_{\substack{1 \le l \le \ell \\ \rho_l = i}} (y_1 - \mathsf{c}(l, 0)) 1_{\mathbf{i}} = 0 \qquad \Longrightarrow \qquad \prod_{\substack{1 \le l \le \ell \\ \rho_l = i}} (y_1 - \mathsf{c}(l, 0)) 1_{\mathbf{i}} V = 0.$$

In view of Definition 3A.1(c) and Lemma 3B.4(a), there is a self-evident bijection between the sets of standard tableaux $\mathrm{Std}(\mathcal{P}_1^\ell)$ and contents $\{\mathsf{c}(l, 0) \,|\, 1 \le l \le \ell\}$. Hence, the module V is 1-content reduced by Lemma 3E.3. This establishes the base case of our induction.

Let $1 \le m < n$. By induction, we assume that V is m-content reduced. For the inductive step we show that $V = \bigoplus_{\mathsf{t} \in \mathrm{Std}(\mathcal{P}_{m+1}^\ell)} V_{\mathsf{t}}$. Fix $\mathsf{s} \in \mathrm{Std}(\mathcal{P}_m^\ell)$ and $j \in I$ and set $V_{\mathsf{s},j} = 1_{\mathsf{r}(\mathsf{s})j,n} V_{\mathsf{s}}$. To show that V is $(m+1)$-content reduced it is enough to prove that

$$V_{\mathsf{s},j} = \sum_{\substack{\mathsf{t} \in \mathrm{Std}(\mathcal{P}_{m+1}^\ell) \\ \mathsf{t}_{\downarrow m} = \mathsf{s} \text{ and } \mathsf{r}_{m+1}(\mathsf{t}) = j}} V_{\mathsf{t}}, \qquad \text{for all } \mathsf{s} \in \mathrm{Std}(\mathcal{P}_m^\ell) \text{ and } j \in I. \qquad (3\text{E}.5)$$

Let $\mathrm{Add}_j(\mathsf{s}) = \{\mathsf{t}^{-1}(m+1) | \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell), \mathsf{t}_{\downarrow m} = \mathsf{s} \text{ and } \mathsf{r}_{m+1}(\mathsf{t}) = j\}$ be the set of addable j-nodes for s. By Lemma 3E.3, to prove (3E.5) it suffices to show that

$$\prod_{(l,r,c) \,\in\, \mathrm{Add}_j(\mathsf{s})} (\mathsf{c}(l, c-r) - y_{m+1}) V_{\mathsf{s},j} = 0, \tag{3E.6}$$

since the contents $\mathsf{c}(l, c-r)$ in this product are distinct by Lemma 3B.4. By convention, empty products are 1, so the last displayed equation includes the claim that $V_{\mathsf{s},j} = 0$ if there are no standard tableaux with residue sequence $\mathbf{i} = \mathsf{r}(\mathsf{s})j$.

Let $(k, a, b) = \mathsf{s}^{-1}(m)$ and set $\mathsf{u} = \mathsf{s}_{\downarrow(m-1)} \in \mathrm{Std}(\mathcal{P}_{m-1}^\ell)$. Define $\mathrm{Add}_j(\mathsf{u})$ as above.
We consider two cases.

Case 1. $j = \mathsf{r}_m(\mathsf{s})$:   By assumption, $\mathrm{Add}_j(\mathsf{u}) = \mathrm{Add}_j(\mathsf{s}) \sqcup \{(k, a, b)\}$. Hence, in view of Lemma 2D.2 and Lemma 2D.3, it follows by induction that

$$\prod_{(l,r,c) \,\in\, \mathrm{Add}_j(\mathsf{u}) \setminus \{(k,a,b)\}} (\mathsf{c}(l, c-r) - y_{m+1}) V_{\mathsf{s},j} = 0.$$

Hence, (3E.6) holds when $j = \mathsf{r}_m(\mathsf{s})$.

Case 2. $j \neq \mathsf{r}_m(\mathsf{s})$:   Set $A = \{(k, r, c) \in \mathcal{N}_n^\ell | \mathsf{r}(k, r, c) = j \text{ and } (r, c) = (a+1, b) \text{ or } (r, c) = (a, b+1)\}$. Then $|A| = -\langle \alpha_{\mathsf{r}_m(\mathsf{s})}, \alpha_j \rangle$ and $\mathrm{Add}_j(\mathsf{s}) \subseteq \mathrm{Add}_j(\mathsf{u}) \sqcup A$ (disjoint union). By Definition 3A.1 (b),

$$Q_{\mathsf{r}_m(\mathsf{s}),j}^{\underline{x}}(\mathsf{c}_m(\mathsf{s}), v) = \epsilon \prod_{(k,r,c) \,\in\, A} (\mathsf{c}(k, c-r) - v), \qquad \text{for some } \epsilon \in \mathbb{k}^\times.$$

Hence, by induction, if $v \in V_{\mathsf{s},j}$ then $\psi_m^2 v = \epsilon \prod_{(k,r,c) \in A} (\mathsf{c}(k, c-r) - y_{m+1}) v$. Therefore,

$$\prod_{(l,r,c) \,\in\, \mathrm{Add}_j(\mathsf{u}) \sqcup A} (\mathsf{c}(l, c-r) - y_{m+1}) V_{\mathsf{s},j} = \prod_{(l,r,c) \,\in\, \mathrm{Add}_j(\mathsf{u})} (\mathsf{c}(l, c-r) - y_{m+1}) \cdot \psi_m^2 V_{\mathsf{s},j}$$

$$= \psi_m \prod_{(l,r,c) \,\in\, \mathrm{Add}_j(\mathsf{u})} (\mathsf{c}(l, c-r) - y_m) \cdot \psi_m V_{\mathsf{s},j}$$

$$\subseteq \psi_m \prod_{(l,r,c) \,\in\, \mathrm{Add}_j(\mathsf{u})} (\mathsf{c}(l, c-r) - y_m) \cdot 1_{\mathsf{r}(\mathsf{u})j\mathsf{r}_m(\mathsf{s}),n} V_{\mathsf{u}}$$

$$= 0,$$

where the second equality uses (KLR6) and the last equality follows by induction. In particular, (3E.6) holds by Lemma Lemma 3E.3 whenever $\mathrm{Add}_j(\mathsf{s}) = \mathrm{Add}_j(\mathsf{u}) \sqcup A$. We need to consider the cases when $\mathrm{Add}_j(\mathsf{s})$ is properly contained in $\mathrm{Add}_j(\mathsf{u}) \sqcup A$, where Lemma 3E.3 potentially gives weight spaces of V_{s} that are not indexed by standard tableaux.

Suppose first that $(k, a, b+1) \in A$ and $(k, a, b+1) \notin \mathrm{Add}_j(\mathsf{s})$. Define $c_l = \mathsf{c}_l(\mathsf{s})$ and $i_l = \mathsf{r}_l(\mathsf{s})$, for $1 \leq l \leq m$ and set $c_{m+1} = \mathsf{c}(k, b+1-a)$ and $i_{m+1} = \mathsf{r}(k, b+1-a)$. Let $\mathsf{c} = (c_1, \ldots, c_{m+1})$ and $\mathbf{i} = (i_1, \ldots, i_{m+1})$. By Lemma 3E.3, $V_{\mathsf{c},\mathbf{i}}$ is a (possibly zero) summand of V_{s}. By way of contradiction, suppose that $V_{\mathsf{c},\mathbf{i}} \neq 0$ and fix a nonzero homogeneous vector $v \in V_{\mathsf{c},\mathbf{i}}$. Let $\boldsymbol{\lambda} = \mathrm{Shape}(\mathsf{s})$. Then $(k, a, b+1)$ is not an addable node of $\boldsymbol{\lambda}$, so $(k, a-1, b) \in \boldsymbol{\lambda}$. By induction, V is m-content reduced, so $V_{\boldsymbol{\lambda}} \cong \mathsf{R}_m^\Lambda(\mathbb{K}[\underline{x}^\pm]) v$ as an $\mathsf{R}_m^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module by (the proof of) Lemma 3E.1. Therefore, without loss of generality, we can assume that $\mathsf{s}(k, a-1, b) = m-1$. In particular, $c_{m+1} = c_{m-1}$ and $i_{m+1} = i_{m-1}$. Moreover, $\psi_{m-1} v = 0$ by Proposition 3C.2, since $\sigma_{m-1}\mathsf{s} \notin \mathrm{Std}(\boldsymbol{\lambda})$ by Lemma 3B.4 (b). Similarly, $\psi_m v = 0$ because V is m-content reduced and no tableau in $\mathrm{Std}(\mathcal{P}_m^\ell)$ has content sequence $(c_1, \ldots, c_{m-1}, c_{m-1})$ and residue sequence $(i_1, \ldots, i_{m-1}, i_{m-1})$. Consequently,

$(\psi_m \psi_{m-1} \psi_m - \psi_{m-1} \psi_m \psi_{m-1})v = 0$. Therefore, $Q^{\underline{x}}_{i_{m-1},i_m,i_{m+1}}(y_{m-1}, y_m, y_{m+1})v = 0$ by (KLR8). However, $Q_{i_{m-1},i_m}(c_{m-1}, c_m) = 0$, so

$$
\begin{aligned}
Q^{\underline{x}}_{i_{m-1},i_m,i_{m+1}}(c_{m-1}, c_m, y_{m+1}) &= \frac{Q^{\underline{x}}_{i_{m-1},i_m}(y_{m+1}, c_m)}{y_{m+1} - c_{m-1}} \\
&= \begin{cases} \epsilon\big(\mathsf{c}(k, b-1-a) - y_{m+1}\big) & \text{if } \mathsf{r}(k, b-1-a) = i_{m-1}, \\ \epsilon & \text{otherwise,} \end{cases}
\end{aligned}
$$

where $\epsilon \in \Bbbk^\times$ and the last equality follows by Definition 3A.1(b). By Definition 3A.1(c), $\mathsf{c}(k, b-1-a) \neq c_{m+1}$, so $Q_{i_{m-1},i_m,i_{m+1}}(y_{m-1}, y_m, y_{m+1})v \neq 0$, giving a contradiction! Hence, $V_{\mathsf{c},\mathbf{i}} = 0$.

Similarly, if $(k, a, b+1) \notin A$ and $(k, a, b+1) \in \mathrm{Add}_j(\mathsf{s})$ then let $\mathsf{c}' = (c_1, \ldots, c_m, c'_{m+1})$ and $\mathbf{i}' = (i_1, \ldots, i_m, i'_{m+1})$, where $c'_{m+1} = \mathsf{c}(k, b-a-1)$ and $i'_{m+1} = \mathsf{r}(k, b-a-1)$. Then $(k, a, b-1) \in \boldsymbol{\lambda}$ and $V_{\mathsf{c}',\mathbf{i}'}$ is a summand of V_s by Lemma 3E.3. Arguing as in the last paragraph, we deduce that $V_{\mathsf{c}',\mathbf{i}'} = 0$.

Consequently, if $j \neq \mathsf{r}_m(\mathsf{s})$ then the last displayed equation, combined with Lemma 3E.3, shows that (3E.6) holds.

We have now established (3E.5) in all cases, so V is $(m+1)$-content reduced. This completes the proof of the inductive step and, hence, the proof of the proposition. $\square$

Applying Theorem 3E.4 to the regular representation, and using Lemma 3E.1, shows that the algebra $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm])$ is completely reducible. Proposition 3G.4 makes this more explicit.

Corollary 3E.7. *Let V be a $\Bbbk[\underline{x}^\pm]$-free $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm])$-module. Then $V = \bigoplus_\mathsf{t} F_\mathsf{t} V$ as a $\Bbbk[\underline{x}^\pm]$-module, where the sum is over $\mathsf{t} \in \mathrm{Std}(\mathcal{P}^\ell_n)$ such that $F_\mathsf{t} V \neq 0$.*

Proof. By Definition 3C.1, if $\mathsf{t} \in \mathrm{Std}(\mathcal{P}^\ell_n)$ then $V_\mathsf{t} \subseteq \{v \in V \mid v = F_\mathsf{t} v\}$. On the other hand, $V = \bigoplus_\mathsf{t} V_\mathsf{t}$ by Theorem 3E.4. Therefore, $V_\mathsf{t} = \{v \in V \mid v = F_\mathsf{t} v\}$ since $F_\mathsf{s} V \cap F_\mathsf{t} V = \delta_{\mathsf{st}} F_\mathsf{t} V$ by Lemma 3B.4. $\square$

Corollary 3E.8. *Suppose that $\mathsf{t} \in \mathrm{Std}(\mathcal{P}^\ell_n)$ and $1 \leq m \leq n$. Then $y_m F_\mathsf{t} = \mathsf{c}_m(\mathsf{t}) F_\mathsf{t}$ in $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm])$.*

Proof. Take $V = \mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm])$ to be the regular representation, which is free as a $\Bbbk[\underline{x}^\pm]$-module by base change from Proposition 2C.6 since $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm]) \cong \Bbbk[\underline{x}^\pm] \otimes_{\Bbbk[\underline{x}]} \mathsf{R}^\Lambda_n(\Bbbk[\underline{x}])$. First note that $F_\mathsf{t} \neq 0$ by (3C.5). By Corollary 3E.7, $V_\mathsf{t} = F_\mathsf{t} \mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm])$. As $F_\mathsf{t} = F_\mathsf{t} \cdot 1 \in F_\mathsf{t} \mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm]) = V_\mathsf{t}$, this implies the result. $\square$

Hence, using Lemma 3B.4 and Definition 3C.1, we obtain:

Corollary 3E.9. *Let $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\mathcal{P}^\ell_n)$. Then $F_\mathsf{s} F_\mathsf{t} = \delta_{\mathsf{st}} F_\mathsf{t}$ in $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm])$.*

Corollary 3E.10. *Suppose that $\mathbf{i} \in I^n$. Then, in $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm])$,*

$$1_\mathbf{i} = \sum_{\mathsf{t} \in \mathrm{Std}(\mathbf{i})} F_\mathsf{t}.$$

In particular, $1_\mathbf{i} = 0$ if and only if $\mathbf{i} \notin I^m_{\mathrm{Std}}$.

Proof. Take $V = \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ to be the regular representation of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$. By Corollary 3E.7,

$$\mathsf{1}_\mathbf{i}\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm]) = \bigoplus_{\mathsf{t} \in \mathrm{Std}(\mathbf{i})} F_\mathsf{t}\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm]).$$

Hence, the element $\mathsf{1}_\mathbf{i} - \sum_{\mathsf{t} \in \mathrm{Std}(\mathbf{i})} F_\mathsf{t}$ acts on $\mathsf{1}_\mathbf{i}\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ as multiplication by zero by Corollary 3E.9. Therefore, by (KLR1), this element acts on $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ as zero. Hence, $\mathsf{1}_\mathbf{i} = \sum_{\mathsf{t} \in \mathrm{Std}(\mathbf{i})} F_\mathsf{t}$ by the faithfulness of the regular representation. Finally, these arguments show that if $\mathrm{Std}(\mathbf{i}) = \emptyset$, then $\mathsf{1}_\mathbf{i} = 0$. That is, $\mathsf{1}_\mathbf{i} = 0$ if and only if $\mathbf{i} \notin I_{\mathrm{Std}}^m$. $\qquad\square$

Remark 3E.11. The last two corollaries are the main results of this subsection. Rather than the approach we have taken, these results can also be deduced from Proposition 3C.2 by first showing that $V = \bigoplus_\lambda V_\lambda$ is a faithful $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module, which can be proved after computing the (graded) dimension of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ using ideas from [8, 11]. That the representation V is faithful now follows from Corollary 3E.10. The next section gives a different take on this description of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ as the endomorphism algebra of V.

Corollary 3E.12. Suppose that $\mathsf{r}_k(\mathsf{t}) \neq \mathsf{r}_{k+1}(\mathsf{t})$ for $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and $1 \leq r < n$. Then $y_m\psi_k F_\mathsf{t} = \mathsf{c}_{\sigma_k(m)}(\mathsf{t})\psi_k F_\mathsf{t}$ whenever $1 \leq m \leq n$. In particular, $\psi_k F_\mathsf{t} = 0$ if $\sigma_k\mathsf{t}$ is not standard.

Proof. Suppose that $\mathsf{r}_k(\mathsf{t}) \neq \mathsf{r}_{k+1}(\mathsf{t})$. The claim that $y_m\psi_k F_\mathsf{t} = \mathsf{c}_{\sigma_k(m)}(\mathsf{t})\psi_k F_\mathsf{t}$ follows immediately from (KLR6) and Corollary 3E.8. For the second statement, if $\sigma_k\mathsf{t} \notin \mathrm{Std}(\mathcal{P}_n^\ell)$ then the node $\mathsf{t}^{-1}(k+1)$ is either directly to the right of, or directly below, $\mathsf{t}^{-1}(k)$. Therefore, $\mathsf{r}_k(\mathsf{t}) \neq \mathsf{r}_{k+1}(\mathsf{t})$ by Lemma 3A.4. Consequently, by Lemma 3B.4(b), there is no element in $\mathrm{Std}(\mathcal{P}_n^\ell)$ with residue sequence $\sigma_k\mathsf{r}(\mathsf{t})$ and content sequence $\sigma_k\mathsf{c}(\mathsf{t})$. Hence, $\psi_k F_\mathsf{t} = F_{\sigma_k\mathsf{t}}\psi_k = 0$ by Corollary 3E.10. $\qquad\square$

3F. The algebra $\mathcal{S}_n^\ell$. This subsection introduces the algebra $\mathcal{S}_n^\ell$, which is the "universal" *semisimple* cyclotomic KLR algebra of level ℓ. In the next subsection we show if $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ has a content system then it is isomorphic to $\mathcal{S}_n^\ell$. We maintain the notation of the previous subsections except we work over the field $\mathbb{K}$.

   Recall from subsection 3A that Γ_ℓ is the quiver of type $A_\infty^{\times\ell}$, with vertex set $J_\ell = \{1, \ldots, \ell\} \times \mathbb{Z}$. Let $\mathcal{S}_n^\ell(\mathbb{K})$ be the standard cyclotomic KLR algebra defined using the (standard) Q-polynomials and weight polynomials of Example 3A.2(a). Let $(\mathsf{c}^{J_\ell}, \mathsf{r}^{J_\ell})$ be the content system for $\mathcal{S}_n^\ell(\mathbb{K})$ given in Example 3A.2(a), so that c^{J_ℓ} is identically zero and r^{J_ℓ} is the identity map on J_ℓ. By assumption, $\underline{x}$ is the empty sequence for $\mathcal{S}_n^\ell$ so, by convention, $\mathbb{K}[\underline{x}^\pm] = \mathbb{K}$.

   To avoid confusion, if $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ let $\mathsf{r}^{J_\ell}(\mathsf{t})$ be the residue sequence of t with respect to the content system $(\mathsf{c}^{J_\ell}, \mathsf{r}^{J_\ell})$. Explicitly, $\mathsf{r}^{J_\ell}(\mathsf{t}) = (\mathsf{r}_1^{J_\ell}(\mathsf{t}), \ldots, \mathsf{r}_n^{J_\ell}(\mathsf{t})) \in J_\ell^n$ where $\mathsf{r}_m^{J_\ell}(\mathsf{t}) = \rho_k + b - a$ if $\mathsf{t}^{-1}(m) = (k, a, b)$. For convenience, set $J_{\mathrm{Std}}^n = \{\mathsf{r}^{J_\ell}(\mathsf{t}) \mid \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)\}$. By Lemma 3B.4, if $\mathbf{j} \in J_{\mathrm{Std}}^n$ then there exists a unique standard tableau $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ such that $\mathsf{r}^{J_\ell}(\mathsf{t}) = \mathbf{j}$ since c^{J_ℓ} is identically zero.

Lemma 3F.1. Suppose that $1 \leq k < n$ and $\mathbf{j} \in J_\ell^n$. Then $y_1 = \cdots = y_n = 0$ and $\mathsf{1}_\mathbf{j} \neq 0$ if and only if $\mathsf{1}_\mathbf{j} = F_\mathsf{t}$ for some $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$. Consequently, $\psi_k\mathsf{1}_\mathbf{j} = 0$ if $j_k - j_{k+1}$ and $\mathsf{1}_\mathbf{j} = 0$ if $j_k = j_{k+1}$ or $j_k = j_{k+2}$ for $1 \leq k < n - 1$).

Proof. Let V be the left regular representation of $\mathcal{S}_n^\ell(\mathbb{K})$. Then $V = \bigoplus_{\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)} V_\mathsf{t}$ by Theorem 3E.4. Since c^{J_ℓ} is identically zero, y_m acts as multiplication by zero on V_t, for $1 \leq m \leq n$ and $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$. Hence, $y_1 = \cdots = y_n = 0$ proving the first claim.

Next, we show that $1_{\mathsf{j}} \neq 0$ if and only if $1_{\mathsf{i}} = F_{\mathsf{t}}$, for some $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$. Observe that if $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ then $\mathsf{s} = \mathsf{t}$ if and only if $\mathsf{r}^{J_\ell}(\mathsf{s}) = \mathsf{r}^{J_\ell}(\mathsf{t})$ by Lemma 3B.4 since c_{J_ℓ} is identically zero. Hence, $1_{\mathsf{j}} = F_{\mathsf{t}}$ for some $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ by Corollary 3E.10. The remaining statements now follow by Corollary 3E.10 and Corollary 3E.12. □

Definition 3F.2. Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. For $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ set $\Psi_{\mathsf{st}} = \psi_w 1_{\mathsf{r}^{J_\ell}(\mathsf{t})}$, where $w \in \mathfrak{S}_n$ is the unique permutation such that $\mathsf{s} = w\mathsf{t}$.

Corollary 3F.3. The algebra $\mathcal{S}_n^\ell(\Bbbk)$ is spanned by $\{\Psi_{\mathsf{st}} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$.

Proof. By Theorem 2D.1 and Lemma 3F.1, $\mathcal{S}_n^\ell$ is spanned by the set

$$\left\{ \psi_w 1_{\mathsf{r}^{J_\ell}(\mathsf{t})} \,\middle|\, w \in \mathfrak{S}_n \text{ and } \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell) \right\}.$$

Hence, it is enough to show that if $\psi_w 1_{\mathsf{r}^{J_\ell}(\mathsf{t})} \neq 0$, for $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and $w \in \mathfrak{S}_n$, then $w\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$. Since w is a product of simple reflections, it is enough to consider the case when $w = \sigma_k = (k, k+1)$, for $1 \leq k < n$. If t is standard then $\sigma_k \mathsf{t}$ is standard unless k and $k+1$ are in the same row, or the same column of t, in which case $\psi_k 1_{\mathsf{j}} = 0$ by Lemma 3F.1. Hence, if $\psi_k 1_{\mathsf{i}} \neq 0$ then $\sigma_k \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ as we needed to show. □

Arguing by induction on n, it is easy to see that if $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and $\mathsf{r}^{J_\ell}(\mathsf{s}) = w\mathsf{r}^{J_\ell}(\mathsf{t})$, for some $w \in \mathfrak{S}_n$, then $\mathrm{Shape}(\mathsf{s}) = \mathrm{Shape}(\mathsf{t})$.

Given $u, w \in \mathfrak{S}_n$, write $u \preceq w$ if there is a reduced expression $w = \sigma_{a_1} \ldots \sigma_{a_l}$ such that $u = \sigma_{a_1} \ldots \sigma_{a_k}$, for some $0 \leq k < l$. (This is the right weak Bruhat order on $\mathfrak{S}_n$.)

Lemma 3F.4. Let $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and suppose that $w\mathsf{t}$ is standard, for some $w \in \mathfrak{S}_n$. Then $u\mathsf{t}$ is standard whenever $u \preceq w$.

Proof. If $1 \leq r < t \leq n$ and $u(r) > u(t)$ then $w(r) > w(t)$ since $u \preceq w$. The result follows easily from this observation. □

Lemma 3F.5. Let $\boldsymbol{\lambda} \in \mathrm{Std}(\mathcal{P}_n^\ell)$. Then there exists an irreducible left $\mathcal{S}_n^\ell(\Bbbk)$-module $W_{\boldsymbol{\lambda}}$ with basis $\{w_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ and where the $\mathcal{S}_n^\ell(\Bbbk)$-action is determined by

$$1_{\mathsf{j}} w_{\mathsf{t}} = \delta_{\mathsf{j}, \mathsf{r}^{J_\ell}(\mathsf{t})} w_{\mathsf{t}}, \quad y_m w_{\mathsf{t}} = 0, \quad \psi_k w_{\mathsf{t}} = \begin{cases} w_{\sigma_k \mathsf{t}} & \text{if } \sigma_k \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \\ 0 & \text{otherwise,} \end{cases}$$

for all $\mathsf{j} \in J_\ell^n$ and all admissible k and m.

Proof. By Lemma 3F.1, the map $\mathsf{t} \mapsto \mathsf{r}^{J_\ell}(\mathsf{t})$ gives a bijection $\mathrm{Std}(\mathcal{P}_n^\ell) \xrightarrow{\sim} J_{\mathrm{Std}}^n$ such that $F_{\mathsf{t}} = 1_{\mathsf{i}}$, where $\mathsf{i} = \mathsf{r}^{J_\ell}(\mathsf{t})$. Moreover, by (3B.5) and Lemma 3F.1,

$$Q_k(\mathsf{t}) = \begin{cases} 1 & \text{if } \sigma_k \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, in view of (3C.7), the lemma is a special case of Proposition 3C.2. □

Remark 3F.6. The $\mathsf{R}_n^\Lambda(\Bbbk[x^\pm])$-module $V_{\boldsymbol{\lambda}}$ is irreducible only over $\Bbbk[x^\pm]$. In contrast, it is easy to see that the module $W_{\boldsymbol{\lambda}}$ is irreducible over any field.

Remark 3F.7. Lemma 3A.4 is also a consequence of [41, Theorem 3.4]. By Lemma 3F.1, the natural grading on $W_{\boldsymbol{\lambda}}$ concentrates everything in degree 0.

We now prove that $\mathcal{S}_n^\ell(\Bbbk)$ is a split semisimple algebra.

Theorem 3F.8. The algebra $\mathcal{S}_n^\ell(\mathbb{K})$ is a split semisimple algebra and $\{W_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ is a complete set of pairwise non-isomorphic irreducible $\mathcal{S}_n^\ell$-modules, up to shift.

Proof. Recall from Corollary 3F.3 that the elements $\{\Psi_{\mathsf{st}} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ span $\mathcal{S}_n^\ell(\Bbbk)$. By Lemma 3F.4 and Lemma 3F.5, if $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\mu})$ then the action of Ψ_{st} on the module $W_{\boldsymbol{\lambda}}$ is given by $\Psi_{\mathsf{st}} w_{\mathsf{u}} = \delta_{\mathsf{tu}} w_{\mathsf{s}}$, for $\mathsf{u} \in \mathrm{Std}(\boldsymbol{\lambda})$. In particular, if $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$ then Ψ_{st} acts as zero on $W_{\boldsymbol{\lambda}}$. Moreover, this implies that the set $\{\Psi_{\mathsf{st}} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ is linearly independent, and so is a basis of $\mathcal{S}_n^\ell(\Bbbk)$ by Corollary 3F.3. Extending scalars to $\mathbb{K}$, there is a well-defined algebra isomorphism

$$\mathcal{E} \colon \mathcal{S}_n^\ell(\mathbb{K}) \longrightarrow \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}_n^\ell} \mathrm{End}_{\mathbb{K}}(W_{\boldsymbol{\lambda}}); \Psi_{\mathsf{st}} \mapsto e_{\mathsf{st}},$$

where e_{st} is the matrix unit given by $e_{\mathsf{st}}(w_{\mathsf{u}}) = \delta_{\mathsf{tu}} w_{\mathsf{s}}$. It follows that $\mathcal{E}$ is an algebra isomorphism since $\{\Psi_{\mathsf{st}}\}$ is a basis of $\mathcal{S}_n^\ell(\mathbb{K}) = \mathbb{K} \otimes_{\Bbbk} \mathcal{S}_n^\ell(\Bbbk)$, completing the proof. $\square$

Remark 3F.9. As in Remark 3F.7, the grading on $\mathcal{S}_n^\ell(\mathbb{K})$ puts everything in degree zero. The complete set of irreducible graded $\mathcal{S}_n^\ell(\mathbb{K})$-modules is $\{q^d W_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $d \in \mathbb{Z}\}$. In contrast, if x is an indeterminate in degree 1, then the complete set of irreducible graded $\mathcal{S}_n^\ell(\mathbb{K}[x^\pm])$-modules is $\{\mathbb{K}[x^\pm] \otimes_{\mathbb{K}} W_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$, since $\mathbb{K}[x^\pm]$ is the unique irreducible graded $\mathbb{K}[x^\pm]$-module.

The proof of Theorem 3F.8 and Corollary 3F.3 gives a basis of $\mathcal{S}_n^\ell(\Bbbk)$.

Corollary 3F.10. The algebra $\mathcal{S}_n^\ell(\Bbbk)$ is free as a $\Bbbk$-module with basis

$$\{\Psi_{\mathsf{st}} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}.$$

3G. **Semisimplicity of deformed cyclotomic KLR algebras.** This subsection returns to the framework of subsection 3A. In particular, we assume that $(\mathbf{Q}_I^x, \mathbf{W}_I^x)$ is a $\Bbbk[x]$-deformation $(\mathbf{Q}_I, \mathbf{W}_I)$ and that (c, r) is a content system for R_n^Λ with values in $\Bbbk[x]$. This subsection proves that the algebras $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ and $\mathcal{S}_n^\ell(\mathbb{K}[\underline{x}^\pm])$ are isomorphic as ungraded algebras, where $\mathbb{K}$ is the field of fractions of $\Bbbk$.

Recall the elements $\varphi_1, \ldots, \varphi_{n-1} \in \mathsf{R}_n^\Lambda(K)$ defined in (2D.4).

Lemma 3G.1. Suppose that $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and $1 \leq k < n$. Then, in $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$,

$$\varphi_k F_{\mathsf{t}} = \begin{cases} F_{\sigma_k \mathsf{t}} \varphi_k & \text{if } \sigma_k \mathsf{t} \text{ is standard,} \\ 0 & \text{otherwise,} \end{cases}$$

Proof. By Lemma 2D.5(d), if $1 \leq m < n$ then $\varphi_k(y_m - c) = (y_{\sigma_k(m)} - c)\varphi_k$. Hence, the result follows by Definition 3C.1 (and Lemma 3B.4). $\square$

Let $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $1 \leq m < n$. Note that if $\mathbf{j} = \mathsf{r}^{J_\ell}(\mathsf{t})$ then $\mathsf{r}_m^{J_\ell}(\mathsf{t}) \neq \mathsf{r}_{m+1}^{J_\ell}(\mathsf{t})$ by Lemma 3F.1. Recall the scalar $Q_m(\mathsf{t})$ for $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ from (3B.5). Set

$$q_m(\mathsf{t}) = \begin{cases} Q_m(\mathsf{t})^{-1} & \text{if } \mathsf{r}_m^{J_\ell}(\mathsf{t}) \neq \mathsf{r}_{m+1}^{J_\ell}(\mathsf{t}), \ \mathsf{r}_m^{J_\ell}(\mathsf{t}) \neq \mathsf{r}_{m+1}^{J_\ell}(\mathsf{t}) \text{ and } \sigma_m \mathsf{t} \rhd \mathsf{t}, \\ 1 & \text{otherwise.} \end{cases} \tag{3G.2}$$

Note that $q_m(\mathsf{t})$ is well-defined because $Q_m(\mathsf{t}) \neq 0$ by Lemma 3B.6. Moreover,

$$\text{if } \mathsf{r}_m^{J_\ell}(\mathsf{t}) \neq \mathsf{r}_{m+1}^{J_\ell}(\mathsf{t}) \text{ and } \mathsf{r}_m^{J_\ell}(\mathsf{t}) \neq \mathsf{r}_{m+1}^{J_\ell}(\mathsf{t}), \text{ then } q_m(\mathsf{t}) q_m(\sigma_m \mathsf{t}) = Q_m(\mathsf{t})^{-1}. \tag{3G.3}$$

Let $\mathcal{S}_n^\ell(\mathbb{K}[\underline{x}^\pm]) = \mathbb{K}[\underline{x}^\pm] \otimes_{\mathbb{K}} \mathcal{S}_n^\ell(\mathbb{K})$. Recall that if A is graded then $\underline{A}$ forgets the grading on A.

Proposition 3G.4. There is an (ungraded) algebra isomorphism

$$\Theta \colon \underline{\mathcal{S}_n^\ell(\mathbb{K}[\underline{x}^\pm])} \xrightarrow{\;\sim\;} \underline{\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])}$$

such that $\Theta(y_m) = 0$,

$$\Theta(1_{\mathbf{j}}) = \begin{cases} F_{\mathsf{t}} & \text{if } \mathbf{j} = \mathsf{r}^{J_\ell}(\mathsf{t}) \in J_{\mathrm{Std}}^n, \\ 0 & \text{if } \mathbf{j} \notin J_{\mathrm{Std}}^n, \end{cases},$$

$$\Theta(\psi_k 1_{\mathbf{j}}) = \begin{cases} q_k(\mathsf{t}) \varphi_k F_{\mathsf{t}} & \text{if } \mathbf{j} = \mathsf{r}^{J_\ell}(\mathsf{t}) \in J_{\mathrm{Std}}^n, \\ 0 & \text{if } \mathbf{j} \notin J_{\mathrm{Std}}^n. \end{cases}$$

for all $\mathbf{j} \in J_\ell^n$ and all admissible m and r.

Proof. First, note that $\Theta(\psi_k) = \sum_{\mathbf{j}} \Theta(\psi_k 1_{\mathbf{j}})$, so the images of the generators of $\mathcal{S}_n^\ell$ under Θ are uniquely determined. Hence, once we show that Θ is a homomorphism it is necessarily unique. If $1 \le m \le n$ then $y_m = 0$, by Lemma 3F.1, so the assumption that $y_m \in \ker \Theta$ does not prevent Θ from being an isomorphism. Similarly, by Lemma 3F.1, if $\mathbf{j} \in J_\ell^n$ then $1_{\mathbf{j}} \ne 0$ if and only if $\mathbf{j} \in J_{\mathrm{Std}}^n$.

To show that Θ is an algebra homomorphism it is enough to check that it respects the KLR relations (KLR1)–(KLR8) and the cyclotomic relation (2C.3). The cyclotomic relation (2C.3) is trivially satisfied and checking relations (KLR1)–(KLR4) and (KLR6) is easy, so these are left to the reader. Relation (KLR5) is routine using Lemma 3G.1. For relation (KLR7) it is enough to show that if $\mathbf{j} \in J_\ell^n$ and $1 \le k < n$ then

$$\Theta\left(\psi_k^2 1_{\mathbf{j}}\right) = \Theta\left(Q_{j_k, j_{k+1}}(y_k, y_{k+1}) 1_{\mathbf{j}}\right)$$

By definition, the right-hand side is equal to

$$\Theta\left(Q_{j_k, j_{k+1}}(y_k, y_{k+1}) 1_{\mathbf{j}}\right) = \begin{cases} F_{\mathsf{t}} & \text{if } \mathbf{j} = \mathsf{r}^{J_\ell}(\mathsf{t}) \in J_{\mathrm{Std}}^n \text{ and } j_k \not\!\!-\, j_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbf{j} \notin J_{\mathrm{Std}}^n$ then $\Theta(1_{\mathbf{j}}) = 0$, so we may assume that $\mathbf{j} = \mathsf{r}^{J_\ell}(\mathsf{t})$, for some $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$. If $j_k \not\!\!-\, j_{k+1}$, then

$$\Theta\left(\psi_k^2 1_{\mathbf{j}}\right) = q_k(\mathsf{t}) q_k(\sigma_k \mathsf{t}) \varphi_k^2 F_{\mathsf{t}}$$

$$= q_k(\mathsf{t}) q_k(\sigma_k \mathsf{t}) \left(Q_{\mathsf{r}_k^{J_\ell}(\mathsf{t}), \mathsf{r}_{k+1}^{J_\ell}(\mathsf{t})}(y_k, y_{k+1}) + \delta_{\mathsf{r}_k^{J_\ell}(\mathsf{t}), \mathsf{r}_{k+1}^{J_\ell}(\mathsf{t})} \right) F_{\mathsf{t}}$$

$$= q_k(\mathsf{t}) q_k(\sigma_k \mathsf{t}) Q_k(\mathsf{t}) F_{\mathsf{t}}$$

$$= F_{\mathsf{t}},$$

where we have used Lemma 2D.5 (f) for the second equality and (3G.3) for the last equality. On the other hand, if $j_k -\!\!- j_{k+1}$ then $\Theta(\psi_k^2 1_{\mathbf{j}}) = \Theta(\psi_k 1_{\sigma_k \mathbf{j}}) \Theta(\psi_k 1_{\mathbf{j}}) = 0$ since $\sigma_k \mathbf{j} \notin J_{\mathrm{Std}}^n$ (compare with Lemma 3F.1). Hence, Θ respects the quadratic relation (KLR7).

Now consider the deformed braid relation (KLR8). Since $y_m = 0$ for $1 \le m \le n$, we need to verify that if $1 \le k < n$ and $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and

$$\Theta\left(\psi_k \psi_{k+1} \psi_k 1_{\mathsf{r}^{J_\ell}(\mathsf{t})}\right) = \Theta\left(\psi_{k+1} \psi_k \psi_{k+1} 1_{\mathsf{r}^{J_\ell}(\mathsf{t})}\right)$$

If $\sigma_k \sigma_{k+1} \sigma_k \mathsf{t} = \sigma_{k+1} \sigma_k \sigma_{k+1} \mathsf{t}$ is not standard then both sides are zero, so we can assume that this tableau is standard. By Lemma 2D.5 (b) and Lemma 3F.4, it is enough to show that

$$q_k(\sigma_{k+1} \sigma_k \mathsf{t}) q_{k+1}(\sigma_k \mathsf{t}) q_k(\mathsf{t}) = q_{k+1}(\sigma_k \sigma_{k+1} \mathsf{t}) q_k(\sigma_{k+1} \mathsf{t}) q_{k+1}(\mathsf{t}).$$

It follows from (3G.2) that $q_k(\sigma_{k+1}\sigma_k\mathsf{t}) = q_{k+1}(\mathsf{t})$, $q_{k+1}(\sigma_k\mathsf{t}) = q_k(\sigma_{k+1}\mathsf{t})$ and $q_k(\mathsf{t}) = q_{k+1}(\sigma_k\sigma_{k+1}\mathsf{t})$, so (KLR8) is satisfied.

We have now proved that Θ is an algebra homomorphism. By Corollary 3E.10, to show that Θ is surjective it is enough to check that $1_{\mathbf{i}}F_{\mathsf{t}}$, y_kF_{t} and ψ_kF_{t} belong to the image of Θ, for all $\mathbf{i} \in I^n$, $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ and all admissible k. Certainly, $1_{\mathbf{i}}F_{\mathsf{t}} = \delta_{\mathbf{i}\,\mathsf{r}^{J_\ell}(\mathsf{t})}F_{\mathsf{t}} = \delta_{\mathbf{i}\,\mathsf{r}^{J_\ell}(\mathsf{t})}\Theta(1_{\mathsf{r}^{J_\ell}(\mathsf{t})}) \in \mathrm{im}\,\Theta$. Hence, $y_kF_{\mathsf{t}} \in \mathrm{im}\,\Theta$ by Corollary 3E.8. Finally, consider ψ_kF_{t}. If $\sigma_k\mathsf{t}$ is not standard, then $\psi_kF_{\mathsf{t}} = 0$ by Corollary 3E.12. Otherwise, by (2D.4) we have

$$q_k(\mathsf{t})^{-1}\Theta\left(\psi_k 1_{\mathsf{r}^{J_\ell}(\mathsf{t})}\right) = \varphi_k F_{\mathsf{t}} = \begin{cases} (\mathsf{c}_k(\mathsf{t}) - \mathsf{c}_{k+1}(\mathsf{t}))\psi_k F_{\mathsf{t}} + F_{\mathsf{t}} & \text{if } \mathsf{r}_k^{J_\ell}(\mathsf{t}) = \mathsf{r}_{k+1}^{J_\ell}(\mathsf{t}), \\ \psi_k F_{\mathsf{t}} & \text{if } \mathsf{r}_k^{J_\ell}(\mathsf{t}) \neq \mathsf{r}_{k+1}^{J_\ell}(\mathsf{t}). \end{cases}$$

In both cases it follows that $\psi_k F_{\mathsf{t}} \in \mathrm{im}\,\Theta$, where we use Definition 3A.1 (c) when $\mathsf{r}_k^{J_\ell}(\mathsf{t}) = \mathsf{r}_{k+1}^{J_\ell}(\mathsf{t})$. Hence, Θ is surjective.

We have now shown that Θ is a surjective algebra homomorphism from $\mathcal{S}_n^\ell(\mathbb{K}[\underline{x}^\pm])$ to $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$. Let K be any field containing $\mathbb{K}[\underline{x}^\pm]$. Extending scalars to K and using Proposition 3C.2, Corollary 3C.10 and Theorem 3F.8, the algebra $\mathsf{R}_n^\Lambda(K)$ has at least as many isomorphism classes of (ungraded) simple modules as $\mathcal{S}_n^\ell(K)$. Hence, by a dimension count, the induced map Θ_K from $\mathcal{S}_n^\ell(K)$ to $\mathcal{R}_n^\Lambda(K)$ is an isomorphism. Therefore, Θ_K, and hence Θ, is injective. It follows that $\Theta\colon \mathcal{S}_n^\ell(\mathbb{K}[\underline{x}^\pm]) \longrightarrow \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ is an isomorphism of ungraded algebras, so the proof is complete. $\square$

Remark 3G.5. The isomorphism Θ of Proposition 3G.4 is not homogeneous because, in general, the elements $\psi_k 1_{\mathbf{j}}$ and $\Theta(\psi_k 1_{\mathbf{j}})$ have different degrees.

Recall the irreducible graded $\mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm])$-module $V_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, defined before Corollary 3C.10. Combining Theorem 3F.8 and Proposition 3G.4 shows that $\mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm])$ is isomorphic to a direct sum of matrix algebras over $\mathbb{K}[x^\pm]$. Hence, we have:

Corollary 3G.6. The algebra $\mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm])$ is a split semisimple algebra over $\mathbb{K}[x^\pm]$ and $\{V_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ is a complete set of pairwise non-isomorphic irreducible graded $\mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm])$-modules.

In particular, up to isomorphism, the irreducible module $V_{\boldsymbol{\lambda}}$ does not depend on the choice of content system (c, r), for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. We already knew from Corollary 3C.10 that $V_{\boldsymbol{\lambda}}$ is independent of the choice of β-coefficients in Proposition 3C.2.

4. Cellular bases of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$

The main results of this paper follow from the construction of cellular bases for the algebra $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}])$, which is the focus of this section. The cellular bases that we construct are analogues of the ψ-bases of [24]. Using the results of section 3 it is easy to see that the ψ-bases are linearly independent. The main difficulty is showing that the ψ-bases span the algebra $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}])$.

Throughout the section, we continue to assume that $(\Gamma, \mathbf{Q}_I^{\frac{x}{}}, \mathbf{W}_I^x)$ is a $\mathbb{k}[\underline{x}]$-deformation of a standard cyclotomic KLR datum $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$ and (c, r) is a (graded) content system with values in $\mathbb{k}[\underline{x}]$ and we let $\mathbb{K}$ be the field of fractions of $\mathbb{k}$. section 3 studied the semisimple representation theory of the algebra $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$.

4A. **Integral and seminormal bases.** Partly inspired by [24, 51], this subsection defines the two new bases of $R_n^\Lambda(\Bbbk[\underline{x}])$ that will ultimately allow us to prove our main results. Defining these bases is easy, but it will take some time to prove that they are both (cellular) bases over $\Bbbk[\underline{x}]$.

Recall from subsection 3B that $\unrhd$ is the dominance order on $\mathcal{P}_n^\ell$. If $\mathsf{s} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ is a standard tableau and $1 \le m \le n$ then $\mathsf{s}_{\downarrow m}$ is the subtableau of s that contains the numbers in $\{1, \ldots, m\}$. Extend the dominance order to $\mathrm{Std}(\mathcal{P}_n^\ell)$ by defining $\mathsf{s} \unrhd \mathsf{t}$ if $\mathrm{Shape}(\mathsf{s}_{\downarrow m}) \unrhd \mathrm{Shape}(\mathsf{t}_{\downarrow m})$, for $1 \le m \le n$. Write $\mathsf{s} \rhd \mathsf{t}$ if $\mathsf{s} \unrhd \mathsf{t}$ and $\mathsf{s} \ne \mathsf{t}$. Similarly, given $(\mathsf{s}, \mathsf{t}), (\mathsf{u}, \mathsf{v}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)$ write $(\mathsf{s}, \mathsf{t}) \unrhd (\mathsf{u}, \mathsf{v})$ if $\mathsf{s} \unrhd \mathsf{u}$ and $\mathsf{t} \unrhd \mathsf{v}$. As before, write $(\mathsf{s}, \mathsf{t}) \rhd (\mathsf{u}, \mathsf{v})$ if $(\mathsf{s}, \mathsf{t}) \unrhd (\mathsf{u}, \mathsf{v})$ and $(\mathsf{s}, \mathsf{t}) \ne (\mathsf{u}, \mathsf{v})$.

Definition 4A.1 (Residue dominance). Let s and t be two standard tableaux. Write $\mathsf{s} \blacktriangleright \mathsf{t}$ if $r(\mathsf{s}) = r(\mathsf{t})$ and $\mathsf{s} \unrhd \mathsf{t}$. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, write $\boldsymbol{\lambda} \blacktriangleright \boldsymbol{\mu}$ if there exist $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\mu})$ such that $\mathsf{s} \blacktriangleright \mathsf{t}$.

In what follows we could replace the posets $(\mathcal{P}_n^\ell, \lhd)$ and $(\mathcal{P}_n^\ell, \rhd)$ with $(\mathcal{P}_n^\ell, \blacktriangleleft)$ and $(\mathcal{P}_n^\ell, \blacktriangleright)$, respectively. However, doing this does not give very much additional information because all of our definitions are compatible with the block decompositions $R_n^\Lambda = \bigoplus_\alpha R_\alpha^\Lambda$ and the residue dominance orderings are just the dominance ordering restricted to these subalgebras. We remark that in type $A_{e-1}^{(1)}$ the algebras R_α^Λ are indecomposable by [11, (1.4)] (and [48]). In type $C_{e-1}^{(1)}$ it is not known if R_α^Λ is indecomposable, although we expect this to be the case.

Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. The *conjugate* of $\boldsymbol{\lambda}$ is the ℓ-partition $\boldsymbol{\lambda}' = \{(\ell - k + 1, c, r) \mid (k, r, c) \in \boldsymbol{\lambda}\}$. That is, $\boldsymbol{\lambda}'$ is the ℓ-partition obtained from $\boldsymbol{\lambda}$ by reversing the order of the components and then swapping the rows and columns in each component. As is well-known, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$ then $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda}' \unlhd \boldsymbol{\mu}'$. Similarly, the *conjugate tableau* to $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ is the standard $\boldsymbol{\lambda}'$-tableau t' with $\mathsf{t}'(k, r, c) = \mathsf{t}(\ell - k + 1, c, r)$, for $(k, r, c) \in \boldsymbol{\lambda}'$.

It is well-known that there exist unique tableaux $\mathsf{t}_{\boldsymbol{\lambda}}^{\rhd}$ and $\mathsf{t}_{\boldsymbol{\lambda}}^{\lhd}$ such that $\mathsf{t}_{\boldsymbol{\lambda}}^{\lhd} \unlhd \mathsf{s} \unlhd \mathsf{t}_{\boldsymbol{\lambda}}^{\rhd}$, for all $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$. Explicitly, $\mathsf{t}_{\boldsymbol{\lambda}}^{\rhd} = (\mathsf{t}^{\rhd \lambda^{(1)}} | \ldots | \mathsf{t}^{\rhd \lambda^{(\ell)}})$ is the standard $\boldsymbol{\lambda}$-tableau with the numbers $1, 2, \ldots, n$ entered in order from left to right along the rows of $\mathsf{t}^{\rhd \lambda^{(1)}}$, and then the rows of $\mathsf{t}^{\rhd \lambda^{(2)}}$ and so on. Similarly, $\mathsf{t}_{\boldsymbol{\lambda}}^{\lhd} = (\mathsf{t}^{\lhd \lambda^{(1)}} | \ldots | \mathsf{t}^{\lhd \lambda^{(\ell)}})$ is the standard $\boldsymbol{\lambda}$-tableau with numbers $1, 2, \ldots, n$ entered in order down the columns of the tableaux $\mathsf{t}^{\lhd \lambda^{(\ell)}}, \ldots, \mathsf{t}^{\lhd \lambda^{(1)}}$. By construction, $\mathsf{t}_{\boldsymbol{\lambda}}^{\lhd} = (\mathsf{t}_{\boldsymbol{\lambda}'}^{\rhd})'$.

Definition 4A.2. For each standard tableau $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$, let $d_\mathsf{t}^{\rhd}, d_\mathsf{t}^{\lhd} \in \mathfrak{S}_n$ be the unique permutations such that $d_\mathsf{t}^{\lhd} \mathsf{t}_{\boldsymbol{\lambda}}^{\lhd} = \mathsf{t} = d_\mathsf{t}^{\rhd} \mathsf{t}_{\boldsymbol{\lambda}}^{\rhd}$. As important special cases, set $d_{\boldsymbol{\lambda}}^{\lhd} = d_{\mathsf{t}_{\boldsymbol{\lambda}}^{\rhd}}^{\lhd}$ and $d_{\boldsymbol{\lambda}}^{\rhd} = d_{\mathsf{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}$.

Recall from subsection 2B that $L \colon \mathfrak{S}_n \longrightarrow \mathbb{N}$ is the length function on $\mathfrak{S}_n$. Although normally stated using slightly different language, the following lemma is well-known and easy to prove. See, for example, [40, Lemma 2.18].

Lemma 4A.3. Suppose $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Then $d_{\boldsymbol{\lambda}}^{\rhd} = (d_{\boldsymbol{\lambda}}^{\lhd})^{-1}$. Moreover, if $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ then

$$d_{\boldsymbol{\lambda}}^{\lhd} = (d_\mathsf{t}^{\rhd})^{-1} d_\mathsf{t}^{\lhd}, \qquad d_{\boldsymbol{\lambda}}^{\rhd} = (d_\mathsf{t}^{\lhd})^{-1} d_\mathsf{t}^{\rhd}, \qquad \text{and} \qquad d_\mathsf{t}^{\lhd} = d_{\mathsf{t}'}^{\rhd},$$

with $L(d_{\boldsymbol{\lambda}}^{\lhd}) = L(d_\mathsf{t}^{\lhd}) + L(d_\mathsf{t}^{\rhd}) = L(d_{\boldsymbol{\lambda}}^{\rhd})$.

In subsection 2D, we fixed a preferred reduced expression $w = \sigma_{a_1} \ldots \sigma_{a_l}$, for each $w \in \mathfrak{S}_n$, and we defined $\psi_w = \psi_{a_1} \ldots \psi_{a_l}$. In particular, we have preferred reduced expressions for the permutations $d_\mathsf{t}^{\lhd}, d_{\boldsymbol{\lambda}}^{\lhd}, d_\mathsf{t}^{\rhd}$ and $d_{\boldsymbol{\lambda}}^{\rhd}$ that define elements $\psi_{d_\mathsf{t}^{\lhd}}, \psi_{d_{\boldsymbol{\lambda}}^{\lhd}}, \psi_{d_\mathsf{t}^{\rhd}}, \psi_{d_{\boldsymbol{\lambda}}^{\rhd}} \in R_n^\Lambda(\Bbbk[\underline{x}])$.

Recall from subsection 3B that $\mathcal{N}_n^\ell = \{(k,r,c) \,|\, 1 \le k \le \ell, r, c \ge 1\}$ is the set of nodes, which we consider as a totally ordered set under the lexicographic order, and that we identify an ℓ-partition with its diagram $\{(k,r,c) \in \mathcal{N}_n^\ell \,|\, 1 \le c \le \lambda_r^{(k)}\}$.

Fix $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. An *addable* node of $\boldsymbol{\lambda}$ is a node $A = (k,r,c) \in \mathcal{N}_n^\ell \setminus \boldsymbol{\lambda}$ such that $\boldsymbol{\lambda} \cup \{A\} \in \mathcal{P}_{n+1}^\ell$. Similarly, a *removable* node of $\boldsymbol{\lambda}$ is a node $A \in \boldsymbol{\lambda}$ such that $\boldsymbol{\lambda} \setminus \{A\} \in \mathcal{P}_{n-1}^\ell$. If $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ let $\mathrm{Add}(\mathsf{t}) = \mathrm{Add}(\boldsymbol{\lambda})$ and $\mathrm{Rem}(\mathsf{t}) = \mathrm{Rem}(\boldsymbol{\lambda})$ be the sets of addable and removable nodes of $\boldsymbol{\lambda}$.

Let $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $1 \le m \le n$ and define:

$$
\begin{aligned}
\mathrm{Add}_m^\lhd(\mathsf{t}) &= \left\{ A \in \mathrm{Add}(\mathsf{t}_{\downarrow m}) \,\Big|\, \mathsf{r}(A) = \mathsf{r}_m(\mathsf{t}) \text{ and } A < \mathsf{t}^{-1}(m) \right\} \\
\mathrm{Rem}_m^\lhd(\mathsf{t}) &= \left\{ A \in \mathrm{Rem}(\mathsf{t}_{\downarrow m}) \,\Big|\, \mathsf{r}(A) = \mathsf{r}_m(\mathsf{t}) \text{ and } A < \mathsf{t}^{-1}(m) \right\} \\
\mathrm{Add}_m^\rhd(\mathsf{t}) &= \left\{ A \in \mathrm{Add}(\mathsf{t}_{\downarrow m}) \,\Big|\, \mathsf{r}(A) = \mathsf{r}_m(\mathsf{t}) \text{ and } A > \mathsf{t}^{-1}(m) \right\} \\
\mathrm{Rem}_m^\rhd(\mathsf{t}) &= \left\{ A \in \mathrm{Rem}(\mathsf{t}_{\downarrow m}) \,\Big|\, \mathsf{r}(A) = \mathsf{r}_m(\mathsf{t}) \text{ and } A > \mathsf{t}^{-1}(m) \right\}.
\end{aligned}
\tag{4A.4}
$$

Recall from subsection 2C that $*$ is the unique anti-isomorphism of R_n^Λ that fixes the generators of Definition 2C.2.

Definition 4A.5 (Integral bases). Let $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Define

$$
\psi_{\mathsf{st}}^\lhd = \psi_{d_{\mathsf{s}}^\lhd} y_{\boldsymbol{\lambda}}^\lhd 1_{\mathbf{i}_{\boldsymbol{\lambda}}^\lhd} \psi_{d_{\mathsf{t}}^\lhd}^* \qquad \text{and} \qquad \psi_{\mathsf{st}}^\rhd = \psi_{d_{\mathsf{s}}^\rhd} y_{\boldsymbol{\lambda}}^\rhd 1_{\mathbf{i}_{\boldsymbol{\lambda}}^\rhd} \psi_{d_{\mathsf{t}}^\rhd}^*,
$$

where $\mathbf{i}_{\boldsymbol{\lambda}}^\lhd = \mathsf{r}(\mathsf{t}_{\boldsymbol{\lambda}}^\lhd)$, $\mathbf{i}_{\boldsymbol{\lambda}}^\rhd = \mathsf{r}(\mathsf{t}_{\boldsymbol{\lambda}}^\rhd)$ and

$$
y_{\boldsymbol{\lambda}}^\lhd = \prod_{m=1}^n \prod_{A \in \mathrm{Add}_m^\lhd(\mathsf{t}_{\boldsymbol{\lambda}}^\lhd)} (y_m - \mathsf{c}(A)) \qquad \text{and} \qquad y_{\boldsymbol{\lambda}}^\rhd = \prod_{m=1}^n \prod_{A \in \mathrm{Add}_m^\rhd(\mathsf{t}_{\boldsymbol{\lambda}}^\rhd)} (y_m - \mathsf{c}(A)).
$$

By definition, if $(\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)$ then $\psi_{\mathsf{st}}^\lhd$ and $\psi_{\mathsf{st}}^\rhd$ are elements of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$, which depend on the choices of reduced expressions for $d_{\mathsf{s}}^\lhd$, $d_{\mathsf{t}}^\lhd$, $d_{\mathsf{s}}^\rhd$ and $d_{\mathsf{t}}^\rhd$. We will abuse notation and consider $\psi_{\mathsf{st}}^\lhd$ and $\psi_{\mathsf{st}}^\rhd$ as elements of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$, $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ and of $\mathscr{R}_n^\Lambda(\Bbbk)$. It is not yet clear that the elements $\psi_{\mathsf{st}}^\lhd$ and $\psi_{\mathsf{st}}^\rhd$ are nonzero but, if they are, they are homogeneous.

To prove that $\{\psi_{\mathsf{st}}^\lhd\}$ and $\{\psi_{\mathsf{st}}^\rhd\}$ are bases of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ we will use some closely related *seminormal bases* of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$. As we will see, the seminormal bases give other realisations of the graded $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-modules $V_{\boldsymbol{\lambda}}$ from Proposition 3C.2. In fact, this is the key to proving that the ψ-bases are linearly independent.

Definition 4A.6 (Seminormal bases). Let $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Set

$$
f_{\mathsf{st}}^\lhd = F_{\mathsf{s}} \psi_{\mathsf{st}}^\lhd F_{\mathsf{t}} \qquad \text{and} \qquad f_{\mathsf{st}}^\rhd = F_{\mathsf{s}} \psi_{\mathsf{st}}^\rhd F_{\mathsf{t}}.
$$

By definition, $f_{\mathsf{st}}^\lhd, f_{\mathsf{st}}^\rhd \in \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ and these elements do not typically belong to $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$. We will show that $\{f_{\mathsf{st}}^\lhd\}$ and $\{f_{\mathsf{st}}^\rhd\}$ are cellular bases of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$. Since $\psi_{\mathsf{st}}^\lhd$ and $\psi_{\mathsf{st}}^\rhd$ are both homogeneous so are $f_{\mathsf{st}}^\lhd$ and $f_{\mathsf{st}}^\rhd$.

Below we prove many parallel results for the elements $\{\psi_{\mathsf{st}}^\lhd\}$ and $\{f_{\mathsf{st}}^\lhd\}$, and for the elements $\{\psi_{\mathsf{st}}^\rhd\}$ and $\{f_{\mathsf{st}}^\rhd\}$. In almost every case, the proofs are identical except that the $\psi^\lhd$-basis and $f^\lhd$-basis use the poset $(\mathcal{P}_n^\ell, \unlhd)$ whereas the $\psi^\rhd$-basis and $f^\rhd$-basis use the poset $(\mathcal{P}_n^\ell, \unrhd)$. For this reason, we work with a generic symbol $\triangle \in \{\lhd, \rhd\}$ and write $\psi_{\mathsf{st}}^\triangle$, $f_{\mathsf{st}}^\triangle$, $\mathsf{t}_{\boldsymbol{\lambda}}^\triangle$, $d_{\mathsf{t}}^\triangle$, ... in place of $\psi_{\mathsf{st}}^\lhd$, $f_{\mathsf{st}}^\lhd$, $\mathsf{t}_{\boldsymbol{\lambda}}^\lhd$, $d_{\mathsf{t}}^\lhd$, ... and $\psi_{\mathsf{st}}^\rhd$, $f_{\mathsf{st}}^\rhd$, $\mathsf{t}_{\boldsymbol{\lambda}}^\rhd$, $d_{\mathsf{t}}^\rhd$, ..., respectively.

Lemma 4A.7. *Let $\mathsf{s}, \mathsf{t}, \mathsf{u}, \mathsf{v} \in \mathrm{Std}(\mathcal{P}_n^\ell)$. Then $\delta_{\mathsf{su}} \delta_{\mathsf{tv}} f_{\mathsf{st}}^\lhd = F_{\mathsf{u}} f_{\mathsf{st}}^\lhd F_{\mathsf{v}}$ and $\delta_{\mathsf{su}} \delta_{\mathsf{tv}} f_{\mathsf{st}}^\rhd = F_{\mathsf{u}} f_{\mathsf{st}}^\rhd F_{\mathsf{v}}$.*

Proof. This is immediate from Corollary 3E.9 and Definition 4A.6. $\qquad\square$

In contrast, it is rarely true that $F_{\mathsf{u}}\psi_{\mathsf{st}}^{\triangle}F_{\mathsf{v}} = \delta_{\mathsf{su}}\delta_{\mathsf{tv}}\psi_{\mathsf{st}}^{\triangle}$, for $(\mathsf{s},\mathsf{t}),(\mathsf{u},\mathsf{v}) \in \mathrm{Std}^2(\mathcal{P}_n^{\ell})$.

We want to show that the sets $\{\psi_{\mathsf{st}}^{\triangle}\}$ and $\{f_{\mathsf{st}}^{\triangle}\}$ are bases of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}^{\pm}])$ and that the transition matrices between the ψ-bases and the corresponding f-bases are unitriangular. Before we can prove this we need a better understanding of how $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}^{\pm}])$ acts on the f-bases and to do this we connect these bases to the seminormal representations of section 3. Motivated by (3C.7), for $\mathsf{s} \in \mathrm{Std}(\mathcal{P}_n^{\ell})$ and $1 \le k < n$ define scalars $\beta_k^{\triangleleft}(\mathsf{s}), \beta_k^{\triangleright}(\mathsf{s}) \in \mathbb{k}[\underline{x}]$ by

$$\beta_k^{\triangleleft}(\mathsf{s}) = \begin{cases} 1 & \text{if } \mathsf{s} \triangleleft \sigma_k\mathsf{s}, \\ Q_k(\mathsf{s}) & \text{if } \sigma_k\mathsf{s} \triangleleft \mathsf{s}, \end{cases} \quad \text{and} \quad \beta_k^{\triangleright}(\mathsf{s}) = \begin{cases} 1 & \text{if } \mathsf{s} \triangleright \sigma_k\mathsf{s}, \\ Q_k(\mathsf{s}) & \text{if } \sigma_k\mathsf{s} \triangleright \mathsf{s}. \end{cases} \quad (4\mathrm{A}.8)$$

Repeating the argument of Lemma 3C.8 shows that:

Lemma 4A.9. *The coefficients $\{\beta_r^{\triangleleft}(\mathsf{s})\}$ and $\{\beta_r^{\triangleright}(\mathsf{s})\}$ satisfy conditions (a)–(c) of Proposition 3C.2.*

Hence, the coefficients $\{\beta_r^{\triangleleft}(\mathsf{s})\}$ and $\{\beta_r^{\triangleright}(\mathsf{s})\}$ both determine irreducible graded $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}^{\pm}])$-modules $V_{\boldsymbol{\lambda}}^{\triangleleft}$ and $V_{\boldsymbol{\lambda}}^{\triangleright}$, respectively. By Corollary 3C.10, $V_{\boldsymbol{\lambda}}^{\triangleleft} \cong V_{\boldsymbol{\lambda}}^{\triangleright}$. Let $\{v_{\mathsf{t}}^{\triangle} \mid \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ be the basis of $V_{\boldsymbol{\lambda}}^{\triangle}$ from Proposition 3C.2. More explicitly, fix a nonzero vector $v_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}} \in F_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}V_{\boldsymbol{\lambda}}^{\triangle}$ and define $v_{\mathsf{t}}^{\triangle}$ by induction on $L(d_{\mathsf{t}}^{\triangle})$ by setting

$$v_{\mathsf{t}}^{\triangle} = \left(\psi_k - \frac{\delta_{\mathsf{r}_k(\mathsf{s})}\delta_{k+1}(\mathsf{s})}{\rho_k(\mathsf{s})}\right) v_{\mathsf{s}}^{\triangle}$$

where $d_{\mathsf{t}}^{\triangle} = s_k d_{\mathsf{s}}^{\triangle}$ with $L(d_{\mathsf{t}}^{\triangle}) = L(d_{\mathsf{s}}^{\triangle}) + 1$, and we set $\rho_k(\mathsf{s}) = \mathsf{c}_{k+1}(\mathsf{s}) - \mathsf{c}_k(\mathsf{s}) \in \mathbb{k}[\underline{x}]$.

The next result should be compared with Proposition 3C.2.

Proposition 4A.10. *Let $(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^{\ell})$ and suppose that $1 \le k < n$, $1 \le m \le n$ and $\mathbf{i} \in I^n$. Then the elements $f_{\mathsf{st}}^{\triangleleft}$ and $f_{\mathsf{st}}^{\triangleright}$ are nonzero and*

$$1_{\mathbf{i}}f_{\mathsf{st}}^{\triangleleft} = \delta_{\mathbf{i}\,\mathsf{r}(\mathsf{s})}f_{\mathsf{st}}^{\triangleleft} \qquad y_m f_{\mathsf{st}}^{\triangleleft} = \mathsf{c}_m(\mathsf{s})f_{\mathsf{st}}^{\triangleleft} \qquad \psi_k f_{\mathsf{st}}^{\triangleleft} = \frac{\delta_{\mathsf{r}_k(\mathsf{s}),\mathsf{r}_{k+1}(\mathsf{s})}}{\rho_k(\mathsf{s})}f_{\mathsf{st}}^{\triangleleft} + \beta_k^{\triangleleft}(\mathsf{s})f_{\mathsf{ut}}^{\triangleleft},$$

$$1_{\mathbf{i}}f_{\mathsf{st}}^{\triangleright} = \delta_{\mathbf{i}\,\mathsf{r}(\mathsf{s})}f_{\mathsf{st}}^{\triangleright} \qquad y_m f_{\mathsf{st}}^{\triangleright} = \mathsf{c}_m(\mathsf{s})f_{\mathsf{st}}^{\triangleright} \qquad \psi_k f_{\mathsf{st}}^{\triangleright} = \frac{\delta_{\mathsf{r}_k(\mathsf{s}),\mathsf{r}_{k+1}(\mathsf{s})}}{\rho_k(\mathsf{s})}f_{\mathsf{st}}^{\triangleright} + \beta_k^{\triangleright}(\mathsf{s})f_{\mathsf{ut}}^{\triangleright},$$

where $\mathsf{u} = \sigma_k\mathsf{s}$.

Proof. Let $\triangle \in \{\triangleleft,\triangleright\}$. Since $f_{\mathsf{st}}^{\triangle} = F_{\mathsf{s}}\psi_{\mathsf{st}}^{\triangle}F_{\mathsf{t}}$, the formulas for $1_{\mathbf{i}}f_{\mathsf{st}}^{\triangle}$ and $y_m f_{\mathsf{st}}^{\triangle}$ follow from Corollary 3E.10 and Corollary 3E.8, respectively. We use these formulas below without mention.

To prove the remaining claims, fix $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and let $W_{\mathsf{t}}^{\triangle}$ be the $\mathbb{K}[\underline{x}^{\pm}]$-submodule of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}^{\pm}])$ spanned by $\{f_{\mathsf{st}}^{\triangle} \mid \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})\}$. Let $\Theta_{\mathsf{t}} \colon W_{\mathsf{t}}^{\triangle} \to V^{\triangle}$ be the map given by $\Theta_{\mathsf{t}}(w) = wv_{\mathsf{t}}^{\triangle}$, for $w \in W_{\mathsf{t}}^{\triangle}$. We prove by induction on dominance order for t that there exists a nonzero scalar a_{t}, which depends only on t, such that $\Theta_{\mathsf{t}}(f_{\mathsf{st}}^{\triangle}) = a_{\mathsf{t}}v_{\mathsf{s}}^{\triangle}$, for $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$. To prove this, first consider the special case when $\mathsf{t} = \mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}$. By Proposition 3C.2,

$$\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle}v_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} = y_{\boldsymbol{\lambda}}^{\triangle}1_{\mathbf{i}_{\boldsymbol{\lambda}}^{\triangle}}v_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} = \prod_{m=1}^{n}\prod_{A \in \mathrm{Add}_m^{\triangle}(\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle})}\left(\mathsf{c}_m\left(\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}\right) - \mathsf{c}(A)\right) \cdot v_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} = a_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}v_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle},$$

where $a_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}} = \prod_m\prod_A(\mathsf{c}_m(\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}) - \mathsf{c}(A)) \in \mathbb{k}[\underline{x}]$. If $A \in \mathrm{Add}_m^{\triangle}(\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle})$ then $\mathsf{r}(A) = \mathsf{r}_m(\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle})$, so each factor of $a_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}$ is nonzero by Definition 3A.1 (c). Consequently, $a_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}} \ne 0$. Moreover,

$f^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} v^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} = a_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} v^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}}$ since $F_{\mathsf{s}}v^{\triangle}_{\mathsf{s}} = v^{\triangle}_{\mathsf{t}}$, for all $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$. In view of (4A.8) and Proposition 3C.2, if $\mathsf{y} \in \mathrm{Std}(\boldsymbol{\lambda})$ then

$$f^{\triangle}_{\mathsf{yt}^{\triangle}_{\boldsymbol{\lambda}}} v^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} = F_{\mathsf{y}}\psi_{d_{\mathsf{y}}} f^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} v^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} = a_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} F_{\mathsf{y}}\psi_{d_{\mathsf{y}}} v^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} = a_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} v^{\triangle}_{\mathsf{y}},$$

where the last equality uses Lemma 4A.7. It follows that $\Theta_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}}$ is multiplication by $a_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}}$. In particular, the map $\Theta_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}}$ is an $\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}])$-module isomorphism and $W^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} \cong V^{\triangle}_{\boldsymbol{\lambda}}$, which implies the desired formulas for $\psi_k f^{\triangle}_{\mathsf{st}^{\triangle}_{\boldsymbol{\lambda}}}$ by Proposition 3C.2 and (4A.8).

Finally, suppose that $\mathsf{t} \neq \mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}$ and let $d_{\mathsf{t}}^{\triangle} = \sigma_{a_1} \ldots \sigma_{a_k}$ be the preferred reduced expression that we fixed for the permutation $d_{\mathsf{t}}^{\triangle} \in \mathfrak{S}_n$ in subsection 2D. Recalling the definition of $Q_m(\mathsf{t})$ from (3B.5), define

$$Q(\mathsf{t}) = Q_{a_1}(\sigma_{a_1}\mathsf{t}) Q_{a_2}(\sigma_{a_2}\sigma_{a_1}\mathsf{t}) \ldots Q_{a_k}(\sigma_{a_k}\ldots\sigma_{a_1}\mathsf{t}).$$

Then $Q(\mathsf{t}) \neq 0$ by Lemma 3B.6. Applying Proposition 3C.2(b) k times,

$$\psi^{\triangle}_{\mathsf{st}} v^{\triangle}_{\mathsf{t}} = \psi_{d^{\triangle}_{\mathsf{s}}} \psi^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} \psi^{*}_{d^{\triangle}_{\mathsf{t}}} v^{\triangle}_{\mathsf{t}} = Q(\mathsf{t})\psi_{d^{\triangle}_{\mathsf{s}}} \psi^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} v^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} = a_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} Q(\mathsf{t})\psi_{d^{\triangle}_{\mathsf{s}}} v^{\triangle}_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} = a_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} Q(\mathsf{t}) v^{\triangle}_{\mathsf{s}}.$$

Therefore, Θ_{t} is multiplication by the scalar $a_{\mathsf{t}} = a_{\mathsf{t}^{\triangle}_{\boldsymbol{\lambda}}} Q(\mathsf{t})$, so $\Theta_{\mathsf{t}} \colon W^{\triangle}_{\mathsf{t}} \xrightarrow{\sim} V^{\triangle}$ is an isomorphism. Hence, the formula for $\psi_k f^{\triangle}_{\mathsf{st}}$ follows from Proposition 3C.2. The proof of Proposition 4A.10 is complete. $\qquad\square$

Since $f^{\triangle}_{\mathsf{st}} = F_{\mathsf{s}}\psi^{\triangle}_{\mathsf{st}} F_{\mathsf{t}}$, this also shows that $\psi^{\triangleleft}_{\mathsf{st}}$ and $\psi^{\triangleright}_{\mathsf{st}}$ are nonzero, for $(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)$. Although we do not state them explicitly, applying the automorphism $*$ to Proposition 4A.10 gives similar formulas for the right actions of the generators of $\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}])$ on the f-bases.

The first corollary of Proposition 4A.10 was established in its proof.

Corollary 4A.11. Let $\boldsymbol{\lambda} \in \mathcal{P}^{\ell}_n$ and suppose $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then, as $\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}])$-modules.

$$V^{\triangleleft}_{\boldsymbol{\lambda}} \cong \bigoplus_{\mathsf{y}} \mathbb{K}[\underline{x}^{\pm}] f^{\triangleleft}_{\mathsf{yt}} \qquad \text{and} \qquad V^{\triangleright}_{\boldsymbol{\lambda}} \cong \bigoplus_{\mathsf{y}} \mathbb{K}[\underline{x}^{\pm}] f^{\triangleright}_{\mathsf{yt}}.$$

Corollary 4A.12. The sets $\{f^{\triangleleft}_{\mathsf{st}} \mid (\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)\}$ and $\{f^{\triangleright}_{\mathsf{st}} \mid (\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)\}$ are bases of $\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}])$.

Proof. Let $\mathbf{i} \in I^n$. By Corollary 3E.10, $1_{\mathbf{i}} \neq 0$ if and only if $\mathbf{i} \in I^n_{\mathrm{Std}} = \{\mathsf{r}(\mathsf{u}) \mid \mathsf{u} \in \mathrm{Std}(\mathcal{P}^{\ell}_n)\}$. Moreover, if $\mathbf{i} \in I^n_{\mathrm{Std}}$ then $1_{\mathbf{i}} = \sum_{\mathsf{u} \in \mathrm{Std}(\mathbf{i})} F_{\mathsf{u}}$. Hence, as $\mathbb{K}[\underline{x}^{\pm}]$-modules,

$$\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}]) = \bigoplus_{\mathbf{i},\mathbf{j} \in I^m_{\mathrm{Std}}} 1_{\mathbf{i}}\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}])1_{\mathbf{j}} = \sum_{\mathsf{s},\mathsf{t} \in \mathrm{Std}(\mathcal{P}^{\ell}_n)} \mathsf{R}^{\Lambda}_{\mathsf{st}}, \qquad \text{where } \mathsf{R}^{\Lambda}_{\mathsf{st}} = F_{\mathsf{s}}\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}])F_{\mathsf{t}}.$$

If $(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)$ then $f^{\triangle}_{\mathsf{st}} \neq 0$, by Proposition 4A.10, and $f^{\triangle}_{\mathsf{st}} \in \mathsf{R}^{\Lambda}_{\mathsf{st}}$, by Corollary 3E.9. $(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)$. Hence, $\{f^{\triangle}_{\mathsf{st}}\}$ is a basis of $\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}])$ and the last displayed equation becomes $\mathsf{R}^{\Lambda}_n(\mathbb{K}[\underline{x}^{\pm}]) = \bigoplus_{(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)} \mathsf{R}^{\Lambda}_{\mathsf{st}}$. $\qquad\square$

The next result shows that the idempotents F_{t} are scalar multiples of the basis elements $f^{\triangleleft}_{\mathsf{tt}}$ and $f^{\triangleright}_{\mathsf{tt}}$. These scalars, $\gamma^{\triangleleft}_{\mathsf{t}}$ and $\gamma^{\triangleright}_{\mathsf{t}}$, play an important role in what follows.

Corollary 4A.13. Suppose that $\mathsf{t} \in \mathrm{Std}(\mathcal{P}^{\ell}_n)$. Then there exist nonzero homogeneous scalars $\gamma^{\triangleleft}_{\mathsf{t}}, \gamma^{\triangleright}_{\mathsf{t}} \in \mathbb{K}[\underline{x}^{\pm}]$ such that

$$\frac{1}{\gamma^{\triangleleft}_{\mathsf{t}}} f^{\triangleleft}_{\mathsf{tt}} = F_{\mathsf{t}} = \frac{1}{\gamma^{\triangleright}_{\mathsf{t}}} f^{\triangleright}_{\mathsf{tt}}.$$

Proof. Let $\triangle \in \{\triangleleft, \triangleright\}$. By Corollary 4A.12, $F_{\mathsf{t}} = \sum_{\mathsf{u},v} r_{\mathsf{uv}} f_{\mathsf{uv}}^{\triangle}$, for some $r_{\mathsf{uv}} \in \mathbb{K}[\underline{x}^{\pm}]$. Multiplying on the left and right by F_{t} and applying Lemma 4A.7 and Corollary 3C.9 shows that $F_{\mathsf{t}} = r_{\mathsf{tt}} f_{\mathsf{tt}}^{\triangle}$. By Corollary 3C.9, $r_{\mathsf{tt}} \neq 0$. Therefore, setting $\gamma_{\mathsf{t}}^{\triangle} = \frac{1}{r_{\mathsf{tt}}}$ gives the result. $\qquad\square$

Lemma 4A.14. Suppose that $(\mathsf{s}, \mathsf{t}), (\mathsf{u}, \mathsf{v}) \in \mathrm{Std}^2(\mathcal{P}_n^{\ell})$. Then

$$f_{\mathsf{st}}^{\triangleleft} f_{\mathsf{uv}}^{\triangleleft} = \delta_{\mathsf{tu}} \gamma_{\mathsf{t}}^{\triangleleft} f_{\mathsf{sv}}^{\triangleleft} \qquad \text{and} \qquad f_{\mathsf{st}}^{\triangleright} f_{\mathsf{uv}}^{\triangleright} = \delta_{\mathsf{tu}} \gamma_{\mathsf{t}}^{\triangleright} f_{\mathsf{sv}}^{\triangleright}.$$

Proof. Let $\triangle \in \{\triangleleft, \triangleright\}$. If $\mathsf{u} \neq \mathsf{t}$ then $f_{\mathsf{st}}^{\triangle} f_{\mathsf{uv}}^{\triangle} = (f_{\mathsf{st}}^{\triangle} F_{\mathsf{t}}) f_{\mathsf{uv}}^{\triangle} = f_{\mathsf{st}}^{\triangle} (F_{\mathsf{t}} f_{\mathsf{uv}}^{\triangle}) = 0$, where we have used Lemma 4A.7 twice. Hence, it remains to consider the products $f_{\mathsf{st}}^{\triangle} f_{\mathsf{tv}}^{\triangle}$. In particular, s, t and v all have the same shape.

By Proposition 4A.10, for $\mathsf{u} \in \mathrm{Std}(\boldsymbol{\lambda})$ there exist homogeneous elements $p_{\mathsf{u}}, q_{\mathsf{u}} \in \mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}^{\pm}])$, which are independent of t, such that $f_{\mathsf{ut}}^{\triangle} = p_{\mathsf{u}} f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle} \mathsf{t}}^{\triangle}$ and $f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle} \mathsf{t}}^{\triangle} = q_{\mathsf{u}} f_{\mathsf{ut}}^{\triangle}$. Therefore, using Corollary 4A.13 and Lemma 4A.7,

$$f_{\mathsf{st}}^{\triangle} f_{\mathsf{tv}}^{\triangle} = p_{\mathsf{s}} f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle} \mathsf{t}}^{\triangle} f_{\mathsf{tv}}^{\triangle} = p_{\mathsf{s}} q_{\mathsf{t}} f_{\mathsf{tt}}^{\triangle} f_{\mathsf{tv}}^{\triangle} = \gamma_{\mathsf{t}}^{\triangle} p_{\mathsf{s}} q_{\mathsf{t}} F_{\mathsf{t}} f_{\mathsf{tv}}^{\triangle} = \gamma_{\mathsf{t}}^{\triangle} p_{\mathsf{s}} q_{\mathsf{t}} f_{\mathsf{tv}}^{\triangle} = \gamma_{\mathsf{t}}^{\triangle} f_{\mathsf{sv}}^{\triangle},$$

as required. $\qquad\square$

We need to determine the γ-coefficients explicitly, which is possible because they satisfy the following recurrence relation involving the scalars $Q_k(\mathsf{s})$ from (3B.5). Note that $Q_k(\mathsf{s}) \neq 0$ whenever $\sigma_k \mathsf{s}$ is standard by Lemma 3B.6.

Lemma 4A.15. Let $\triangle \in \{\triangleleft, \triangleright\}$ and suppose that $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^{\ell})$ with $\mathsf{s} \triangle \mathsf{t} = \sigma_k \mathsf{s}$, where $1 \leq k < n$. Then $\gamma_{\mathsf{t}}^{\triangle} = Q_k(\mathsf{s}) \gamma_{\mathsf{s}}^{\triangle}$.

Proof. By (4A.8), $\beta_k^{\triangle}(\mathsf{s}) = 1$. Therefore, using Lemma 4A.14 and Proposition 4A.10 several times,

$$\gamma_{\mathsf{t}}^{\triangle} f_{\mathsf{ss}}^{\triangle} = f_{\mathsf{st}}^{\triangle} f_{\mathsf{ts}}^{\triangle} = f_{\mathsf{ss}}^{\triangle} \left(\psi_k - \frac{\delta_{\mathsf{r}_k(\mathsf{s}) \mathsf{r}_{k+1}(\mathsf{s})}}{\rho_k(\mathsf{s})} \right)^2 f_{\mathsf{ss}}^{\triangle}$$

$$= f_{\mathsf{ss}}^{\triangle} \left(\psi_k^2 - \frac{2 \psi_k \delta_{\mathsf{r}_k(\mathsf{s}) \mathsf{r}_{k+1}(\mathsf{s})}}{\rho_k(\mathsf{s})} + \frac{\delta_{\mathsf{r}_k(\mathsf{s}) \mathsf{r}_{k+1}(\mathsf{s})}}{\rho_k(\mathsf{s})^2} \right) f_{\mathsf{ss}}^{\triangle}$$

$$= f_{\mathsf{ss}}^{\triangle} \left(Q_{\mathsf{r}_k(\mathsf{s}) \mathsf{r}_{k+1}(\mathsf{s})}^x (c_k(\mathsf{s}), c_{k+1}(\mathsf{s})) - \frac{\delta_{\mathsf{r}_k(\mathsf{s}) \mathsf{r}_{k+1}(\mathsf{s})}}{\rho_k(\mathsf{s})^2} \right) f_{\mathsf{ss}}^{\triangle}$$

$$= Q_k(\mathsf{s}) \gamma_{\mathsf{s}}^{\triangle} f_{\mathsf{ss}}^{\triangle}.$$

For the third equality, notice that $\psi_k f_{\mathsf{ss}}^{\triangle}$ introduces a term involving $f_{\mathsf{ts}}^{\triangle}$ but this term does not survive because $f_{\mathsf{ss}}^{\triangle} f_{\mathsf{ts}}^{\triangle} = 0$ by Lemma 4A.14. The result now follows by Corollary 4A.12. $\qquad\square$

Lemma 4A.16. Suppose that $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$. Then

$$\gamma_{\mathsf{t}}^{\triangleleft} = \prod_{m=1}^{n} \frac{\prod_{A \in \mathrm{Add}_m^{\triangleleft}(\mathsf{t})} (\mathsf{c}_m(\mathsf{t}) - \mathsf{c}(A))}{\prod_{B \in \mathrm{Rem}_m^{\triangleleft}(\mathsf{t})} (\mathsf{c}_m(\mathsf{t}) - \mathsf{c}(B))} \qquad \text{and} \qquad \gamma_{\mathsf{t}}^{\triangleright} = \prod_{m=1}^{n} \frac{\prod_{A \in \mathrm{Add}_m^{\triangleright}(\mathsf{t})} (\mathsf{c}_m(\mathsf{t}) - \mathsf{c}(A))}{\prod_{B \in \mathrm{Rem}_m^{\triangleright}(\mathsf{t})} (\mathsf{c}_m(\mathsf{t}) - \mathsf{c}(B))}.$$

Proof. We consider only the result for $\gamma_{\mathsf{t}}^{\triangleleft}$ and leave the symmetric case of $\gamma_{\mathsf{t}}^{\triangleright}$ to the reader. We argue by induction on dominance. If $\mathsf{t} = \mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}$ then $f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft} \mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}}^{\triangleleft} = y_{\boldsymbol{\lambda}}^{\triangleleft} 1_{\mathbf{i}_{\boldsymbol{\lambda}}^{\triangleleft}}$. Therefore, by Lemma 4A.14 and Proposition 4A.10,

$$\gamma_{t_\lambda^\lhd}^\lhd f_{t_\lambda^\lhd t_\lambda^\lhd}^\lhd = f_{t_\lambda^\lhd t_\lambda^\lhd}^\lhd f_{t_\lambda^\lhd t_\lambda^\lhd}^\lhd = y_\lambda^\lhd f_{t_\lambda^\lhd t_\lambda^\lhd}^\lhd = \prod_{m=1}^{n} \prod_{A \in \mathrm{Add}_m^\lhd(t_\lambda^\lhd)} (\mathsf{c}_m(t_\lambda^\lhd) - \mathsf{c}(A)) f_{t_\lambda^\lhd t_\lambda^\lhd}^\lhd.$$

As $\mathrm{Rem}_m^\lhd(t_\lambda^\lhd) = \emptyset$, for $1 \le m \le n$, this gives the result when $\mathsf{t} = t_\lambda^\lhd$. If $\mathsf{t} \rhd t_\lambda^\lhd$ then, by Lemma 4A.15, there exists a tableau s and an integer a, with $1 \le a < n$, such that $\mathsf{s} \lhd \mathsf{t} = \sigma_a \mathsf{s}$ and $\gamma_\mathsf{t}^\lhd = Q_a(\mathsf{s}) \gamma_\mathsf{s}^\lhd$. To complete the proof, write $(k, r, c) = \mathsf{t}^{-1}(a)$ and observe that $\mathrm{Add}_m^\lhd(\mathsf{t}) = \mathrm{Add}_m^\lhd(\mathsf{s})$ and $\mathrm{Rem}_m^\lhd(\mathsf{t}) = \mathrm{Rem}_m^\lhd(\mathsf{s})$ if $m \ne a, a+1$. Moreover, $\mathrm{Add}_a^\lhd(\mathsf{t}) = \mathrm{Add}_{a+1}^\lhd(\mathsf{s})$ and $\mathrm{Rem}_a^\lhd(\mathsf{t}) = \mathrm{Rem}_{a+1}^\lhd(\mathsf{s})$ and

$$\mathrm{Add}_{a+1}^\lhd(\mathsf{t}) = \begin{cases} \mathrm{Add}_a^\lhd(\mathsf{s}) \setminus \{(k, r, c)\}, & \text{if } \mathsf{r}_a(\mathsf{s}) = \mathsf{r}(k, r, c), \\ \mathrm{Add}_a^\lhd(\mathsf{s}) \cup A, & \text{otherwise,} \end{cases}$$

where A is the set of addable $\mathsf{r}_a(\mathsf{s})$-nodes in $\{(k, r+1, c), (k, r, c-1)\}$. Similarly,

$$\mathrm{Rem}_{a+1}^\lhd(\mathsf{t}) = \begin{cases} \mathrm{Rem}_a^\lhd(\mathsf{s}) \cup \{(k, r, c)\}, & \text{if } \mathsf{r}_a(\mathsf{s}) = \mathsf{r}(k, r, c), \\ \mathrm{Rem}_a^\lhd(\mathsf{s}) \setminus R, & \text{othewise,} \end{cases}$$

where R is the set of removable $\mathsf{r}_a(\mathsf{s})$-nodes in $\{(k, r+1, c), (k, r, c-1)\}$. By induction, the lemma holds for $\gamma_\mathsf{s}^\lhd$. Hence, recalling the definition of $Q_a(\mathsf{s})$ from (3B.5), the Lemma 4A.16 holds for $\gamma_\mathsf{t}^\lhd$ since $\gamma_\mathsf{t}^\lhd = Q_a(\mathsf{s}) \gamma_\mathsf{s}^\lhd$. This completes the proof. $\square$

We can now compute the transition matrices between the ψ-bases and the corresponding f-bases.

Proposition 4A.17. Suppose that $\mathsf{s}, \mathsf{t} \in \mathrm{Std}^2(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. In $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$,

$$\psi_{\mathsf{st}}^\lhd = f_{\mathsf{st}}^\lhd + \sum_{\substack{\boldsymbol{\mu} \lhd \boldsymbol{\lambda} \\ (\mathsf{u}, \mathsf{v}) \in \mathrm{Std}^2(\boldsymbol{\mu})}} a_{\mathsf{uv}} f_{\mathsf{uv}}^\lhd \qquad \text{and} \qquad \psi_{\mathsf{st}}^\rhd = f_{\mathsf{st}}^\rhd + \sum_{\substack{\boldsymbol{\mu} \rhd \boldsymbol{\lambda} \\ (\mathsf{u}, \mathsf{v}) \in \mathrm{Std}^2(\boldsymbol{\mu})}} b_{\mathsf{uv}} f_{\mathsf{uv}}^\rhd$$

for homogeneous coefficients in $\mathbb{K}[\underline{x}^\pm]$ such that

- $a_{\mathsf{uv}} \ne 0$ only if $\mathsf{r}(\mathsf{u}) = \mathsf{r}(\mathsf{s})$, $\mathsf{r}(\mathsf{v}) = \mathsf{r}(\mathsf{t})$ and either $\boldsymbol{\mu} \lhd \boldsymbol{\lambda}$, or $\boldsymbol{\mu} = \boldsymbol{\lambda}$, $\mathsf{u} \lhd \mathsf{s}$ and $\mathsf{v} = \mathsf{t}$
- $b_{\mathsf{uv}} \ne 0$ only if $\mathsf{r}(\mathsf{u}) = \mathsf{r}(\mathsf{s})$, $\mathsf{r}(\mathsf{v}) = \mathsf{r}(\mathsf{t})$ and either $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$, or $\boldsymbol{\mu} = \boldsymbol{\lambda}$, $\mathsf{u} \rhd \mathsf{s}$ and $\mathsf{v} = \mathsf{t}$.

Proof. Let $\triangle \in \{\lhd, \rhd\}$. By Theorem 3E.4 and Corollary 4A.13,

$$1_{\mathsf{i}_\lambda^\triangle} = \sum_\mathsf{u} F_\mathsf{u} = \sum_\mathsf{u} \frac{1}{\gamma_\mathsf{u}^\triangle} f_{\mathsf{uu}}^\triangle,$$

where both sums are over $\mathsf{u} \in \mathrm{Std}(\mathsf{i}_\lambda^\triangle)$. Using Proposition 4A.10,

$$\psi_{t_\lambda^\triangle t_\lambda^\triangle}^\triangle = y_\lambda^\triangle 1_{\mathsf{i}_\lambda^\triangle} = \sum_{\mathsf{u} \in \mathrm{Std}(\mathsf{i}_\lambda^\triangle)} \frac{1}{\gamma_\mathsf{t}^\triangle} y_\lambda^\triangle f_{\mathsf{uu}}^\triangle = \sum_{\mathsf{u} \in \mathrm{Std}(\mathsf{i}_\lambda^\triangle)} \frac{1}{\gamma_\mathsf{t}^\triangle} \prod_{m=1}^n \prod_{A \in \mathrm{Add}_m^\triangle(t_\lambda^\triangle)} (\mathsf{c}_m(\mathsf{u}) - \mathsf{c}(A)) f_{\mathsf{uu}}^\triangle$$

for some $a_\mathsf{u} \in \mathbb{K}[\underline{x}^\pm]$. If $\mathsf{u} = t_\lambda^\triangle$ then the coefficient of $f_{\mathsf{uu}}^\triangle$ in the displayed equation is 1 by Lemma 4A.16. Now suppose that $\mathsf{u} \in \mathrm{Std}(\mathsf{i}_\lambda^\triangle)$ and $\mathsf{u} \ne t_\lambda^\triangle$. Let m be minimal such that $\mathsf{t}_{\downarrow m} \ne (t_\lambda^\triangle)_{\downarrow m}$. Then $A = \mathsf{u}^{-1}(m) \in \mathrm{Add}_m^\triangle(t_\lambda^\triangle)$, so $f_{\mathsf{uu}}^\triangle$ appears in $y_\lambda^\triangle 1_{\mathsf{i}_\lambda^\triangle}$ with coefficient zero. Hence, $f_{\mathsf{uu}}^\triangle$ appears in $\psi_{t_\lambda^\triangle t_\lambda^\triangle}^\triangle$ with nonzero coefficient only if $\mathsf{u} \underline{\triangle} t_\lambda^\triangle$, so $\boldsymbol{\lambda} \triangle \mathrm{Shape}(\mathsf{u})$ if $\mathsf{u} \ne t_\lambda^\triangle$. This proves the base case of our induction. If $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ then

$$\psi_{\mathsf{st}}^\triangle = \psi_{d_\mathsf{s}^\triangle} \psi_{t_\lambda^\triangle t_\lambda^\triangle}^\triangle \psi_{d_\mathsf{t}^\triangle}^* = \psi_{d_\mathsf{s}^\triangle} \left(f_{t_\lambda^\triangle t_\lambda^\triangle}^\triangle + \sum_{\mathsf{u} \triangle t_\lambda^\triangle} a_\mathsf{u} f_{\mathsf{uu}}^\triangle \right) \psi_{d_\mathsf{t}^\triangle}^*.$$

Hence, the result follows by Proposition 4A.10 and induction on $\boldsymbol{\lambda}$. $\qquad\square$

By Corollary 4A.12, this implies that $\{\psi_{\mathsf{st}}^{\triangleleft}\}$ and $\{\psi_{\mathsf{st}}^{\triangleright}\}$ are both bases of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}^{\pm}])$.

4B. **Cellular algebras.** König and Xi [42] introduced affine cellular algebras, generalising results of Graham and Lehrer [21]. Following [24], this subsection incorporates a grading into this framework and at the same time allows the ground ring K to have a non-trivial grading. The next subsection shows that the $f^{\triangle}$ and $\psi^{\triangle}$-bases induce K-cellular structures on the algebras $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}^{\pm}])$ and $\mathsf{R}_n^{\Lambda}(\mathbb{k}[\underline{x}])$.

Definition 4B.1 (cf. Graham and Lehrer, König and Xi [21, 24, 42])**.** Let K be a graded commutative domain with 1 and suppose that A is a graded K-algebra that is K-free and of finite rank as a K-module. A *graded K-cell datum* for A is an ordered tuple $(P, T, a, \deg)$, where $(P, >)$ is the *weight poset*, $T = \coprod_{\lambda \in P} T_{\lambda}$ is a finite set,

$$a \colon \coprod_{\lambda \in P} T_{\lambda} \times T_{\lambda} \longrightarrow A; (s, t) \mapsto a_{st},$$

is an injective map and $\deg \colon T \longrightarrow \mathbb{Z}$ is a *degree function* such that:

(C$_0$) If $s, t \in T_{\lambda}$ then a_{st} is homogeneous of *degree* $\deg(a_{st}) = \deg(s) + \deg(t)$.
(C$_1$) The set $\{a_{st} \mid s, t \in T_{\lambda} \text{ for } \lambda \in P\}$ is a K-basis of A.
(C$_2$) Let $h \in A$ be homogeneous and fix $s, t \in T_{\lambda}$, for $\lambda \in P$. There exist (homogeneous) scalars $r_{su}(h) \in K$, which do not depend on t, such that
$$ha_{st} = \sum_{u \in T_{\lambda}} r_{us}(h) a_{ut} \pmod{A^{>\lambda}},$$
where $A^{>\lambda}$ is the K-submodule of A spanned by $\{a_{vw} \mid \mu > \lambda \text{ and } v, w \in T(\mu)\}$.
(C$_3$) The K-linear map $*\colon A \longrightarrow A$ determined by $(a_{st})^* = a_{ts}$, for all $\lambda \in P$ and $s, t \in T_{\lambda}$, is an anti-isomorphism of A.

A *graded K-cellular algebra* is an algebra that has a graded K-cell datum. A *K-cellular algebra* is an algebra that has a graded K-cell datum such that $\deg(t) = 0$ for all $t \in T$. A *(graded) cellular algebra* is an algebra that has a (graded) K-cell datum when $K = K_0$ is concentrated in degree 0.

Remark 4B.2. If $K = K_0$ is concentrated in degree 0 then a graded K-cellular algebra is a graded cellular algebra in the sense of [24]. If $K = K_0$ and $\deg(t) = 0$ for all $t \in T$ we recover the cellular algebras of Graham and Lehrer [21]. A K-cellular algebra is a graded analogue of the affine cellular algebras of König and Xi [42] in the special case where their affine commutative algebra B is K considered as a K_0-algebra.

If L is a K-algebra, define $A(L) = L \otimes_K A$. Then $A(L)$ is a (graded) L-cellular algebra.

Let $A = A(K)$ be a graded K-cellular algebra with graded K-cell datum $(P, T, c, \deg)$. As in (C$_2$), for $\lambda \in P$ let $A^{\geq \lambda}(K)$ be the K-submodule of A spanned by $\{a_{st} \mid s, t \in T(\mu) \text{ for } \mu \geq \lambda\}$. By (C$_2$) and (C$_3$), $A^{\geq \lambda}(K)$ and $A^{>\lambda}(K) = \bigoplus_{\mu > \lambda} A^{\geq \mu}(K)$ are two-sided ideals of A. Set $A_{\lambda}(K) = A^{\geq \lambda}(K)/A^{>\lambda}(K)$.

For $\lambda \in P$, the *cell module* $S_{\lambda}(K)$ is the free K-module with basis $\{a_s \mid s \in T(\lambda)\}$, where a_s is homogeneous of degree $\deg(s)$, and where the A-action on $S_{\lambda}(K)$ is given by

$$ha_s = \sum_{u \in T(\lambda)} r_{us}(h) a_u, \qquad \text{for } h \in A \text{ and } s \in T(\lambda),$$

where $r_{us}(h) \in K$ is the scalar from (C$_2$). If $t \in T(\lambda)$ then $q^{\deg t} S_{\lambda}(K)$ is isomorphic to the A-submodule of $A_{\lambda}(K)$ with basis $\{a_{st} + A^{>\lambda}(K) \mid s \in T(\lambda)\}$.

If L is a (graded) K-module set $S_\lambda(L) = S_\lambda(K) \otimes_K L$. For example, if $K = \mathbb{K}[x]$ and $L = q^d\mathbb{K}$, which is the $\mathbb{K}[x]$-module concentrated in degree d on which x acts as 0, then $S_\lambda(L) \cong q^d S_\lambda(\mathbb{K})$.

By (C_2) and (C_3), there is a unique symmetric bilinear form $\langle \ , \ \rangle_\lambda \colon S_\lambda(L) \times S_\lambda(L) \longrightarrow L$ such that

$$\langle a_s, a_t \rangle_\lambda a_u = a_{us} a_t \qquad \text{for } s, t, u \in T(\lambda). \tag{4B.3}$$

Moreover, $\langle \ , \ \rangle_\lambda$ is homogeneous and $\langle ax, y \rangle_\lambda = \langle x, a^*y \rangle_\lambda$, for all $a \in A$ and $x, y \in S_\lambda(L)$. In particular, if L is concentrated in degree zero then $\langle \ , \ \rangle_\lambda$ is homogeneous of degree zero. Furthermore,

$$\operatorname{rad} S_\lambda(L) = \{x \in S_\lambda(L) \mid \langle x, y \rangle = 0 \text{ for all } y \in S_\lambda(L)\}$$

is a graded A-module of $S_\lambda(L)$, so that $D_\lambda(L) = S_\lambda(L)/\operatorname{rad} S_\lambda(L)$ is a graded A-module.

Suppose that $K = \bigoplus_d K_d$ is a graded commutative ring such that K_0 is a field. Then K_d is a finite dimensional K_0-vector space. Let $\operatorname{Irr}(K)$ be a complete set of irreducible graded K-modules, up to isomorphism. Recall from subsection 2A that q is the grading shift functor.

Lemma 4B.4. Suppose that $K = \mathbb{K}[\underline{x}]$. Then $\operatorname{Irr}(\mathbb{K}[\underline{x}]) = \{q^d\mathbb{K} \mid d \in \mathbb{Z}\}$.

Proof. Any irreducible graded $\mathbb{K}[\underline{x}]$-module is a $\mathbb{K}$-vector space on which each $x \in \underline{x}$ acts as multiplication by 0. (Compare Remark 2A.2.) $\qquad\square$

Example 4B.5. Suppose that $\mathbb{K}$ is a field and x is an indeterminate over $\mathbb{K}$. Then $\mathbb{K}[x]$ is a graded field and $q^d\mathbb{K}[x^\pm] = \mathbb{K}[x^\pm]$, for $d \in \mathbb{Z}$, since x has degree 1. Hence, $\mathbb{K}[x^\pm]$ is the unique irreducible graded $\mathbb{K}[x^\pm]$-module. In contrast, if $\deg y = 2$ the $\operatorname{Irr}(\mathbb{K}[y^\pm] = \{\mathbb{K}[y^\pm], q\mathbb{K}[y^\pm]\}$. (This is why we define each indeterminate $x \in \underline{x}$ to have degree 1.)

Now consider $\mathbb{K}[x^\pm, y^\pm]$, where y be a second indeterminate over $\mathbb{K}$. Then $L = \mathbb{K}[z^\pm]$ becomes an irreducible graded $\mathbb{K}[x^\pm, y^\pm]$-module by letting x act as multiplication by $c_1 z$ and y act as multiplication by $c_2 z$, for nonzero scalars $c_1, c_2 \in \mathbb{K}^\times$. Equivalently, the module $L \cong \mathbb{K}[x^\pm, y^\pm]/(c_2 x - c_1 y)$ is uniquely determined by the fact that $x - \frac{c_1}{c_2}y$ acts on L as multiplication by 0. Hence, this makes L into an irreducible graded $\mathbb{K}[x^\pm, y^\pm]$-module for each $c \in \mathbb{K}^\times$.

Assume that K_0 is a field. If $L \in \operatorname{Irr}(K)$ set $P_0(L) = \{\lambda \in P \mid D_\lambda(L) \neq 0\}$.

Theorem 4B.6. Let K be a graded commutative domain such that K_0 is a field. Suppose that A be a graded K-cellular algebra. Then

$$\{D_\lambda(L) \mid \lambda \in P_0(L) \text{ and } L \in \operatorname{Irr}(K)\}$$

is a complete set of pairwise non-isomorphic irreducible graded A-modules. Moreover, $D_\lambda(L)$ is self-dual as an A-module if and only if $L \in \operatorname{Irr}(K)$ is self-dual as a K-module.

Proof. By Lemma 4B.4, up to shift the irreducible graded A-modules are irreducible $A(M)$-modules. The result now follows by repeating the standard arguments for classifying the simple modules of cellular algebras; see [42, Theorem 3.12], [21, Theorem 3.4], or [49, Theorem 2.16]. $\qquad\square$

Example 4B.7. Suppose that A is a graded $\mathbb{K}[\underline{x}]$-cellular algebra, where $\mathbb{K}$ is a field. Define P_0 as above. By Lemma 4B.4, $\operatorname{Irr}(\mathbb{K}[x]) = \{q^d\mathbb{K} \mid q \in \mathbb{Z}\}$. So

$$\{D_\lambda(L) \mid \lambda \in P_0 \text{ and } L \in \operatorname{Irr}(\mathbb{K}[\underline{x}])\} = \{q^d D_\lambda(\mathbb{K}) \mid \lambda \in P_0 \text{ and } d \in \mathbb{Z}\}$$

is a complete set of pairwise non-isomorphic irreducible graded A-modules. Let $A(\mathbb{K}[x^{\pm}])$ $= \mathbb{K}[x^{\pm}] \otimes_{\mathbb{K}[x]} A$ be the corresponding graded $\mathbb{K}[x^{\pm}]$-cellular algebra over $\mathbb{K}[x^{\pm}]$. Then $\{D_\lambda(\mathbb{K}[x^{\pm}]) \,|\, \lambda \in P_0\}$ is a complete set of pairwise non-isomorphic irreducible graded $A(\mathbb{K}[x^{\pm}])$-modules.

Definition 4B.8. Suppose that $K = \mathbb{K}[\underline{x}]$ and let A be a graded $\mathbb{K}[\underline{x}]$-cellular algebra. Let $\lambda \in P$ and $\mu \in P_0 = P_0(\mathbb{K})$ and set $S_\lambda = S_\lambda(\mathbb{K})$ and $D_\mu = D_\mu(\mathbb{K})$. Then $\mathsf{D} = ([S_\lambda : D_\mu]_q)$ is the *graded decomposition matrix* of A, where

$$[S_\lambda : D_\mu]_q = \sum_{k \in \mathbb{Z}} \left[S_\lambda : q^k D_\mu \right] q^k \quad \in \mathbb{N}\left[q, q^{-1}\right],$$

and $[S_\lambda : q^k D_\mu]$ is the multiplicity of $q^k D_\mu$ as a composition factor of S_λ.

Standard arguments from the theory of cellular algebras now prove the following:

Corollary 4B.9. Suppose that A is a graded $\mathbb{K}[\underline{x}]$-cellular algebra. Then
   (a) If $\lambda \in P$ and $\mu \in P_0$ then $[S_\mu : D_\mu]_q = 1$ and $[S_\lambda : D_\mu]_q \neq 0$ only if $\lambda \geq \mu$.
   (b) The Cartan matrix of A is $\mathsf{D}^T \mathsf{D}$.

4C. **Cellular bases for $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$.** This subsection applies the results of the last two subsections to show that $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$ is a $\mathbb{K}[\underline{x}^{\pm}]$-cellular algebra. We have to wait until subsection 4F to prove that $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}])$ is a $\mathbb{k}[\underline{x}]$-cellular algebra.

We have most of the data we need to define graded $\mathbb{K}[\underline{x}^{\pm}]$-cell data for $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^{\pm}])$: we have posets $(\mathcal{P}_n^\ell, \lhd)$ and $(\mathcal{P}_n^\ell, \rhd)$ and sets of standard tableaux $\mathrm{Std}(\mathcal{P}_n^\ell) = \coprod_{\lambda \in \mathcal{P}_n^\ell} \mathrm{Std}(\boldsymbol{\lambda})$. Moreover, by the results of subsection 4A, we have bases $\{f_{\mathsf{st}}^\lhd\}$, $\{\psi_{\mathsf{st}}^\lhd\}$, $\{f_{\mathsf{st}}^\rhd\}$ and $\{\psi_{\mathsf{st}}^\rhd\}$, which we view as being given by injective maps

$$f^\lhd \to \mathsf{R}_n^\Lambda\left(\mathbb{K}[\underline{x}^{\pm}]\right), \quad \psi^\lhd \to \mathsf{R}_n^\Lambda\left(\mathbb{K}[\underline{x}^{\pm}]\right), \quad f^\rhd \to \mathsf{R}_n^\Lambda\left(\mathbb{K}[\underline{x}^{\pm}]\right) \quad \text{and} \quad \psi^\lhd \to \mathsf{R}_n^\Lambda\left(\mathbb{K}[\underline{x}^{\pm}]\right),$$

which send (s, t) to $f_{\mathsf{st}}^\lhd$, $\psi_{\mathsf{st}}^\lhd$, $f_{\mathsf{st}}^\rhd$ and $\psi_{\mathsf{st}}^\rhd$, respectively. We still need to define corresponding degree functions on $\mathrm{Std}(\mathcal{P}_n^\ell)$.

For $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$, recall the homogeneous scalars $\gamma_{\mathsf{t}}^\lhd, \gamma_{\mathsf{t}}^\rhd \in \mathbb{K}[\underline{x}^{\pm}]$ from Corollary 4A.13. As $\mathbb{K}[\underline{x}^{\pm}]$ is a graded ring, both of these scalars have a degree in $\mathbb{Z}$. Recall that $\deg \colon \mathbb{K}[\underline{x}^{\pm}] \longrightarrow \mathbb{Z}$ is the degree function on $\mathbb{K}[\underline{x}^{\pm}]$ and that $\deg(x) = 1$, for all $x \in \underline{x}$. By Lemma 4A.16, the scalars $\gamma_{\mathsf{t}}^\lhd$ and $\gamma_{\mathsf{t}}^\rhd$ depend on the content function c and are polynomials in x^2 and, in particular, have even degree.

Definition 4C.1. Let $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$. Define *degree functions*,

$$\deg^\lhd \colon \mathrm{Std}(\mathcal{P}_n^\ell) \longrightarrow \mathbb{Z} \qquad \text{and} \qquad \deg^\rhd \colon \mathrm{Std}(\mathcal{P}_n^\ell) \longrightarrow \mathbb{Z},$$

with respect to the posets $(\mathcal{P}_n^\ell, \lhd)$ and $(\mathcal{P}_n^\ell, \rhd)$, respectively, by

$$\deg^\lhd(\mathsf{t}) = \frac{1}{2} \deg \gamma_{\mathsf{t}}^\lhd \qquad \text{and} \qquad \deg^\rhd(\mathsf{t}) = \frac{1}{2} \deg \gamma_{\mathsf{t}}^\rhd, \qquad \text{for } \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell).$$

When (c, r) is a graded content system both of these degree functions already exist in the literature. In type $A_{e-1}^{(1)}$, Brundan, Kleshchev and Wang [13] call $\deg^\rhd$ the degree of a tableau and $\deg^\lhd$ its codegree. In type $C_{e-1}^{(1)}$ Ariki, Park and Speyer [8] use $\deg^\rhd$ to define the degrees of the basis elements of their candidates for homogeneous Specht modules. Using Definition 4C.1 it is not clear that these degree functions coincide with those given in [8, 13], however, this is immediate from the next result.

Recall from subsection 2B that $D = \mathrm{diag}(\mathsf{d}_i | i \in I)$ is the symmetriser of the Cartan matrix of Γ.

Lemma 4C.2. Suppose that $t \in \mathrm{Std}(\mathcal{P}_n^\ell)$. Then

$$\deg^\lhd(t) = \sum_{m=1}^n d_{r_m(t)}\big(\# \mathrm{Add}_m^\lhd(t) - \# \mathrm{Rem}_m^\lhd(t)\big)$$

and

$$\deg^\rhd(t) = \sum_{m=1}^n d_{r_m(t)}\big(\# \mathrm{Add}_m^\rhd(t) - \# \mathrm{Rem}_m^\rhd(t)\big).$$

Proof. Apply Lemma 4A.16, using the fact that $\gamma_t^\triangle \neq 0$ and $\deg c_m(t) = 2d_{r_m(t)}$, which follows from Definition 3A.1 (c) because (c, r) is a graded content system. □

We can now show that $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ is a (graded) $\mathbb{K}[\underline{x}^\pm]$-cellular algebra.

Theorem 4C.3. Suppose that (c, r) is a graded content system for $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}])$. Then the algebra $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ is a $\mathbb{K}[\underline{x}^\pm]$-cellular algebra with cellular bases:

  (a) $\{f_{st}^\lhd \mid (s,t) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \trianglelefteq)$ and degree function $\deg^\lhd$.
  (b) $\{f_{st}^\rhd \mid (s,t) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \trianglerighteq)$ and degree function $\deg^\rhd$.
  (c) $\{\psi_{st}^\lhd \mid (s,t) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \trianglelefteq)$ and degree function $\deg^\lhd$.
  (d) $\{\psi_{st}^\rhd \mid (s,t) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \trianglerighteq)$ and degree function $\deg^\rhd$.

Proof. Let $\triangle \in \{\lhd, \rhd\}$. By Corollary 4A.12, $\{f_{st}^\triangle\}$ is a $\mathbb{K}[\underline{x}^\pm]$-basis of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ and by Proposition 4A.10 the $f^\triangle$-basis satisfies (C_2). Recall that $*$ is unique anti-isomorphism of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ that fixes each of its generators. By construction, $(\psi_{st}^\triangle)^* = \psi_{ts}^\triangle$ and $F_s^* = F_s$, so $(f_{st}^\triangle)^* = f_{ts}^\triangle$ for $(s,t) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)$. Hence, $\{f_{st}^\triangle\}$ is a $\mathbb{K}[\underline{x}^\pm]$-cellular basis of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$.

Next, consider $\{\psi_{st}^\triangle\}$. This is a basis of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ by Proposition 4A.17, so (C_1) is satisfied. We have already seen that $(\psi_{st}^\triangle)^* = \psi_{ts}^\triangle$, verifying (C_3), so it remains to check (C_2). By Proposition 4A.17,

$$\psi_{st_\lambda^\triangle}^\triangle \equiv f_{st_\lambda^\triangle}^\triangle + \sum_{u \triangle s} r_u f_{ut_\lambda^\triangle}^\triangle \ \big(\mathrm{mod}\ (\mathsf{R}_n^\Lambda)^{\triangle\lambda}\big)$$

for some $r_u \in \mathbb{K}[\underline{x}^\pm]$ and where $(\mathsf{R}_n^\Lambda)^{\triangle\lambda}$ is the two-sided ideal of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ spanned by $\{f_{uv}^\triangle\}$ where $\mathrm{Shape}(u) = \mathrm{Shape}(v) \triangle \lambda$. By Proposition 4A.17, $(\mathsf{R}_n^\Lambda)^{\triangle\lambda}$ is also spanned by $\{\psi_{uv}^\triangle\}$. Multiplying the last displayed equation on the left by $a \in \mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$, and using Proposition 4A.10 and Proposition 4A.17,

$$a\psi_{st_\lambda^\triangle}^\triangle \equiv a\left(f_{st_\lambda^\triangle}^\triangle + \sum_{u \triangle s} a_u f_{ut_\lambda^\triangle}^\triangle\right) = \sum_{x \in \mathrm{Std}(\lambda)} b_x f_{xt_\lambda^\triangle}^\triangle \equiv \sum_{x \in \mathrm{Std}(\lambda)} c_x \psi_{xt_\lambda^\triangle}^\triangle \quad \mathrm{mod}\ (\mathsf{R}_n^\Lambda)^{\triangle\lambda},$$

for some homogeneous scalars $a_u, b_x, t_x \in \mathbb{K}[\underline{x}^\pm]$. Multiplying on the right by $\psi_{d_t^\triangle}^*$ shows that $\psi_{st}^\triangle$ satisfies (C_2). Hence, $\{\psi_{st}^\triangle\}$ is a $\mathbb{K}[\underline{x}^\pm]$-cellular basis of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$.

It remains to show that each of these bases is a graded $\mathbb{K}[\underline{x}^\pm]$-cellular basis of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ when (c, r) is a graded content system. By Definition 4A.5, $\psi_{st}^\triangle$ is homogeneous, for $(s,t) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)$. By Definition 3C.1, F_t is homogeneous of degree 0, and $f_{st}^\triangle = F_s \psi_{st}^\triangle F_t$. Hence, $f_{st}^\triangle$ is homogeneous and $\deg f_{st}^\triangle = \deg \psi_{st}^\triangle$. Therefore, it is enough to show that $\deg f_{st}^\triangle = \deg^\triangle(s) + \deg^\triangle(t)$. Further, since $*$ is homogeneous, $\deg f_{st}^\triangle = \deg f_{ts}^\triangle$. So, using Lemma 4A.14,

$$\deg f_{st}^\triangle = \tfrac{1}{2} \deg\left(f_{st}^\triangle f_{ts}^\triangle\right) = \tfrac{1}{2} \deg\left(\gamma_t^\triangle f_{ss}^\triangle\right) = \tfrac{1}{2} \deg\left(\gamma_s^\triangle \gamma_t^\triangle F_s\right) = \deg(s) + \deg(t),$$

as we wanted to show. This completes the proof. □

Proving that $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$ is a $\mathbb{K}[\underline{x}^\pm]$-cellular algebra is nice but it does not directly help us in constructing a cellular basis for the KLR algebras $\mathscr{R}_n^\Lambda(\Bbbk)$ and $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$. We prove that $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ is $\Bbbk[\underline{x}]$-cellular in the next subsection. As a prelude to doing this, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ define $S_{\boldsymbol{\lambda}}^{\lhd}(\mathbb{K}[\underline{x}^\pm])$ and $S_{\boldsymbol{\lambda}}^{\rhd}(\mathbb{K}[\underline{x}^\pm])$ to be the graded cell modules for $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$-determined by the seminormal bases $\{f_{\mathsf{st}}^{\lhd}\}$ and $\{f_{\mathsf{st}}^{\rhd}\}$, respectively. Let $\triangle \in \{\lhd, \rhd\}$. By Proposition 4A.10, $S_{\boldsymbol{\lambda}}^{\triangle}(\mathbb{K}[\underline{x}^\pm])$ has basis $\{f_{\mathsf{s}}^{\triangle}\}$ and there is an isomorphism

$$q^{\deg^{\triangle} \mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}} S_{\boldsymbol{\lambda}}^{\triangle}\left(\mathbb{K}[\underline{x}^\pm]\right) \cong \left(\mathsf{R}_n^\Lambda\left(\mathbb{K}[\underline{x}^\pm]\right) f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} + (\mathsf{R}_n^\Lambda)^{\triangle\boldsymbol{\lambda}}\right) / (\mathsf{R}_n^\Lambda)^{\triangle\boldsymbol{\lambda}}; f_{\mathsf{s}}^{\triangle} \mapsto f_{\mathsf{st}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} + (\mathsf{R}_n^\Lambda)^{\triangle\boldsymbol{\lambda}}.$$

For $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$, let $\psi_{\mathsf{s}}^{\triangle} = \psi_{d_{\mathsf{s}}^{\triangle}} f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle}$ be the element of $S_{\boldsymbol{\lambda}}^{\triangle}(\mathbb{K}[\underline{x}^\pm])$ that is sent to $\psi_{\mathsf{st}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} + (\mathsf{R}_n^\Lambda)^{\triangle\boldsymbol{\lambda}}$ under this isomorphism. In view of Corollary 3C.10 and Proposition 4A.17, we have:

Lemma 4C.4. Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. As $\mathbb{K}[\underline{x}^\pm]$-modules,

$$S_{\boldsymbol{\lambda}}^{\lhd}\left(\mathbb{K}[\underline{x}^\pm]\right) = \bigoplus_{\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})} \mathbb{K}[\underline{x}^\pm]\psi_{\mathsf{s}}^{\lhd} \quad \text{and} \quad S_{\boldsymbol{\lambda}}^{\rhd}\left(\mathbb{K}[\underline{x}^\pm]\right) = \bigoplus_{\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})} \mathbb{K}[\underline{x}^\pm]\psi_{\mathsf{s}}^{\rhd}.$$

By Lemma 4A.7, if $\theta \colon S_{\boldsymbol{\lambda}}^{\lhd}(\mathbb{K}[\underline{x}^\pm]) \longrightarrow S_{\boldsymbol{\lambda}}^{\rhd}(\mathbb{K}[\underline{x}^\pm])$ is an isomorphism then $\theta(f_{\mathsf{s}}^{\lhd}) = af_{\mathsf{s}}^{\rhd}$, for some $a \in \mathbb{K}[\underline{x}^\pm]$. Comparing degrees, a is homogeneous of degree $\deg^{\lhd}(\mathsf{s}) - \deg^{\rhd}(\mathsf{s})$. In particular, such an isomorphism and its inverse are defined over $\Bbbk[\underline{x}]$ if and only if $\deg^{\lhd}(\mathsf{s}) = \deg^{\rhd}(\mathsf{s})$ for all $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$.

Let $S_{\boldsymbol{\lambda}}^{\lhd}(\Bbbk[\underline{x}]) = \bigoplus_{\mathsf{s}} \Bbbk[\underline{x}]\psi_{\mathsf{s}}^{\lhd}$ and $S_{\boldsymbol{\lambda}}^{\rhd}(\Bbbk[\underline{x}]) = \bigoplus_{\mathsf{s}} \Bbbk[\underline{x}]\psi_{\mathsf{s}}^{\rhd}$, where in both sums $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$. By definition, $S_{\boldsymbol{\lambda}}^{\lhd}(\Bbbk[\underline{x}])$ and $S_{\boldsymbol{\lambda}}^{\rhd}(\Bbbk[\underline{x}])$ are free $\Bbbk[\underline{x}]$-modules and $S_{\boldsymbol{\lambda}}^{\triangle}(\mathbb{K}[\underline{x}^\pm]) \cong \mathbb{K}[\underline{x}^\pm] \otimes_{\Bbbk[\underline{x}]} S_{\boldsymbol{\lambda}}^{\triangle}(\Bbbk[\underline{x}])$ by Lemma 4C.4. In fact, $S_{\boldsymbol{\lambda}}^{\lhd}(\Bbbk[\underline{x}])$ and $S_{\boldsymbol{\lambda}}^{\rhd}(\Bbbk[\underline{x}])$ are both $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$-modules.

Proposition 4C.5. Suppose that $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Then:

(a) If $1 \le k < n$ then $\psi_k \psi_{\mathsf{s}}^{\lhd} \in S_{\boldsymbol{\lambda}}^{\lhd}(\Bbbk[\underline{x}])$ and $\psi_k \psi_{\mathsf{s}}^{\rhd} \in S_{\boldsymbol{\lambda}}^{\rhd}(\Bbbk[\underline{x}])$.

(b) If $1 \le m \le n$ then $y_m \psi_{\mathsf{s}}^{\lhd} \in S_{\boldsymbol{\lambda}}^{\lhd}(\Bbbk[\underline{x}])$ and $y_m \psi_{\mathsf{s}}^{\rhd} \in S_{\boldsymbol{\lambda}}^{\rhd}(\Bbbk[\underline{x}])$.

(c) If $\sigma_{b_1} \ldots \sigma_{b_l}$ is a reduced expression for $d_{\mathsf{s}}^{\triangle}$ then

$$\psi_{\mathsf{s}}^{\lhd} - \psi_{b_1} \ldots \psi_{b_l} \psi_{\mathsf{t}_{\boldsymbol{\lambda}}}^{\lhd} \in \bigoplus_{\mathsf{u} \lhd \mathsf{s}} \Bbbk[\underline{x}]\psi_{\mathsf{u}}^{\lhd} \qquad \text{and} \qquad \psi_{\mathsf{s}}^{\rhd} - \psi_{b_1} \ldots \psi_{b_l} \psi_{\mathsf{t}_{\boldsymbol{\lambda}}}^{\rhd} \in \bigoplus_{\mathsf{u} \rhd \mathsf{s}} \Bbbk[\underline{x}]\psi_{\mathsf{u}}^{\rhd}.$$

Proof. Let $\triangle \in \{\lhd, \rhd\}$. To prove the proposition we argue by induction on the length $L(d_{\mathsf{s}}^{\triangle})$ of the permutation $d_{\mathsf{s}}^{\triangle}$. To start the induction, suppose that $\mathsf{s} = \mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}$, so that $d_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} = 1$. Then (c) is vacuously true and,

$$y_m \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} = y_m f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} = \mathsf{c}_m(\mathsf{s}) f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle}$$

by Proposition 4A.10, so (b) holds. To prove (a), applying Proposition 4A.10 shows that

$$\psi_k \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} = \psi_k f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}}^{\triangle} = \begin{cases} f_u^{\triangle} = \psi_u^{\triangle} & \text{if } \mathsf{u} = \sigma_k \mathsf{t}_{\boldsymbol{\lambda}}^{\triangle} \in \mathrm{Std}(\boldsymbol{\lambda}), \\ 0 & \text{if } \sigma_k \mathsf{t}_{\boldsymbol{\lambda}}^{\triangle} \notin \mathrm{Std}(\boldsymbol{\lambda}). \end{cases}$$

Hence, the proposition is true when $\mathsf{s} = \mathsf{t}_{\boldsymbol{\lambda}}^{\triangle}$.

Now suppose that $\mathsf{t}_{\boldsymbol{\lambda}}^{\triangle} \triangle \mathsf{s}$. First, consider (c). Let $d_{\mathsf{s}}^{\triangle} = \sigma_{a_1} \ldots \sigma_{a_l}$ be the preferred reduced expression for $d_{\mathsf{s}}^{\triangle}$ that we fixed after Lemma 4A.3. If $\sigma_{b_1} \ldots \sigma_{b_l}$ is a second reduced expression for $d_{\mathsf{s}}^{\triangle}$ then, by Matsumoto's theorem (see, for example, [49, Theorem 1.8]), we can convert the reduced expression $\sigma_{a_1} \ldots \sigma_{a_l}$ into our preferred reduced expression $\sigma_{b_1} \ldots \sigma_{b_l}$ using only the braid relations of $\mathfrak{S}_n$. The ψ_k satisfy the commuting braid relations and by (KLR8) they satisfy the braid relations of length 3 plus or minus an

"error term" of the form $\delta_{i_k i_{k+2}} Q^x_{i_k i_{k+1} i_{k+1}}(y_k, y_{k+1}, y_{k+1})\psi_u$, where u is smaller than $d^\triangle_{\sf s}$ in the Bruhat order and so, in particular, $L(u) < L(d^\triangle_{\sf s})$. Hence, by induction, part (c) holds for $\psi^\triangle_{\sf s}$.

Now consider $\psi_k \psi^\triangle_{\sf s}$ as in (a). If $L(\sigma_k d^\triangle_{\sf s}) < L(d^\triangle_{\sf s})$ then $d^\triangle_{\sf s}$ has a reduced expression of the form $\sigma_k \sigma_{a_2} \dots \sigma_{a_l}$. Therefore, using (c), which we have already proved,

$$\psi_k \psi^\triangle_{\sf s} = \psi_k \left(\psi_k \psi_{a_2} \dots \psi_{a_l} \psi^\triangle_{{\sf t}^\triangle_\lambda} + \sum_{{\sf u} \,\triangle\, {\sf s}} r_{\sf u} \psi^\triangle_{\sf u} \right), \qquad \text{for some } r_{\sf u} \in \Bbbk[\underline{x}],$$

$$= \psi_k^2 \psi_{a_2} \dots \psi_{a_l} \psi^\triangle_{{\sf t}^\triangle_\lambda} + \sum_{{\sf u} \,\triangle\, {\sf s}} r_{\sf u} \psi_k \psi^\triangle_{\sf u}$$

$$= Q^{\underline{x}}_{r_{k+1}({\sf s}), r_k({\sf s})}(y_k, y_{k+1}) \psi_{a_2} \dots \psi_{a_l} \psi^\triangle_{{\sf t}^\triangle_\lambda} + \sum_{{\sf u} \,\triangle\, {\sf s}} r_{\sf u} \psi_k \psi^\triangle_{\sf u}.$$

By induction, all of these terms belong to $S^\triangle_\lambda(\Bbbk[\underline{x}])$, showing that $\psi^\triangle_{\sf s}$ satisfies (a).

Finally, consider $y_m \psi^\triangle_{\sf s}$. Let ${\sf v} \in \mathrm{Std}(\boldsymbol{\lambda})$ be the unique standard tableau such that ${\sf s} = \sigma_{r_1} {\sf v}$. Then $L(d^\triangle_{\sf s}) = L(d^\triangle_{\sf v}) + 1$, so by part (c) and induction,

$$\psi^\triangle_{\sf v} = \psi_{a_2} \dots \psi_{a_l} \psi^\triangle_{{\sf t}^\triangle_\lambda} + \sum_{{\sf u} \,\triangle\, {\sf v}} r_{\sf u} \psi^\triangle_{\sf u},$$

for some $r_{\sf u} \in \Bbbk[\underline{x}]$ (these scalars are different from those in the last paragraph). Therefore,

$$y_m \psi^\triangle_{\sf s} = y_m \psi_{r_1} \left(\psi^\triangle_{\sf v} - \sum_{{\sf u} \,\triangle\, {\sf v}} r_{\sf u} \psi^\triangle_{\sf u} \right).$$

Applying (KLR6) and induction now completes the proof of Proposition 4C.5. $\qquad\square$

4D. **Defect polynomials.** The algebra $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}^\pm])$ is a split semisimple graded algebra, so it is naturally a symmetric algebra with symmetrising form given by taking the matrix trace on the regular representation. This form does not restrict to give a trace on $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}])$, so the aim of this subsection is to show how to use this trace form to give an "integral" trace form" on $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}])$. In later subsections, these results will be used to understand the duals of some $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}])$-modules.

We continue to assume that (c, r) is a graded content system for $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}])$ with values in $\Bbbk[\underline{x}]$. The following innocuous lemma is the key to constructing our trace form and to understanding the defect of the blocks of $\mathsf{R}^\Lambda_n(\Bbbk[\underline{x}])$.

Lemma 4D.1. Suppose that $\boldsymbol{\lambda} \in \mathcal{P}^\ell_n$. Then $\gamma^\triangleleft_{\sf s} \gamma^\triangleright_{\sf s} = \gamma^\triangleleft_{\sf t} \gamma^\triangleright_{\sf t}$ for all ${\sf s}, {\sf t} \in \mathrm{Std}(\boldsymbol{\lambda})$.

Proof. It is enough to consider the case when ${\sf s} \triangleright {\sf t} = \sigma_k {\sf s}$, for some $1 \le k < n$. In this case we have that $\gamma^\triangleright_{\sf t} = Q_k({\sf s}) \gamma^\triangleright_{\sf s}$ and $\gamma^\triangleleft_{\sf s} = Q_k({\sf t}) \gamma^\triangleleft_{\sf t}$ by Lemma 4A.15. By (3B.5) and the symmetry of Rouquier's Q-polynomials, $Q_k({\sf s}) = Q_k({\sf t})$. Hence,

$$\gamma^\triangleleft_{\sf s} \gamma^\triangleright_{\sf s} = \frac{Q_k({\sf t})}{Q_k({\sf s})} \cdot \gamma^\triangleleft_{\sf t} \gamma^\triangleright_{\sf t} = \gamma^\triangleleft_{\sf t} \gamma^\triangleright_{\sf t},$$

as required. $\qquad\square$

Definition 4D.2. Let $\boldsymbol{\lambda} \in \mathcal{P}^\ell_n$. The $\boldsymbol{\lambda}$-*defect polynomial* is $\gamma^\Phi_{\boldsymbol{\lambda}} = \gamma^\triangleleft_{\sf t} \gamma^\triangleright_{\sf t}$, for any ${\sf t} \in \mathrm{Std}(\boldsymbol{\lambda})$.

By Lemma 4D.1, the defect polynomial $\gamma^\Phi_{\boldsymbol{\lambda}}$ depends only on $\boldsymbol{\lambda}$, and not on the choice of ${\sf t} \in \mathrm{Std}(\boldsymbol{\lambda})$. We will show in Corollary 4D.7 that the degree of the defect polynomial

is a block invariant. That is, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_\alpha^\ell$ then $\deg \gamma_{\boldsymbol{\lambda}}^{\diamond} = \deg \gamma_{\boldsymbol{\mu}}^{\diamond}$. To prove this, and to explain why we call this the *defect* polynomial we need some more notation.

For $i \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ let $\mathrm{Add}_i(\boldsymbol{\lambda})$ and $\mathrm{Rem}_i(\boldsymbol{\lambda})$ be the sets of addable and removable i-nodes of $\boldsymbol{\lambda}$, respectively. Recall from subsection 2B that $\{\mathsf{d}_i \mid i \in I\}$ is the set of symmetrisers of Γ.

Definition 4D.3. Let $\alpha \in Q^+$.

  (a) For $\alpha \in Q_n^+$ let $\mathcal{P}_\alpha^\ell = \{\boldsymbol{\lambda} \in \mathcal{P}_n^\ell \mid \alpha_{\boldsymbol{\lambda}} = \alpha\}$.
  (b) The Λ-*defect* of $\alpha \in Q_n^+$ is $\mathrm{def}(\alpha) = (\Lambda|\alpha) - \frac{1}{2}(\alpha|\alpha)$.
  (c) The $\boldsymbol{\lambda}$-*positive root* is $\alpha_{\boldsymbol{\lambda}} = \sum_{A \in \boldsymbol{\lambda}} \alpha_{\mathsf{r}(A)} \in Q_n^+$.
  (d) The Λ-*defect* of $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$ is $\mathrm{def}(\boldsymbol{\lambda}) = \mathrm{def}(\alpha_{\boldsymbol{\lambda}})$.
  (e) Motivated by (4A.4), given an addable or removable i-node A of $\boldsymbol{\lambda}$ define

$$d_A^{\triangleleft}(\boldsymbol{\lambda}) = \mathsf{d}_i \times \left(\#\{B \in \mathrm{Add}_i(\boldsymbol{\lambda}) \mid B < A\} - \#\{B \in \mathrm{Rem}_i(\boldsymbol{\lambda}) \mid B < A\} \right),$$

$$d_A^{\triangleright}(\boldsymbol{\lambda}) = \mathsf{d}_i \times \left(\#\{B \in \mathrm{Add}_i(\boldsymbol{\lambda}) \mid B > A\} - \#\{B \in \mathrm{Rem}_i(\boldsymbol{\lambda}) \mid B > A\} \right),$$

$$d_i(\boldsymbol{\lambda}) = \mathsf{d}_i \times \left(\# \mathrm{Add}_i(\boldsymbol{\lambda}) - \# \mathrm{Rem}_i(\boldsymbol{\lambda}) \right).$$

By definition, $\mathrm{def}(\alpha) \in \mathbb{Z}$. We show in Corollary 6E.21 that, in fact, $\mathrm{def}(\alpha) \in \mathbb{N}$. Generalising [13, Lemma 3.11], we give some standard facts about defect.

Lemma 4D.4. Let $\boldsymbol{\lambda} = \boldsymbol{\mu} + A$, where $A \in \mathrm{Add}_i(\boldsymbol{\mu})$ for $i \in I$. Then $\alpha_{\boldsymbol{\lambda}} = \alpha_{\boldsymbol{\mu}} + \alpha_i$ and

$$d_i(\boldsymbol{\lambda}) = d_i(\boldsymbol{\mu}) - 2\mathsf{d}_i = d_A^{\triangleleft}(\boldsymbol{\lambda}) + d_A^{\triangleright}(\boldsymbol{\lambda}) + \mathsf{d}_i \tag{4.32a}$$

$$d_i(\boldsymbol{\lambda}) = (\Lambda - \alpha_{\boldsymbol{\lambda}}|\alpha_i) \tag{4.32b}$$

$$\mathrm{def}(\boldsymbol{\lambda}) = \mathrm{def}(\boldsymbol{\mu}) + d_i(\boldsymbol{\lambda}) + \mathsf{d}_i = \mathrm{def}(\boldsymbol{\mu}) + d_A^{\triangleleft}(\boldsymbol{\lambda}) + d_A^{\triangleright}(\boldsymbol{\lambda}). \tag{4.32c}$$

Proof. (4.32a) is just a rephrasing of Definition 4D.3 (e).

To prove (4.32b) we argue by induction on n. If $n = 0$ then $\boldsymbol{\lambda} = \underline{\mathbf{0}}_\ell$, $\alpha_{\boldsymbol{\lambda}} = 0$ and $(\Lambda|\alpha_i) = d_i(\boldsymbol{\lambda})$ is the number of addable i-nodes of $\underline{\mathbf{0}}_\ell$. If $n > 0$ then

$$(\Lambda - \alpha_{\boldsymbol{\lambda}}|\alpha_i) = (\Lambda - \alpha_{\boldsymbol{\mu}}|\alpha_i) - (\alpha_i|\alpha_i) = d_i(\boldsymbol{\mu}) - 2\mathsf{d}_i = d_i(\boldsymbol{\lambda}),$$

where the second equality follows by induction and the third equality from (a). This proves (4.32b).

Now consider (4.32c). As $\boldsymbol{\lambda}$ has a removable i-node, $\alpha_{\boldsymbol{\mu}} = \alpha_{\boldsymbol{\lambda}} - \alpha_i \in Q_{n-1}^+$ and

$$\mathrm{def}(\boldsymbol{\lambda}) = \mathrm{def}(\alpha_{\boldsymbol{\mu}} + \alpha_i) = (\Lambda|\alpha_{\boldsymbol{\mu}}) + (\Lambda|\alpha_i) - \tfrac{1}{2}\big((\alpha_{\boldsymbol{\mu}}|\alpha_{\boldsymbol{\mu}}) + 2(\alpha_{\boldsymbol{\mu}}|\alpha_i) + (\alpha_i|\alpha_i)\big)$$

$$= \mathrm{def}(\boldsymbol{\mu}) + (\Lambda - \alpha_{\boldsymbol{\mu}}|\alpha_i) - \mathsf{d}_i, \qquad\qquad \text{by induction,}$$

$$= \mathrm{def}(\boldsymbol{\mu}) + d_i(\boldsymbol{\mu}) - \mathsf{d}_i, \qquad\qquad \text{by (4.32b),}$$

$$= \mathrm{def}(\boldsymbol{\mu}) + d_i(\boldsymbol{\lambda}) + \mathsf{d}_i,$$

where the last equality follows by (4.32a). The second equality in (4.32c) follows by a second application of (4.32a). $\qquad\square$

Corollary 4D.6. Suppose that $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Then $\deg^{\triangleleft}(\mathsf{t}) + \deg^{\triangleright}(\mathsf{t}) = \mathrm{def}(\boldsymbol{\lambda})$.

Proof. This follows by induction on n. If $n = 0$ then $\deg^{\triangleleft}(\mathsf{t}) = \deg^{\triangleright}(\mathsf{t}) = \mathrm{def}(\boldsymbol{\lambda}) = 0$, so the result holds. Suppose that $n > 0$ and let $A = \mathsf{t}^{-1}(n)$. Set $\mathsf{s} = \mathsf{t}_{\downarrow(n-1)}$, $\boldsymbol{\mu} = \mathrm{Shape}(\mathsf{s})$ and $i = \mathsf{r}_n(\mathsf{t}) = \mathsf{r}(A)$. Then,

$$\deg^{\triangleleft}(\mathsf{t}) + \deg^{\triangleright}(\mathsf{t}) = \deg^{\triangleleft}(\mathsf{s}) + d_A^{\triangleleft}(\boldsymbol{\lambda}) + \deg^{\triangleright}(\mathsf{s}) + d_A^{\triangleright}(\boldsymbol{\lambda}) \qquad \text{by Lemma 4C.2,}$$
$$= \operatorname{def}(\boldsymbol{\mu}) + d_A^{\triangleleft}(\boldsymbol{\lambda}) + d_A^{\triangleright}(\boldsymbol{\lambda}) \qquad \text{by induction,}$$
$$= \operatorname{def}(\boldsymbol{\lambda}),$$

with the last equality coming from Equation 4.32c. $\qquad\square$

We can now explain the origin of the name *defect* polynomial. In view of Corollary 6E.21 below, this shows that $\gamma_{\boldsymbol{\lambda}}^{\diamondsuit} \in \Bbbk[\underline{x}]$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$. It would be interesting to determine these polynomials explicitly; compare [15].

Corollary 4D.7. *Let $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$. Then $\gamma_{\boldsymbol{\lambda}}^{\diamondsuit}$ is a homogeneous polynomial of degree $2 \operatorname{def}(\boldsymbol{\lambda})$.*

Proof. If $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then, by Definition 4C.1 and Corollary 4D.6, the defect polynomial $\gamma_{\boldsymbol{\lambda}}^{\diamondsuit}$ is homogeneous of degree $\deg \gamma_{\mathsf{t}}^{\triangleleft} + \deg \gamma_{\mathsf{t}}^{\triangleright} = 2\big(\deg^{\triangleleft}(\mathsf{t}) + \deg^{\triangleright}(\mathsf{t})\big) = 2 \operatorname{def}(\boldsymbol{\lambda})$. $\qquad\square$

Although we do not need this, we note that the defect polynomial, or more correctly Lemma 4D.1, allows us to describe the transition matrix between the $f^{\triangleleft}$-basis and the $f^{\triangleright}$-basis, generalising Corollary 4A.13.

Proposition 4D.8. *Let $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$. Then $f_{\mathsf{st}}^{\triangleleft} = \frac{\gamma_{\boldsymbol{\lambda}}^{\diamondsuit}}{\gamma_{\mathsf{s}}^{\triangleright}\gamma_{\mathsf{t}}^{\triangleright}} f_{\mathsf{st}}^{\triangleright}$ in $\mathsf{R}_n^{\Lambda}(\Bbbk[\underline{x}^{\pm}])$.*

Proof. By Lemma 4D.1, $\gamma_{\mathsf{s}}^{\triangleleft}/\gamma_{\mathsf{t}}^{\triangleright} = \gamma_{\mathsf{t}}^{\triangleleft}/\gamma_{\mathsf{s}}^{\triangleright}$, so the statement of the proposition is equivalent to the equivalent claims that $\frac{\gamma_{\mathsf{s}}^{\triangleleft}}{\gamma_{\mathsf{t}}^{\triangleright}} f_{\mathsf{st}}^{\triangleright} = f_{\mathsf{st}}^{\triangleleft} = \frac{\gamma_{\mathsf{t}}^{\triangleleft}}{\gamma_{\mathsf{s}}^{\triangleright}} f_{\mathsf{st}}^{\triangleright}$. Since $f_{\mathsf{st}}^{\triangle} = (f_{\mathsf{ts}}^{\triangle})^{*}$, it is enough to show that

$$f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}\mathsf{t}}^{\triangleleft} = \frac{\gamma_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}}^{\triangleleft}}{\gamma_{\mathsf{t}}^{\triangleright}} f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleright}\mathsf{t}}^{\triangleright}$$

by Lemma 4A.14. We show this by arguing by induction on $L(d_{\mathsf{t}}^{\triangleleft})$, the length of the permutation $d_{\mathsf{t}}^{\triangleleft}$. When $\mathsf{t} = \mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}$ the result follows from Corollary 4A.13. If $\mathsf{t} \neq \mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}$ then we can write $\mathsf{t} = \sigma_k \mathsf{v}$ with $\mathsf{v} \triangleleft \mathsf{t}$ and $L(d_{\mathsf{v}}^{\triangleleft}) = L(d_{\mathsf{t}}^{\triangleleft}) - 1$. Hence, by two applications of Proposition 4A.10, and induction,

$$f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}\mathsf{t}}^{\triangleleft} = f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}\mathsf{v}}^{\triangleleft}\left(\psi_k - \frac{\delta_{r_k(\mathsf{v}),r_{k+1}(\mathsf{v})}}{\mathsf{c}_{k+1}(\mathsf{v}) - c_k(\mathsf{v})}\right) = \frac{\gamma_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}}^{\triangleleft}}{\gamma_{\mathsf{v}}^{\triangleright}} f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleright}\mathsf{v}}^{\triangleright}\left(\psi_k - \frac{\delta_{r_k(\mathsf{v}),r_{k+1}(\mathsf{v})}}{\mathsf{c}_{k+1}(\mathsf{v}) - c_k(\mathsf{v})}\right) = \frac{\gamma_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft}}^{\triangleleft}}{\gamma_{\mathsf{v}}^{\triangleright}} f_{\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleright}\mathsf{t}}^{\triangleright} Q_k(\mathsf{v}).$$

This completes the proof of the inductive step, and the proposition, since $\gamma_{\mathsf{v}}^{\triangleright} = Q_k(\mathsf{v})\gamma_{\mathsf{t}}^{\triangleright}$ by Lemma 4A.15. $\qquad\square$

By the proposition, $f_{\mathsf{st}}^{\triangleleft} = \frac{\gamma_{\boldsymbol{\lambda}}^{\diamondsuit}}{\gamma_{\mathsf{s}}^{\triangleright}\gamma_{\mathsf{t}}^{\triangleright}} f_{\mathsf{st}}^{\triangleright} = \frac{\gamma_{\mathsf{s}}^{\triangleleft}}{\gamma_{\mathsf{t}}^{\triangleright}} f_{\mathsf{st}}^{\triangleright} = \frac{\gamma_{\mathsf{t}}^{\triangleleft}}{\gamma_{\mathsf{s}}^{\triangleright}} f_{\mathsf{st}}^{\triangleright}$. In particular, the four terms in this equation have the same degree, which is easily checked using Corollary 4D.6.

4E. **A symmetrising form.** This subsection uses the defect polynomials to define a symmetrising form on the algebra $\mathsf{R}_n^{\Lambda}(\Bbbk[\underline{x}]) = \bigoplus_{\alpha \in Q_n^{+}} \mathsf{R}_{\alpha}^{\Lambda}(\Bbbk[\underline{x}])$, and hence shows that it is a graded symmetric algebra. This symmetrising form specialises to give a non-degenerate symmetrising form on the cyclotomic KLR algebra $\mathscr{R}_n^{\Lambda}(\Bbbk)$.

This subsection is partly inspired by [50], where similar results were obtained for the Ariki-Koike algebras. The arguments given here are much shorter than those in [50], which is surprising both because the results here are stronger and because we need to prove everything from the ground up.

Definition 4E.1. Let $\alpha \in Q_n^+$. For $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$ let $\chi_{\boldsymbol{\lambda}}$ be the character of the irreducible $R_\alpha^\Lambda(\mathbb{K}[\underline{x}^\pm])$-module $V_{\boldsymbol{\lambda}}(\mathbb{K}[\underline{x}^\pm])$. The α-*trace form* is the map $\tau_\alpha \colon R_\alpha^\Lambda(\mathbb{K}[\underline{x}^\pm]) \longrightarrow \mathbb{K}[\underline{x}^\pm]$ given by

$$\tau_\alpha = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell} \frac{1}{\gamma_{\boldsymbol{\lambda}}^{\Phi}} \chi_{\boldsymbol{\lambda}}.$$

By Corollary 4D.7, the trace form τ_α is homogeneous of degree $-2\operatorname{def}(\alpha)$ and takes values in $\mathbb{K}[\underline{x}^\pm]$.

We use the characters of $V_{\boldsymbol{\lambda}}(\mathbb{K}[\underline{x}^\pm])$, for $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$, in this definition to emphasise that τ_α does not depend on a choice of basis. Note that if $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ then $S_{\boldsymbol{\lambda}}^\lhd(\mathbb{K}[\underline{x}^\pm]) \cong S_{\boldsymbol{\lambda}}(\mathbb{K}[\underline{x}^\pm]) \cong S_{\boldsymbol{\lambda}}^\rhd(\mathbb{K}[\underline{x}^\pm])$ by Corollary 3C.10.

Example 4E.2. Let $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$ and $\alpha \in Q_n^+$. Using Lemma 4A.14,

$$\tau_\alpha(F_\mathsf{t}) = \frac{1}{\gamma_{\boldsymbol{\lambda}}^{\Phi}}, \qquad \tau_\alpha\left(f_{\mathsf{st}}^\lhd\right) = \frac{\delta_{\mathsf{st}}}{\gamma_\mathsf{t}^\rhd} \qquad \text{and} \qquad \tau_\alpha\left(f_{\mathsf{st}}^\rhd\right) = \frac{\delta_{\mathsf{st}}}{\gamma_\mathsf{t}^\lhd}.$$

To study $R_n^\Lambda(\mathbb{k}[\underline{x}])$, we use τ_α to define an "integral" bilinear form. If $f(\underline{x}) \in \mathbb{k}[\underline{x}^\pm]$ is a homogeneous polynomial let $f_0 \in \mathbb{k}$ be the constant term of $f(\underline{x})$.

Definition 4E.3. Let $\alpha \in Q_n^+$. Let $\langle\ ,\ \rangle_\alpha \colon R_\alpha^\Lambda(\mathbb{k}[\underline{x}]) \times R_\alpha^\Lambda(\mathbb{k}[\underline{x}]) \longrightarrow \mathbb{k}$ be the homogeneous bilinear form on $R_\alpha^\Lambda(\mathbb{k}[\underline{x}])$ of degree $-2\operatorname{def}(\alpha)$ given by $\langle a, b \rangle_\alpha = \tau_\alpha(ab)_0$.

We leave the proof of the following easy facts about τ_α and $\langle\ ,\ \rangle_\alpha$ to the reader.

Lemma 4E.4. Suppose that $a, b \in R_\alpha^\Lambda(\mathbb{k}[\underline{x}])$, for $\alpha \in Q_n^+$. Then

(a) Let $a, b, \in R_\alpha^\Lambda(\mathbb{k}[\underline{x}])$. Then $\tau_\alpha(ab) = \tau_\alpha(ba)$, $\tau_\alpha(a) = \tau_\alpha(a^*)$, and $\langle a, b \rangle_\alpha = \langle b, a \rangle_\alpha$.
(b) If $a, b, c \in R_\alpha^\Lambda(\mathbb{k}[\underline{x}])$ then $\langle ab, c \rangle_\alpha = \langle a, bc \rangle_\alpha$.

We want to show that $\langle\ ,\ \rangle_\alpha$ is a homogeneous non-degenerate bilinear form on $R_\alpha^\Lambda(\mathbb{k}[\underline{x}])$. The next results pave the way to proving this. The first result is similar in spirit to [24, Lemma 4.11].

Lemma 4E.5. Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Then there exist $r_\mathsf{t}, s_\mathsf{t} \in \mathbb{K}[\underline{x}^\pm]$ such that

$$\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\lhd = f_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\lhd + \sum_{\mathsf{t} \lhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd} r_\mathsf{t} f_{\mathsf{tt}}^\lhd \qquad \text{and} \qquad \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\rhd = f_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\rhd + \sum_{\mathsf{t} \rhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd} s_\mathsf{t} f_{\mathsf{tt}}^\rhd.$$

Proof. Let $\triangle \in \{\lhd, \rhd\}$. By definition and Corollary 3E.10,

$$\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\triangle \mathsf{t}_{\boldsymbol{\lambda}}^\triangle}^\triangle = y_{\boldsymbol{\lambda}}^\triangle 1_{\mathsf{i}_{\boldsymbol{\lambda}}^\triangle} = y_{\boldsymbol{\lambda}}^\triangle \sum_{\mathsf{t} \in \operatorname{Std}(\mathsf{i}_{\boldsymbol{\lambda}}^\triangle)} \frac{1}{\gamma_\mathsf{t}^\triangle} F_\mathsf{t} = \sum_{\mathsf{t} \in \operatorname{Std}(\mathsf{i}_{\boldsymbol{\lambda}}^\triangle)} \frac{1}{\gamma_\mathsf{t}^\triangle} \prod_{A \in \operatorname{Add}^\triangle(\mathsf{t}_{\boldsymbol{\lambda}}^\triangle)} \left(\mathsf{c}_m(\mathsf{t}) - \mathsf{c}(A)\right) F_\mathsf{t}$$

Suppose that $\mathsf{t} \in \operatorname{Std}(\mathsf{i}_{\boldsymbol{\lambda}}^\triangle)$ and that $\mathsf{t} \not\triangle \mathsf{t}_{\boldsymbol{\lambda}}^\triangle$. Let $1 \le k < n$ be minimal such that $\mathsf{t}_{\downarrow k} \triangle \mathsf{t}_{\boldsymbol{\lambda}, \downarrow k}^\triangle$ and $\mathsf{t}_{\downarrow(k+1)} \not\triangle \mathsf{t}_{\boldsymbol{\lambda}, \downarrow(k+1)}^\triangle$. Let $A = \mathsf{t}^{-1}(k+1)$ and $B = (\mathsf{t}_{\boldsymbol{\lambda}}^\triangle)^{-1}(k+1)$. Abusing notation slightly, $B \triangle A$, so $A \in \operatorname{Add}_k^\triangle(\mathsf{t}_{\boldsymbol{\lambda}}^\triangle)$. That is, $A \in \operatorname{Add}^\triangle(\mathsf{t}_{\boldsymbol{\lambda}}^\triangle)$ appears in the product above, contributing the factor $\mathsf{c}_{k+1}(\mathsf{t}) - \mathsf{c}(A) = 0$. Hence, $f_{\mathsf{tt}}^\triangle = \frac{1}{\gamma_\mathsf{t}^\triangle} F_\mathsf{t}$ appears in $\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\triangle \mathsf{t}_{\boldsymbol{\lambda}}^\triangle}^\triangle$ only if $\mathsf{t} \triangle \mathsf{t}_{\boldsymbol{\lambda}}^\triangle$, where dominance holds because $\mathsf{r}(\mathsf{t}) = \mathsf{i}_{\boldsymbol{\lambda}}^\triangle$. $\qquad\square$

The next result strengthens Proposition 4A.17. Recall from subsection 4A that $(\mathsf{s}, \mathsf{t}) \trianglelefteq (\mathsf{u}, \mathsf{v})$ if $\mathsf{s} \trianglelefteq \mathsf{u}$ and $\mathsf{t} \trianglelefteq \mathsf{v}$.

Lemma 4E.6. Let $s, t \in \mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Then

$$\psi_{st}^\lhd = f_{st}^\lhd + \sum_{\substack{(u,v) \in \mathrm{Std}^2(\mathcal{P}_n^\ell) \\ (u,v) \lhd (s,t)}} a_{uv} f_{uv}^\lhd \qquad \psi_{st}^\rhd = f_{st}^\rhd + \sum_{\substack{(u,v) \in \mathrm{Std}^2(\mathcal{P}_n^\ell) \\ (u,v) \rhd (s,t)}} b_{uv} f_{uv}^\rhd$$

$$f_{st}^\lhd = \psi_{st}^\lhd + \sum_{\substack{(u,v) \in \mathrm{Std}^2(\mathcal{P}_n^\ell) \\ (u,v) \lhd (s,t)}} c_{uv} \psi_{uv}^\lhd \qquad f_{st}^\rhd = f_{st}^\rhd + \sum_{\substack{(u,v) \in \mathrm{Std}^2(\mathcal{P}_n^\ell) \\ (u,v) \rhd (s,t)}} d_{uv} \psi_{uv}^\rhd,$$

for some scalars $a_{uv}, b_{uv}, c_{uv}, d_{uv} \in \mathbb{K}[\underline{x}^\pm]$.

Proof. Let $\triangle \in \{\lhd, \rhd\}$. We argue by induction on the dominance order $\triangle$ on $\mathcal{P}_n^\ell$. Let $\boldsymbol{\lambda}$ be maximal with respect to $\triangle$. Then $\boldsymbol{\lambda} = (0|\ldots|0|1^n)$ if $\triangle = \lhd$ and $\boldsymbol{\lambda} = (n|0|\ldots|0)$ if $\triangle = \rhd$. In this case, $f_{t_{\boldsymbol{\lambda}}^\triangle t_{\boldsymbol{\lambda}}^\triangle}^\triangle = \psi_{t_{\boldsymbol{\lambda}}^\triangle t_{\boldsymbol{\lambda}}^\triangle}^\triangle$, so the result holds.

Now suppose that $\boldsymbol{\lambda}$ is not maximal. By Lemma 4E.5 the proposition holds for $f_{t_{\boldsymbol{\lambda}}^\triangle t_{\boldsymbol{\lambda}}^\triangle}^\triangle$ so, by induction, the result holds for $\psi_{t_{\boldsymbol{\lambda}}^\triangle t_{\boldsymbol{\lambda}}^\triangle}^\triangle$. Now suppose that $(t_{\boldsymbol{\lambda}}^\triangle, t_{\boldsymbol{\lambda}}^\triangle) \triangle (s, t)$, for $s, t \in \mathrm{Std}(\boldsymbol{\lambda})$. We can assume that $s \neq t_{\boldsymbol{\lambda}}^\triangle$ by applying $*$, if necessary. Pick k such that $y = \sigma_k s \triangle s$. By Proposition 4C.5 (c) and induction,

$$\psi_{st}^\triangle = \psi_k \psi_{yt}^\triangle = \psi_k \left(f_{yt}^\triangle + \sum_{(u,v) \triangle (y,t)} r_{uv} f_{uv}^\triangle \right) = f_{st}^\triangle + \sum_{(u,v) \triangle (y,t)} r_{uv} \psi_k f_{uv}^\triangle,$$

for some $r_{uv} \in \mathbb{K}[\underline{x}^\pm]$. Consider a term $\psi_k f_{uv}^\triangle$ on the right-hand side and let $w = \sigma_k u$. If $L(d_w^\triangle) = L(d_u^\triangle) + 1$ then $d_w^\triangle$ is a subexpression of $d_s^\triangle$ since $u \triangle y$ and $L(d_s^\triangle) = L(d_y^\triangle) + 1$, so $w \triangle s$. If $L(d_w^\triangle) = L(d_u^\triangle) + 1$ then $w \triangle u \triangle y \triangle s$. Therefore, $\psi_{st}^\triangle$ can be written in the required form by Proposition 4A.10. Inverting this equation, $f_{st}^\triangle$ can also be written in the required form. This completes the proof of the inductive step and hence the lemma. $\square$

Corollary 4E.7. Let $(s, t), (u, v) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)$. Then $\psi_{st}^\lhd \psi_{uv}^\rhd \neq 0$ only if $t \trianglerighteq u$, and $\psi_{uv}^\rhd \psi_{st}^\lhd \neq 0$ only if $s \trianglerighteq v$. Moreover, $\psi_{st}^\lhd \psi_{ts}^\rhd = f_{st}^\lhd f_{ts}^\rhd$ and $\psi_{ts}^\rhd \psi_{st}^\lhd = f_{ts}^\rhd f_{st}^\lhd$ are homogeneous of degree $2 \operatorname{def}(\boldsymbol{\lambda})$.

Proof. Consider the first statement. Using Lemma 4E.6,

$$\psi_{st}^\lhd \psi_{uv}^\rhd = \left(f_{st}^\lhd + \sum_{\substack{(w,x) \in \mathrm{Std}^2(\mathcal{P}_n^\ell) \\ (w,x) \lhd (s,t)}} a_{wx} f_{wx}^\lhd \right) \left(f_{uv}^\rhd + \sum_{\substack{(y,z) \in \mathrm{Std}^2(\mathcal{P}_n^\ell) \\ (y,z) \rhd (u,v)}} b_{yz} f_{yz}^\rhd \right)$$

$$= \sum_{\substack{(w,x) \in \mathrm{Std}^2(\mathcal{P}_n^\ell) \\ (w,x) \trianglelefteq (s,t)}} \sum_{\substack{(y,z) \in \mathrm{Std}^2(\mathcal{P}_n^\ell) \\ (y,z) \trianglerighteq (u,v)}} a_{wx} b_{yz} f_{wx}^\lhd f_{yz}^\rhd,$$

where we set $a_{st} = 1 = b_{uv}$. Therefore, $\psi_{st}^\lhd \psi_{uv}^\rhd \neq 0$ only if $f_{wx}^\lhd f_{yz}^\rhd \neq 0$ for some $(w, x), (y, z) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)$ with $w \trianglelefteq s$, $x \trianglelefteq t$, $y \trianglerighteq u$ and $z \trianglerighteq v$. By Lemma 4A.7, $f_{wx}^\lhd f_{yz}^\rhd \neq 0$ only if $x = y$, so this forces $t \trianglerighteq x = y \trianglerighteq u$, as required. Since $\psi_{ts}^\rhd \psi_{st}^\lhd = (\psi_{st}^\lhd \psi_{ts}^\rhd)^*$, this implies that if $\psi_{ts}^\rhd \psi_{vu}^\lhd \neq 0$ then $s \trianglerighteq v$. When $u = t$ and $v = s$ the last displayed equation shows that $\psi_{st}^\lhd \psi_{ts}^\rhd = f_{st}^\lhd f_{ts}^\rhd$. By definition, $\psi_{st}^\lhd \psi_{ts}^\rhd$ is a homogeneous element of $R_n^\Lambda(\mathbb{k}[\underline{x}])$ of degree $2 \operatorname{def}(\boldsymbol{\lambda})$. Similarly, $\psi_{ts}^\rhd \psi_{st}^\lhd = f_{ts}^\rhd f_{st}^\lhd$ is homogeneous of defect $2 \operatorname{def}(\boldsymbol{\lambda})$. $\square$

Definition 4E.8. For $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ set $z_{\boldsymbol{\lambda}}^\lhd = \psi_{t_{\boldsymbol{\lambda}}^\lhd t_{\boldsymbol{\lambda}}^\lhd}^\lhd \psi_{t_{\boldsymbol{\lambda}}^\lhd t_{\boldsymbol{\lambda}}^\lhd}^\rhd$ and $z_{\boldsymbol{\lambda}}^\rhd = \psi_{t_{\boldsymbol{\lambda}}^\rhd t_{\boldsymbol{\lambda}}^\rhd}^\rhd \psi_{t_{\boldsymbol{\lambda}}^\rhd t_{\boldsymbol{\lambda}}^\rhd}^\lhd$.

By Lemma 4A.3 we can also write $z_{\boldsymbol{\lambda}}^{\lhd} = \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}}^{\lhd}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}$ and $z_{\boldsymbol{\lambda}}^{\rhd} = \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}}^{\lhd}$. We will not need this, but it is not difficult to show that $z_{\boldsymbol{\lambda}}^{\lhd} = \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathsf{s}}^{\lhd}\psi_{\mathsf{s}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}$ and $z_{\boldsymbol{\lambda}}^{\rhd} = \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}\mathsf{s}}^{\rhd}\psi_{\mathsf{s}\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}}^{\lhd}$, for any $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$.

In the classical representation theory of the symmetric groups, elements very similar to $z_{\boldsymbol{\lambda}}^{\lhd}$ and $z_{\boldsymbol{\lambda}}^{\rhd}$ are often used as distinguished generators for the semisimple Specht modules. The extra structure provided by the grading shows that these elements are "almost" canonical.

Proposition 4E.9. Let $\boldsymbol{\lambda} \in \mathcal{P}_{\alpha}^{\ell}$, for $\alpha \in Q^{+}$. Then $z_{\boldsymbol{\lambda}}^{\lhd} = \gamma_{\boldsymbol{\lambda}}^{\Phi}F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}$ and $z_{\boldsymbol{\lambda}}^{\rhd} = \gamma_{\boldsymbol{\lambda}}^{\Phi}F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}}$. Consequently, $\frac{1}{\gamma_{\boldsymbol{\lambda}}^{\Phi}}z_{\boldsymbol{\lambda}}^{\lhd}$ and $\frac{1}{\gamma_{\boldsymbol{\lambda}}^{\Phi}}z_{\boldsymbol{\lambda}}^{\rhd}$ are (nonzero) primitive idempotents in $\mathsf{R}_{n}^{\Lambda}(\Bbbk[\underline{x}^{\pm}])$ and $\tau_{\alpha}(z_{\boldsymbol{\lambda}}^{\lhd})_{0} = 1 = \tau_{\alpha}(z_{\boldsymbol{\lambda}}^{\rhd})_{0}$.

Proof. We give the proof only for $z_{\boldsymbol{\lambda}}^{\lhd}$, with the result for $z_{\boldsymbol{\lambda}}^{\rhd}$ following by symmetry. Since $z_{\boldsymbol{\lambda}}^{\lhd} = F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}z_{\boldsymbol{\lambda}}^{\lhd}F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}$ by Lemma 4A.7, it follows that $z_{\boldsymbol{\lambda}}^{\lhd}$ is a scalar multiple of $F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}} = \frac{1}{\gamma_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}}f_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}$ by Corollary 4A.13. Then, there exist scalars $a_{\mathsf{wx}}, b_{\mathsf{yz}} \in \Bbbk[\underline{x}^{\pm}]$ such that

$$(z_{\boldsymbol{\lambda}}^{\lhd})^{2} = \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd} \qquad \text{by Definition 4A.5,}$$

$$= \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}\left(f_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd} + \sum_{(\mathsf{x},\mathsf{w})\rhd(\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd},\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd})}a_{\mathsf{wx}}f_{\mathsf{wx}}^{\rhd}\right)\left(f_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd} + \sum_{(\mathsf{y},\mathsf{z})\lhd(\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd},\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd})}b_{\mathsf{yz}}f_{\mathsf{yz}}^{\lhd}\right)\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}, \quad \text{by Lemma 4E.6,}$$

$$= \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}f_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}f_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}, \qquad \text{by Lemma 4A.7,}$$

$$= \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd} \cdot \gamma_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}} \cdot \gamma_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}} \cdot \psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}, \qquad \text{by Corollary 4A.13,}$$

$$= \gamma_{\boldsymbol{\lambda}}^{\Phi}z_{\boldsymbol{\lambda}}^{\lhd}, \qquad \text{by Lemma 4A.7.}$$

Hence, $\frac{1}{\gamma_{\boldsymbol{\lambda}}^{\Phi}}z_{\boldsymbol{\lambda}}^{\lhd} = F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}$ is a primitive idempotent in $\mathsf{R}_{n}^{\Lambda}(\Bbbk[\underline{x}^{\pm}])$. Finally, $\tau_{\alpha}(z_{\boldsymbol{\lambda}}^{\lhd})_{0} = 1$ by Example 4E.2. $\square$

Although we do not need this, it is not hard to show that $\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}\mathscr{R}_{n}^{\Lambda}(\Bbbk[\underline{x}])\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd} = \Bbbk[\underline{x}]z_{\boldsymbol{\lambda}}^{\lhd}$ is a free $\Bbbk[\underline{x}]$-module of rank 1, giving another way to prove that $S_{\boldsymbol{\lambda}}(\Bbbk[x^{\pm}])$ is an irreducible $\mathsf{R}_{n}^{\Lambda}(\Bbbk[x^{\pm}])$-module.

We have reached the main results of this subsection.

Theorem 4E.10. Suppose that $(\mathsf{s}, \mathsf{t}), (\mathsf{u}, \mathsf{v}) \in \mathrm{Std}^{2}(\mathcal{P}_{\alpha}^{\ell})$, for $\alpha \in Q_{n}^{+}$. Then

$$\langle \psi_{\mathsf{st}}^{\lhd}, \psi_{\mathsf{uv}}^{\rhd}\rangle_{\alpha} = \begin{cases} 1 & \text{if } (\mathsf{s}, \mathsf{t}) = (\mathsf{v}, \mathsf{u}), \\ 0 & \text{if } (\mathsf{s}, \mathsf{t}) \not\unrhd (\mathsf{v}, \mathsf{u}). \end{cases}$$

Proof. By definition and Lemma 4E.4, $\langle \psi_{\mathsf{st}}^{\lhd}, \psi_{\mathsf{uv}}^{\rhd}\rangle_{\alpha} = \tau_{\alpha}(\psi_{\mathsf{st}}^{\lhd}\psi_{\mathsf{uv}}^{\rhd}) = \tau_{\alpha}(\psi_{\mathsf{uv}}^{\rhd}\psi_{\mathsf{st}}^{\lhd})$. Hence, $\langle \psi_{\mathsf{st}}^{\lhd}, \psi_{\mathsf{uv}}^{\rhd}\rangle_{\alpha} = 0$ unless $\mathsf{t} \unrhd \mathsf{u}$ and $\mathsf{s} \unrhd \mathsf{v}$ by Corollary 4E.7. Now suppose that $\mathsf{u} = \mathsf{t}$ and $\mathsf{v} = \mathsf{s}$ and consider the inner product $\langle \psi_{\mathsf{st}}^{\lhd}, \psi_{\mathsf{st}}^{\rhd}\rangle_{\alpha} = \tau_{\alpha}(\psi_{\mathsf{st}}^{\lhd}\psi_{\mathsf{st}}^{\rhd})$. Using Lemma 4E.4,

$$\langle \psi_{\mathsf{st}}^{\lhd}, \psi_{\mathsf{ts}}^{\rhd}\rangle_{\alpha} = \tau_{\alpha}\left(\psi_{\mathsf{st}}^{\lhd}\psi_{\mathsf{ts}}^{\rhd}\right)_{0} = \tau_{\alpha}\left(\psi_{d_{\mathsf{s}}^{\lhd}}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathsf{t}}^{\lhd}\psi_{d_{\mathsf{t}}^{\rhd}}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}\mathsf{s}}^{\rhd}\right)_{0}$$

$$= \tau_{\alpha}\left(\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}\mathsf{s}}^{\rhd}\psi_{d_{\mathsf{s}}^{\lhd}}\right)_{0} = \tau_{\alpha}\left(\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\lhd}\psi_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\rhd}\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}^{\rhd}\right)_{0}, \quad \text{by two applications of Lemma 4A.3,}$$

$$= \tau_{\alpha}\left(z_{\boldsymbol{\lambda}}^{\lhd}\right)_{0} = \gamma_{\boldsymbol{\lambda}}^{\Phi}\tau_{\alpha}\left(F_{\mathfrak{t}_{\boldsymbol{\lambda}}^{\lhd}}\right)_{0}, \qquad \text{by Proposition 4E.9,}$$

$$= 1,$$

where the last equality follows from Example 4E.2. $\square$

4F. **Cellular bases for** $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$**.** We can now prove that $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ is a $\Bbbk[\underline{x}]$-cellular algebra. In particular, this proves a stronger form of Theorem **A**, our first main result from the introduction.

Theorem 4F.1. Suppose that (c, r) is a graded content system with values in $\Bbbk[\underline{x}]$. Then $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ is a graded $\Bbbk[\underline{x}]$-cellular algebra with $\Bbbk[\underline{x}]$-cellular bases:

    (a) $\{\psi_{\mathsf{st}}^\lhd \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \trianglelefteq)$ and degree function $\deg^\lhd$.

    (b) $\{\psi_{\mathsf{st}}^\rhd \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \trianglerighteq)$ and degree function $\deg^\rhd$.

Proof. By Proposition 2C.6, $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ is free as a $\Bbbk[\underline{x}]$-module, so $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ naturally embeds into the $\mathbb{K}[\underline{x}^\pm]$-algebra $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm]) \cong \mathbb{K}[\underline{x}^\pm] \otimes_{\Bbbk[\underline{x}]} \mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$. In particular, the $\Bbbk[\underline{x}]$-rank of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ is equal to the $\mathbb{K}[\underline{x}^\pm]$-rank of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$.

We only show that $\{\psi_{\mathsf{st}}^\lhd\}$ is a $\Bbbk[\underline{x}]$-cellular basis of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$, as the $\Bbbk[\underline{x}]$-cellularity of $\{\psi_{\mathsf{st}}^\rhd\}$ follows by symmetry. Since $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}]) = \bigoplus_{\alpha \in Q_n^+} \mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$, it is enough to show that $\{\psi_{\mathsf{st}}^\lhd \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_\alpha^\ell)\}$ is a $\Bbbk[\underline{x}]$-cellular basis of $\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$, for $\alpha \in Q_n^+$. By Theorem 4C.3, $\{\psi_{\mathsf{st}}^\lhd \mid (\mathsf{s}, \mathsf{t}) \in \mathcal{P}_\alpha^\ell\}$ is a $\mathbb{K}[\underline{x}^\pm]$-cellular basis of $\mathsf{R}_\alpha^\Lambda(\mathbb{K}[\underline{x}^\pm])$. Therefore, to prove the theorem it is enough to show that $\{\psi_{\mathsf{st}}^\lhd(\mathsf{s}, \mathsf{t}) \in \mathcal{P}_\alpha^\ell\}$ spans $\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ and that the structure constants for this basis belong to $\Bbbk[\underline{x}]$.

Let $(\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_\alpha^\ell)$. Using Theorem 4E.10 and Gaussian elimination to argue by induction on dominance, there exist homogeneous elements $\eta_{\mathsf{uv}}^\rhd \in \mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ such that $\langle \psi_{\mathsf{st}}^\lhd, \eta_{\mathsf{uv}}^\rhd \rangle_\alpha = \delta_{(\mathsf{s},\mathsf{t})(\mathsf{v},\mathsf{u})}$ and $\eta_{\mathsf{uv}}^\rhd = \psi_{\mathsf{uv}}^\rhd + \sum_{(\mathsf{x},\mathsf{y}) \rhd (\mathsf{u},\mathsf{v})} e_{\mathsf{xy}}^\rhd \psi_{\mathsf{xy}}^\rhd$, for homogeneous scalars $e_{\mathsf{xy}}^\rhd \in \Bbbk[\underline{x}]$. Therefore, if $h \in \mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ then

$$h = \sum_{(\mathsf{u},\mathsf{v}) \in \mathrm{Std}^2(\mathcal{P}_\alpha^\ell)} \langle h, \eta_{\mathsf{uv}}^\rhd \rangle_\alpha \, \psi_{\mathsf{uv}}^\lhd.$$

In particular, the set $\{\psi_{\mathsf{st}}^\lhd \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_\alpha^\ell)\}$ spans $\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ as a $\Bbbk[\underline{x}]$-module. Hence, $\{\psi_{\mathsf{st}}^\lhd\}$ is a basis of $\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ by Theorem 4C.3. Moreover, if $h \in \mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ then $h\psi_{\mathsf{st}}^\lhd \in \mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$, so $\langle h\psi_{\mathsf{st}}^\lhd, \eta_{\mathsf{uv}}^\rhd \rangle_\alpha \in \Bbbk[\underline{x}]$, for $(\mathsf{u}, \mathsf{v}) \in] \mathrm{Std}^2(\mathcal{P}_\alpha^\ell)$. Therefore,

$$h\psi_{\mathsf{st}}^\lhd = \sum_{(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_\alpha^\ell)} \langle h\psi_{\mathsf{st}}^\lhd, \eta_{\mathsf{uv}}^\rhd \rangle_\alpha \, \psi_{\mathsf{uv}}^\lhd.$$

showing that the structure constants of $\{\psi_{\mathsf{st}}^\lhd \mid (\mathsf{s}, \mathsf{t}) \in \mathcal{P}_\alpha^\ell\}$ belong to $\Bbbk[\underline{x}]$.

Hence, $\{\psi_{\mathsf{st}}^\lhd \mid (\mathsf{s}, \mathsf{t}) \in \mathcal{P}_\alpha^\ell\}$ is a $\Bbbk[\underline{x}]$-cellular basis of $\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ by Theorem 4C.3. $\qquad\square$

The strategy used to prove Theorem 4F.1 is quite general. For example, an easy modifications this argument gives a streamlined proof of the fact that the Murphy basis of [19, Theorem 3.26] is a cellular basis of the cyclotomic Hecke algebras of type A [19, Theorem 3.26].

Remark 4F.2. In type $A_{e-1}^{(1)}$, even in the ungraded world, pairs of dual bases for the algebras $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ are not known. It seems hard to explicitly describe the basis $\{\eta_{\mathsf{st}}^\rhd\}$ that is dual to $\{\psi_{\mathsf{st}}^\lhd\}$. Similarly, it is hard to describe the basis $\{\eta_{\mathsf{st}}^\lhd\}$ that is dual to $\{\psi_{\mathsf{st}}^\rhd\}$. On the other hand, using Theorem 4F.1, it is straightforward to check that $\{\eta_{\mathsf{st}}^\rhd\}$ and $\{\eta_{\mathsf{st}}^\lhd\}$ are $\Bbbk[\underline{x}]$-cellular bases of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$.

As noted in Example 3A.2, content systems (c, r) do not always exist in positive characteristic. Nonetheless, by base-change, Theorem 4F.1 gives cellular bases over other rings. Indeed, since Example 3A.2 gives content systems with values in $\mathbb{Z}[x]$ for quivers of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$, we obtain cellular bases over $\Bbbk[x]$ for arbitrary rings $\Bbbk$.

Corollary 4F.3. Suppose that (c, r) is a graded content system with values in $\Bbbk[\underline{x}]$ and let K be commutative domain with 1 that is a $\Bbbk[\underline{x}]$-algebra. Then $\mathsf{R}_n^\Lambda(K)$ is a graded K-cellular algebra with cellular bases:

(a) $\{\psi_{\mathsf{st}}^{\lhd} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \unlhd)$ and degree function $\deg^{\lhd}$.

(b) $\{\psi_{\mathsf{st}}^{\rhd} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \unrhd)$ and degree function $\deg^{\rhd}$.

Proof. This is immediate from Theorem 4F.1 since $\mathsf{R}_n^\Lambda(K) \cong K \otimes_{\Bbbk[\underline{x}]} \mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$. $\qquad\square$

Essentially as an important special case, this implies that the (standard) cyclotomic KLR algebras $\mathscr{R}_n^\Lambda(K)$ of type $A_{e-1}^{(1)}$ or $C_{e-1}^{(1)}$ are cellular over any ring K.

Corollary 4F.4. Let K be commutative domain with 1 and suppose that $\mathscr{R}_n^\Lambda(K)$ is a cyclotomic KLR algebra of type $A_{e-1}^{(1)}$, A_∞, $C_{e-1}^{(1)}$ or C_∞. Then $\mathscr{R}_n^\Lambda(K)$ is a graded cellular algebra with cellular bases:

(a) $\{\psi_{\mathsf{st}}^{\lhd} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \unlhd)$ and degree function $\deg^{\lhd}$.

(b) $\{\psi_{\mathsf{st}}^{\rhd} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\ell)\}$ with weight poset $(\mathcal{P}_n^\ell, \unrhd)$ and degree function $\deg^{\rhd}$.

Proof. For quivers of type $A_{e-1}^{(1)}$ of $C_{e-1}^{(1)}$, by Lemma 3A.3 there exist graded content system (c, r) with values in $\mathbb{Z}[x]$ for a deformed cyclotomic KLR algebra $\mathsf{R}_n^\Lambda(|Z[x])$. Therefore, $\mathscr{R}_n^\Lambda(K) \cong K \otimes_{\mathbb{Z}[x]} \mathsf{R}_n^\Lambda(\mathbb{Z}[x])$ as K-algebras, where K is considered as a $\mathbb{Z}[x]$-algebra by letting x act as multiplication by 0, so the result follows by Theorem 4F.1. For quivers of type A_∞ of C_∞, by taking e sufficiently large, this implies that the cyclotomic KLR algebras of type A_∞ and C_∞ are cellular; compare with [26, Corollary 2.10]. $\qquad\square$

Remark 4F.5. For the cyclotomic KLR algebras of type $A_{e-1}^{(1)}$ Corollary 4F.4 recovers, with considerably less effort, the main theorem of Li [45], which generalises [24] to give an integral basis of $\mathscr{R}_n^\Lambda(\mathbb{Z})$. The papers [9, 56] use Websters' diagrammatic KLRW algebras to construct different cellular bases for the cyclotomic KLR algebras of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$, which depend on a choice of "loading". In type $A_{e-1}^{(1)}$, Bowman [9, Proposition 7.3] has shown that the transition matrix between the $\psi^{\rhd}$-basis of $\mathscr{R}_n^\Lambda(\Bbbk)$ and the "asymptotic Webster diagram basis" is unitriangular. In type $C_{e-1}^{(1)}$, we do not know the relationship between the cellular bases considered in this paper and those in [56], although it seems likely that Bowman's arguments generalise to show that the transition matrices between these bases is unitriangular in the "asymptotic case".

Remark 4F.6. The cellular bases in Theorem 4F.1 give graded Specht modules for the cyclotomic KLR algebras $\mathscr{R}_n^\Lambda(\Bbbk)$. In type $A_{e-1}^{(1)}$ this recovers the results of [13, 24]. Ariki, Park and Speyer [8] have given a conjectural construction of graded Specht modules in type $C_{e-1}^{(1)}$ using analogues of the homogeneous Garnir relations from [40], and they have proved these conjectures in type C_∞. As shown in [55], it is easy to prove the conjectures of [8] using Theorem 4F.1.

It is very difficult to do calculations with the cyclotomic KLR algebras $\mathscr{R}_n^\Lambda$. In contrast, it is very easy to calculate with the ψ-bases of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ because the transition matrices to the corresponding seminormal bases are unitriangular by Proposition 4A.17 and the action of $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ on the seminormal bases is completely described by Proposition 4A.10. The rest of this paper can be viewed as theoretical applications of this observation. In a different direction, this observation is used in [17, 54] to implement the cyclotomic KLR algebras of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$ in SAGEMATH [66].

An R-algebra A is a *graded symmetric algebra* algebra if there is a non-degenerate homogeneous bilinear form $\langle\ ,\ \rangle\colon A \times A \longrightarrow R$ of degree d such that $\langle ab, c \rangle = \langle a, bc \rangle$, for all $a, b, c \in A$; compare [18, Definition 66.1]. Hence, combining Theorem 4E.10 and Theorem 4F.1 yields:

Corollary 4F.7. *Let $\alpha \in Q_n^+$. Then $\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ is a graded symmetric algebra with homogeneous trace form of degree $-2\operatorname{def}(\alpha)$.*

The bilinear form $\langle\ ,\ \rangle_\alpha$ is defined over $\Bbbk$. So, in view of Lemma 3A.3, we obtain the corresponding results for the cyclotomic KLR algebras $\mathscr{R}_n^\Lambda(\mathbb{Z})$.

Corollary 4F.8. *Let $\alpha \in Q_n^+$. Then $\mathscr{R}_\alpha^\Lambda(\mathbb{Z})$ is a graded symmetric algebra with homogeneous trace form of degree $-2\operatorname{def}(\alpha)$. In particular, the cyclotomic Hecke algebras of type $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$ are graded symmetric algebras over any ring.*

For the cyclotomic KLR algebras of type $A_{e-1}^{(1)}$, Corollary 4F.8 was first proved as [24, Corollary 6.18]. Later, Kashiwara [35] and Webster [69, Remark 3.19] used categorical and diagrammatic arguments, respectively, to show that cyclotomic KLR algebras of symmetrisable type are graded symmetric algebras.

As our first application of the trace form on $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$ we show that the graded Specht modules $S_\lambda^\triangleleft(\Bbbk[\underline{x}])$ and $S_\lambda^\triangleright(\Bbbk[\underline{x}])$ are dual to each other, up to shift.

Proposition 4F.9. *Suppose that K is a $\Bbbk[\underline{x}]$-module and let $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$, for $\alpha \in Q_n^+$. Then*
$$S_\lambda^\triangleleft(K) \cong q^{\operatorname{def}(\boldsymbol{\lambda})} S_\lambda^\triangleright(K)^\circledast \qquad \text{and} \qquad S_\lambda^\triangleright(K) \cong q^{\operatorname{def}(\boldsymbol{\lambda})} S_\lambda^\triangleleft(K)^\circledast$$
as $\mathsf{R}_n^\Lambda(\Bbbk[\underline{x}])$-modules.

Proof. The two isomorphisms are equivalent so we prove only the first one. For $\mathsf{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ let $\theta_\mathsf{s} \in q^{\operatorname{def}(\boldsymbol{\lambda})} S_\lambda^\triangleright(K)^\circledast$ be the unique K-linear map such that
$$\theta_\mathsf{s}(\psi_\mathsf{t}^\triangleright) = \left\langle \psi_{\mathsf{t}_\lambda^\triangleleft \mathsf{s}}^\triangleleft, \psi_{\mathsf{t}\mathsf{t}_\lambda^\triangleleft}^\triangleright \right\rangle_\alpha, \qquad \text{for } \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda}).$$
Define a homomorphism $\Theta\colon S_\lambda^\triangleleft(K) \longrightarrow S_\lambda^\triangleright(K)^\circledast$ by $\Theta(\psi_\mathsf{s}^\triangleleft) = \theta_\mathsf{s}$, for $\mathsf{s} \in \operatorname{Std}(\boldsymbol{\lambda})$. By Corollary 4D.6, $\deg^\triangleleft(\mathsf{t}_\lambda^\triangleleft) + \deg^\triangleright(\mathsf{t}_\lambda^\triangleright) = \operatorname{def}(\boldsymbol{\lambda})$, so Θ is a homogeneous map of degree zero into $q^{\operatorname{def}(\boldsymbol{\lambda})}\bigl(S_\lambda^\triangleright(K)\bigr)^\circledast$. In view of Lemma 4E.4, Θ is an $\mathsf{R}_n^\Lambda(K)$-module homomorphism and, by Theorem 4E.10, it is an isomorphism of K-modules. $\qquad\square$

In particular, the specialisation of $\underline{x}$ to 0, which corresponds to taking $K = \Bbbk$, shows that
$$S_\lambda^\triangleleft(\Bbbk) \cong q^{\operatorname{def}(\boldsymbol{\lambda})} S_\lambda^\triangleright(\Bbbk)^\circledast \qquad \text{and} \qquad S_\lambda^\triangleright(\Bbbk) \cong q^{\operatorname{def}(\boldsymbol{\lambda})} S_\lambda^\triangleleft(\Bbbk)^\circledast$$
as $\mathscr{R}_n^\Lambda(\Bbbk)$-modules. In view of Lemma 3A.3, and base change, $\Bbbk$ can be an arbitrary ring. In type $A_{e-1}^{(1)}$, this recovers [24, Proposition 6.19].

As the last result in this subsection, we note that combining Lemma 4E.6 and Theorem 4F.1 gives the following useful strengthening of Proposition 4C.5 (b).

Corollary 4F.10. *Suppose that $1 \le m \le n$ and $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Then*
$$y_m \psi_{\mathsf{st}}^\triangleleft = \mathsf{c}_m(\mathsf{s})\psi_{\mathsf{st}}^\triangleleft + \sum_{(\mathsf{u},\mathsf{v}) \triangleleft (\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}^\triangleleft \qquad \text{and} \qquad y_m \psi_{\mathsf{st}}^\triangleright = \mathsf{c}_m(\mathsf{s})\psi_{\mathsf{st}}^\triangleright + \sum_{(\mathsf{u},\mathsf{v}) \triangleright (\mathsf{s},\mathsf{t})} d_{\mathsf{uv}}\psi_{\mathsf{uv}}^\triangleright$$
for some $c_{\mathsf{uv}}, d_{\mathsf{uv}} \in \Bbbk[\underline{x}]$ such that
- $c_{\mathsf{uv}} \ne 0$ *only if* $\mathsf{r}(\mathsf{u}) = \mathsf{r}(\mathsf{s})$, $\mathsf{r}(\mathsf{v}) = \mathsf{r}(\mathsf{t})$ *and either* $\boldsymbol{\mu} \triangleleft \boldsymbol{\lambda}$, *or* $\boldsymbol{\mu} = \boldsymbol{\lambda}$, $\mathsf{v} = \mathsf{t}$ *and* $\mathsf{u} \triangleleft \mathsf{s}$,
- $d_{\mathsf{uv}} \ne 0$ *only if* $\mathsf{r}(\mathsf{u}) = \mathsf{r}(\mathsf{s})$, $\mathsf{r}(\mathsf{v}) = \mathsf{r}(\mathsf{t})$ *and either* $\boldsymbol{\mu} \triangleright \boldsymbol{\lambda}$, *or* $\boldsymbol{\mu} = \boldsymbol{\lambda}$, $\mathsf{v} = \mathsf{t}$ *and* $\mathsf{u} \triangleright \mathsf{s}$.

Notice, in particular, that the coefficients of the leading term $\psi_{\mathsf{st}}^{\triangle}$ are zero in the standard KLR algebras $\mathscr{R}_n^{\Lambda}(\Bbbk)$ since $\mathsf{c}_m(\mathsf{s})$ is a polynomial in $\Bbbk[\underline{x}]$ with zero constant term by the degree requirements of Definition 3A.1. Hence, it follows that $y_r^{|\mathrm{Std}(\mathbf{i})|}1_{\mathbf{i}} = 0$ in $\mathsf{R}_n^{\Lambda}(\Bbbk)$ $\cong \mathscr{R}_n^{\Lambda}(\Bbbk)$, generalising [26, Corollary 4.31].

5. Graded Specht and simple modules

This section uses the cellular bases of Theorem 4F.1 to construct complete sets of graded simple modules for $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$. We prove some identities relating the decomposition matrices associated to the different bases and over different fields. Some of these results will be instrumental in the next section when we show than the algebra $\bigoplus_{n \geq 0} \mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$ categorifies the integral highest weight module $L(\Lambda)$ of the corresponding Kac–Moody algebra.

In this section we slightly weaken the assumptions of the last two sections and assume that $(\Gamma, \mathbf{Q}_I^{\underline{x}}, \mathbf{W}_I^{\underline{x}})$ is a $\Bbbk[\underline{x}]$-deformation of a standard cyclotomic KLR datum $(\Gamma, \mathbf{Q}_I, \mathbf{W}_I)$ and (c, r) is a (graded) content system with values in $\Bbbk[\underline{x}]$. Assume that $\mathbb{K}$ is a field that is a $\Bbbk$-algebra, so that $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$ is a graded $\mathbb{K}[\underline{x}]$-cellular algebra by Corollary 4F.3. As explained below, the results in this section apply to the standard cyclotomic KLR algebras of type $A_{e-1}^{(1)}$, A_{∞}, $C_{e-1}^{(1)}$ and C_{∞} since the graded irreducible $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$-modules and the graded irreducible $\mathscr{R}_n^{\Lambda}(\mathbb{K})$-modules coincide.

5A. **Irreducible modules.** This subsection describes the irreducible graded R_n^{Λ}-modules, both as subquotients and as submodules of R_n^{Λ}. Recall that $\mathbb{K}$ is a field that is a $\Bbbk$-algebra.

Let L be a $\Bbbk[\underline{x}]$-module. Fix $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$. Via (4B.3), the $\Bbbk[\underline{x}]$-cellular algebra framework equips the Specht modules $S_{\boldsymbol{\lambda}}^{\triangleleft}(L)$ and $S_{\boldsymbol{\lambda}}^{\triangleright}(L)$ with homogeneous symmetric associative bilinear forms that are characterised by

$$\langle \psi_{\mathsf{s}}^{\triangleleft}, \psi_{\mathsf{t}}^{\triangleleft} \rangle_{\boldsymbol{\lambda}}^{\triangleleft} \, \psi_{\mathsf{u}}^{\triangleleft} = \psi_{\mathsf{us}}^{\triangleleft} \psi_{\mathsf{t}}^{\triangleleft} \qquad \text{and} \qquad \langle \psi_{\mathsf{s}}^{\triangleright}, \psi_{\mathsf{t}}^{\triangleright} \rangle_{\boldsymbol{\lambda}}^{\triangleright} \, \psi_{\mathsf{u}}^{\triangleright} = \psi_{\mathsf{us}}^{\triangleright} \psi_{\mathsf{t}}^{\triangleright}, \tag{5A.1}$$

for $\mathsf{s}, \mathsf{tu} \in \mathrm{Std}(\boldsymbol{\lambda})$. The *radicals* of the graded Specht modules are the submodules:

$$\mathrm{rad}\, S_{\boldsymbol{\lambda}}^{\triangleleft}(L) = \{ a \in S_{\boldsymbol{\lambda}}^{\triangleleft}(L) \mid \langle a, b \rangle_{\boldsymbol{\lambda}}^{\triangleleft} = 0 \text{ for all } b \in S_{\boldsymbol{\lambda}}^{\triangleleft}(L) \}$$
$$\mathrm{rad}\, S_{\boldsymbol{\lambda}}^{\triangleright}(L) = \{ a \in S_{\boldsymbol{\lambda}}^{\triangleleft}(L) \mid \langle a, b \rangle_{\boldsymbol{\lambda}}^{\triangleright} = 0 \text{ for all } b \in S_{\boldsymbol{\lambda}}^{\triangleright}(L) \}.$$

Note that these definitions make sense for any (graded) $\Bbbk[\underline{x}]$-module L.

Definition 5A.2. Let $\boldsymbol{\mu} \in \mathcal{P}_{\alpha}^{\ell}$, for $\alpha \in Q_n^+$. Let L be a $\Bbbk[\underline{x}]$-module and define

$$D_{\boldsymbol{\mu}}^{\triangleleft}(L) = S_{\boldsymbol{\mu}}^{\triangleleft}(L) / \mathrm{rad}\, S_{\boldsymbol{\mu}}^{\triangleleft}(L) \qquad \text{and} \qquad D_{\boldsymbol{\nu}}^{\triangleright}(L) = S_{\boldsymbol{\mu}}^{\triangleright}(L) / \mathrm{rad}\, S_{\boldsymbol{\mu}}^{\triangleright}(L)$$

If $K = \mathbb{K}[\underline{x}]$ then $D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K})$ and $D_{\boldsymbol{\nu}}^{\triangleright}(\mathbb{K})$ are $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$-modules. Set

$$\mathcal{K}_{\alpha}^{\triangleleft} = \left\{ \boldsymbol{\mu} \in \mathcal{P}_{\alpha}^{\ell} \, \middle| \, D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K}) \neq 0 \right\} \qquad \text{and} \qquad \mathcal{K}_{\alpha}^{\triangleright} = \left\{ \boldsymbol{\mu} \in \mathcal{P}_{\alpha}^{\ell} \, \middle| \, D_{\boldsymbol{\nu}}^{\triangleright}(\mathbb{K}) \neq 0 \right\}.$$

Let $\mathcal{K}_n^{\triangleleft} = \bigcup_{\alpha \in Q_n^+} \mathcal{K}_{\alpha}^{\triangleleft}$ and $\mathcal{K}_n^{\triangleright} = \bigcup_{\alpha \in Q_n^+} \mathcal{K}_{\alpha}^{\triangleright}$.

When the choice of L is clear (usually, $L = \mathbb{K}$), then we write $D_{\boldsymbol{\mu}}^{\triangleleft}$ and $D_{\boldsymbol{\nu}}^{\triangleright}$.

As $\mathbb{K}$-vector spaces, with respect to the $\underline{x}$-grading, $D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K})$ is the degree zero component of $D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K}[x])$ and $D_{\boldsymbol{\nu}}^{\triangleright}(\mathbb{K})$ is the degree zero component of $D_{\boldsymbol{\nu}}^{\triangleright}(\mathbb{K}[\underline{x}])$. The modules $D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K}[\underline{x}])$ and $D_{\boldsymbol{\nu}}^{\triangleright}(\mathbb{K}[\underline{x}])$ are free $\mathbb{K}[\underline{x}]$-modules, and so infinite dimensional $\mathbb{K}$-vector spaces if $\underline{x} \neq \emptyset$, whereas $D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K})$ and $D_{\boldsymbol{\nu}}^{\triangleright}(\mathbb{K})$ are finite dimensional $\mathbb{K}$-vector spaces upon which each $x \in \underline{x}$ acts as multiplication by 0.

Even though our notation does not reflect this, the sets $\mathcal{K}_n^\lhd$ and $\mathcal{K}_n^\rhd$ depend on $\boldsymbol{\rho}$ and, a priori, on the field $\mathbb{K}$. In type $A_{e-1}^{(1)}$ the sets $\mathcal{K}_n^\lhd$ and $\mathcal{K}_n^\rhd$ have already been determined [2, 11]. In Theorem 6F.14 below we give a uniform characterisation of $\mathcal{K}_n^\lhd$ and $\mathcal{K}_n^\rhd$ in types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$. In particular, this result shows that the sets $\mathcal{K}_n^\lhd$ and $\mathcal{K}_n^\rhd$ do not depend on the choice of field $\mathbb{K}$.

Combining Theorem 4C.3 and Theorem 4B.6 we obtain:

Theorem 5A.3. Let $\triangle \in \{\lhd, \rhd\}$ and suppose that $K = \mathbb{K}[\underline{x}]$. Then $\{q^z D_{\boldsymbol{\mu}}^\triangle(\mathbb{K}) \mid \boldsymbol{\mu} \in \mathcal{K}_n^\triangle$ and $z \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic irreducible graded $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$-modules. Moreover, $D_{\boldsymbol{\mu}}^\triangle(\mathbb{K})$ is a graded self-dual $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$-module, for $\boldsymbol{\mu} \in \mathcal{K}_n^\triangle$.

By Corollary 4F.4 and Example 4B.7, the set of isomorphism classes of irreducible graded $\mathscr{R}_n^\Lambda(\mathbb{K})$-module coincides with the set of isomorphism classes of irreducible $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$-modules. The point is that if L is a $\mathbb{K}[\underline{x}]$-module and some $x \in \underline{x}$ does not act on L as multiplication by zero then $D_{\boldsymbol{\mu}}^\triangle(L)$ is not irreducible.

We next show how to realise the graded simple modules of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$ as submodules of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$, up to shift. To do this we first need a similar description of the Specht modules, for which we use the elements $z_{\boldsymbol{\lambda}}^\lhd$ and $z_{\boldsymbol{\lambda}}^\rhd$ from Definition 4E.8. Extending the definition of $z_{\boldsymbol{\lambda}}^\triangle$, for $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ set

$$z_{\mathsf{s}}^\lhd = \psi_{d_{\mathsf{s}}^\lhd} z_{\boldsymbol{\lambda}}^\lhd = \psi_{\mathsf{s}\mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\lhd \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\rhd \qquad \text{and} \qquad z_{\mathsf{s}}^\rhd = \psi_{d_{\mathsf{s}}^\rhd} z_{\boldsymbol{\lambda}}^\rhd = \psi_{\mathsf{s}\mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\rhd \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\lhd.$$

Lemma 5A.4. Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Then there are $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}])$-module isomorphisms

$$\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}]) z_{\boldsymbol{\lambda}}^\lhd \cong q^{\mathrm{def}(\boldsymbol{\lambda}) + \mathrm{deg}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd} S_{\boldsymbol{\lambda}}^\lhd \qquad \text{and} \qquad \mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}]) z_{\boldsymbol{\lambda}}^\rhd \cong q^{\mathrm{def}(\boldsymbol{\lambda}) + \mathrm{deg}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd} S_{\boldsymbol{\lambda}}^\rhd.$$

Moreover, these modules have bases $\{z_{\boldsymbol{\lambda}}^\lhd \mid \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ and $\{z_{\boldsymbol{\lambda}}^\rhd \mid \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})\}$, respectively.

Proof. Let $\{\triangle, \triangledown\} = \{\lhd, \rhd\}$. By Corollary 4E.7, there is a well-defined, homogeneous, $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}])$-module homomorphism $\pi_{\boldsymbol{\lambda}}^\triangle : q^{\mathrm{def}(\boldsymbol{\lambda}) + \mathrm{deg}^\triangle \mathsf{t}_{\boldsymbol{\lambda}}^\triangle} S_{\boldsymbol{\lambda}}^\triangle \longrightarrow \mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}]) z_{\boldsymbol{\lambda}}^\triangle$ given by

$$\pi_{\boldsymbol{\lambda}}^\triangle \left(\psi_{\mathsf{s}\mathsf{t}_{\boldsymbol{\lambda}}^\triangle}^\triangle + (\mathsf{R}_n^\Lambda)^{\triangle\boldsymbol{\lambda}} \right) = \psi_{\mathsf{s}\mathsf{t}_{\boldsymbol{\lambda}}^\triangle}^\triangle \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\triangle \mathsf{t}_{\boldsymbol{\lambda}}^\triangle}^\triangledown = z_{\mathsf{s}}^\triangle, \qquad \text{for } \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda}).$$

By Theorem 4C.3, $\pi_{\boldsymbol{\lambda}}^\triangle$ is homogeneous of degree zero. The set $\{z_{\mathsf{s}}^\triangle \mid \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ is a basis for the image of $\pi^\triangle$ since multiplying by the idempotents F_{t}, for $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, shows that these elements are linearly independent. Hence, $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}]) z_{\boldsymbol{\lambda}}^\triangle = \mathrm{im}\, \pi^\triangle$ in view of Proposition 4E.9. The result follows. $\square$

By Definition 4E.8, $\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\rhd z_{\boldsymbol{\lambda}}^\lhd = z_{\boldsymbol{\lambda}}^\rhd \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\lhd$ and $\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\lhd z_{\boldsymbol{\lambda}}^\rhd = z_{\boldsymbol{\lambda}}^\lhd \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\rhd$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Applying Lemma 4A.3,

$$\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\lhd z_{\boldsymbol{\lambda}}^\rhd = \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\lhd \cdot \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\rhd \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\lhd = \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\lhd \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\rhd \cdot \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\lhd = z_{\boldsymbol{\lambda}}^\lhd \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd}^\lhd \tag{5A.5}$$

and, similarly, $\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\rhd z_{\boldsymbol{\lambda}}^\lhd = z_{\boldsymbol{\lambda}}^\rhd \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\rhd$. The next result, which has its origins in the work of James [29, §11], shows that these elements generate the simple $\mathsf{R}_n^\Lambda(\mathbb{k}[\underline{x}])$-modules.

Theorem 5A.6. Suppose $\boldsymbol{\mu} \in \mathcal{K}_\alpha^\lhd$ and $\boldsymbol{\nu} \in \mathcal{K}_\alpha^\rhd$, for $\alpha \in Q^+$. As $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$-modules,

$$q^{2\,\mathrm{def}(\boldsymbol{\mu}) + \mathrm{deg}^\lhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd} D_{\boldsymbol{\mu}}^\lhd(\mathbb{K}) \cong \mathsf{R}_n^\Lambda(\mathbb{K}) z_{\boldsymbol{\mu}}^\lhd \psi_{\mathsf{t}_{\boldsymbol{\mu}}^\lhd \mathsf{t}_{\boldsymbol{\mu}}^\rhd}^\lhd \quad \text{and} \quad q^{2\,\mathrm{def}(\boldsymbol{\nu}) + \mathrm{deg}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\rhd} D_{\boldsymbol{\nu}}^\rhd(\mathbb{K}) \cong \mathsf{R}_n^\Lambda(\mathbb{K}) z_{\boldsymbol{\nu}}^\rhd \psi_{\mathsf{t}_{\boldsymbol{\nu}}^\rhd \mathsf{t}_{\boldsymbol{\nu}}^\lhd}^\rhd$$

In particular, $D_{\boldsymbol{\mu}}^\lhd(\mathbb{K}) \neq 0$ if and only if $z_{\boldsymbol{\mu}}^\lhd \psi_{\mathsf{t}_{\boldsymbol{\mu}}^\lhd \mathsf{t}_{\boldsymbol{\mu}}^\rhd}^\lhd \neq 0$ and $D_{\boldsymbol{\nu}}^\rhd(\mathbb{K}) \neq 0$ if and only if $z_{\boldsymbol{\nu}}^\rhd \psi_{\mathsf{t}_{\boldsymbol{\nu}}^\rhd \mathsf{t}_{\boldsymbol{\nu}}^\lhd}^\rhd \neq 0$ in $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$.

Proof. We prove only the first isomorphism as the second isomorphism follows by symmetry. We first prove some related results over $\Bbbk[\underline{x}]$. As in the proof of Proposition 4F.9, define $\theta_{\mathsf{t}} \in S^{\lhd}_{\boldsymbol{\mu}}(\Bbbk[\underline{x}])^{\circledast}$ by $\theta_{\mathsf{t}}(\psi^{\lhd}_{\mathsf{u}}) = \tau_{\alpha}(\psi^{\rhd}_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}} \psi^{\lhd}_{\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}\mathsf{t}})$, for $\mathsf{t}, \mathsf{u} \in \mathrm{Std}(\boldsymbol{\mu})$. Using (5A.5), Lemma 5A.4 and Proposition 4F.9, there are homogeneous $\mathsf{R}^{\Lambda}_n(\Bbbk[\underline{x}])$-module homomorphisms (the reader is welcome to determine the degrees of these maps),

$$S^{\lhd}_{\boldsymbol{\mu}}(\Bbbk[\underline{x}]) \xrightarrow{f} \mathsf{R}^{\Lambda}_n(\Bbbk[\underline{x}])z^{\lhd}_{\boldsymbol{\mu}} \xrightarrow{g} \mathsf{R}^{\Lambda}_n(\Bbbk[\underline{x}])z^{\rhd}_{\boldsymbol{\mu}} \xrightarrow{h} S^{\lhd}_{\boldsymbol{\mu}}(\Bbbk[\underline{x}])^{\circledast},$$

given by $f(\psi^{\lhd}_{\mathsf{s}}) = z^{\lhd}_{\mathsf{s}} = \psi_{d^{\lhd}_{\mathsf{s}}}z^{\lhd}_{\boldsymbol{\mu}}$, $g(a) = a\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}}$ and $h(z^{\rhd}_{\mathsf{t}}) = \theta_{\mathsf{t}}$, for tableaux $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\mu})$ and $a \in \mathsf{R}^{\Lambda}_n(\Bbbk[\underline{x}]))$. By Lemma 5A.4 and the proof of Proposition 4F.9, f and h are isomorphisms. Let $\theta = h \circ g \circ f$ be the composition of these three maps. To determine θ, for $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\mu})$ write

$$z^{\lhd}_{\mathsf{s}} = \psi^{\lhd}_{\mathsf{s}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}} = \sum_{(\mathsf{u},\mathsf{v}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)} a_{\mathsf{u}\mathsf{v}}\psi^{\rhd}_{\mathsf{u}\mathsf{v}}, \qquad \text{for } a_{\mathsf{u}\mathsf{v}} \in L.$$

By (C_2) and Theorem 4F.1, $a_{\mathsf{u}\mathsf{v}} \neq 0$ only if $\mathrm{Shape}(\mathsf{v}) \unlhd \boldsymbol{\mu}$, with equality only if $\mathsf{v} = \mathsf{t}^{\lhd}_{\boldsymbol{\mu}}$. Therefore,

$$\theta(\psi^{\lhd}_{\mathsf{s}}) = h\left(z^{\lhd}_{\mathsf{s}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}}\right) = h\left(\sum_{(\mathsf{u},\mathsf{v}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)} a_{\mathsf{u}\mathsf{v}}\psi^{\rhd}_{\mathsf{u}\mathsf{v}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}}\right)$$

$$= \sum_{\mathsf{u} \in \mathrm{Std}(\boldsymbol{\mu})} a_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}h\left(\psi^{\rhd}_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}}\right) = \sum_{\mathsf{u} \in \mathrm{Std}(\boldsymbol{\mu})} a_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\theta_{\mathsf{u}},$$

where we have used Corollary 4E.7, for the third equality, and Lemma 4A.3 for the last equality together with the identity $z^{\rhd}_{\mathsf{u}} = \psi^{\rhd}_{\mathsf{u}\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}}\psi^{\lhd}_{\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}} = \psi^{\rhd}_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}}$. Consequently, since τ_{α} is a trace form,

$$\theta(\psi^{\lhd}_{\mathsf{s}})(\psi^{\lhd}_{\mathsf{t}}) = \sum_{\mathsf{u} \in \mathrm{Std}(\boldsymbol{\mu})} a_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\theta_{\mathsf{u}}(\psi^{\lhd}_{\mathsf{t}}) = \sum_{\mathsf{u} \in \mathrm{Std}(\boldsymbol{\mu})} a_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\tau_{\alpha}\left(\psi^{\rhd}_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\lhd}_{\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}\mathsf{t}}\right)$$

$$= \tau_{\alpha}\left(\sum_{\mathsf{u} \in \mathrm{Std}(\boldsymbol{\mu})} a_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\rhd}_{\mathsf{u}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}}\right)$$

$$= \tau_{\alpha}\left(\sum_{(\mathsf{u},\mathsf{v}) \in \mathrm{Std}^2(\mathcal{P}^{\ell}_n)} a_{\mathsf{u}\mathsf{v}}\psi^{\rhd}_{\mathsf{u}\mathsf{v}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}}\right), \qquad \text{by Corollary 4E.7,}$$

$$= \tau_{\alpha}\left(z^{\lhd}_{\mathsf{s}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}}\right) = \tau_{\alpha}\left(\psi^{\lhd}_{\mathsf{s}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}}\right) = \tau_{\alpha}\left(\psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}}\psi^{\lhd}_{\mathsf{s}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\right)$$

$$= \langle\psi^{\lhd}_{\mathsf{t}}, \psi^{\lhd}_{\mathsf{s}}\rangle^{\lhd}_{\boldsymbol{\lambda}}\,\tau_{\alpha}\left(\psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}}\right) = \langle\psi^{\lhd}_{\mathsf{t}}, \psi^{\lhd}_{\mathsf{s}}\rangle^{\lhd}_{\boldsymbol{\lambda}}\,\tau_{\alpha}(z^{\lhd}_{\boldsymbol{\lambda}}) = \langle\psi^{\lhd}_{\mathsf{t}}, \psi^{\lhd}_{\mathsf{s}}\rangle^{\lhd}_{\boldsymbol{\lambda}},$$

where the first equality on the last line uses Corollary 4E.7, and the definition of the inner product on $S^{\lhd}_{\boldsymbol{\mu}}(\Bbbk[\underline{x}]))$, and the last equality follows by Proposition 4E.9. Hence, ignoring the degree shift, θ is the natural $\Bbbk[\underline{x}]$-linear map from $S^{\lhd}_{\boldsymbol{\mu}}(\Bbbk[\underline{x}])) \to S^{\lhd}_{\boldsymbol{\mu}}(\Bbbk[\underline{x}]))^{\circledast}$ induced by the bilinear form $\langle\ ,\ \rangle^{\lhd}_{\boldsymbol{\lambda}}$ on $S^{\lhd}_{\boldsymbol{\mu}}(\Bbbk[\underline{x}]))$.

Finally, to identify $D^{\lhd}_{\boldsymbol{\mu}}(\mathbb{K})$, consider $\mathbb{K}$ as a $\mathbb{K}[\underline{x}]$-module by letting each $x \in \underline{x}$ act as zero. Tensoring with $\mathbb{K}$, the calculations above show that, for the induced maps after base change, $\theta \neq 0$ if and only if $D^{\lhd}_{\boldsymbol{\mu}}(\mathbb{K}) \neq 0$. By construction, the maps f and h are both isomorphisms, so $D^{\lhd}_{\boldsymbol{\mu}}(\mathbb{K}) \neq 0$ if and only if $g \neq 0$, which is if and only if $z^{\lhd}_{\boldsymbol{\mu}}\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\mu}}\mathsf{t}^{\rhd}_{\boldsymbol{\mu}}} \neq 0$.

Further, if $D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{K}) \neq 0$ then $q^d D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{K}) \cong \operatorname{im}(g \circ f) = \mathsf{R}_n^{\Lambda}(\mathbb{K}) z_{\boldsymbol{\mu}}^{\lhd} \psi_{\mathsf{t}_{\boldsymbol{\mu}}^{\lhd} \mathsf{t}_{\boldsymbol{\mu}}^{\rhd}}^{\lhd}$, for some $d \in \mathbb{Z}$. Inspection of the maps, using (4.32a), shows that $d = 2 \operatorname{def}(\boldsymbol{\lambda}) + \deg^{\lhd} \mathsf{t}_{\boldsymbol{\lambda}}^{\rhd}$. $\qquad \square$

Remark 5A.7. If $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ then the simple module $\mathsf{R}_n^{\Lambda}(\mathbb{K}) z_{\boldsymbol{\mu}}^{\lhd} \psi_{\mathsf{t}_{\boldsymbol{\mu}}^{\lhd} \mathsf{t}_{\boldsymbol{\mu}}^{\rhd}}^{\lhd}$ is the socle of a projective cover of $D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{K})$, up to shift. The module $\mathsf{R}_n^{\Lambda}(\mathbb{K}) z_{\boldsymbol{\mu}}^{\lhd} \psi_{\mathsf{t}_{\boldsymbol{\mu}}^{\lhd} \mathsf{t}_{\boldsymbol{\mu}}^{\rhd}}^{\lhd}$ is spanned by $\{z_{\mathsf{s}}^{\lhd} \psi_{\mathsf{t}_{\boldsymbol{\mu}}^{\lhd} \mathsf{t}_{\boldsymbol{\mu}}^{\rhd}}^{\lhd} \mid \mathsf{s} \in \operatorname{Std}(\boldsymbol{\mu})\}$.

5B. **Graded decomposition numbers.** This subsection introduces graded decomposition matrices together with the key result that these matrices are unitriangular. This will be used in the next section to construct bases in the Grothendieck groups of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$, which we use to prove Theorem **C** from the introduction.

If M is an $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$-module and D is an irreducible $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$-module then the *graded decomposition multiplicity* of D in M is the Laurent polynomial

$$[M : D]_q = \sum_{k \in \mathbb{Z}} \left[M : q^k D \right] q^k \quad \in \mathbb{N}\left[q, q^{-1} \right],$$

where $[M : q^k D] \in \mathbb{N}$ is equal to the number of composition factors of M that are isomorphic to $q^k D$.

The *graded decomposition numbers* of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$ are the decomposition multiplicities

$$\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) = \left[S_{\boldsymbol{\lambda}}^{\triangle}(\mathbb{K}) : D_{\boldsymbol{\mu}}^{\triangle}(\mathbb{K}) \right]_q \qquad \text{and} \qquad \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q) = [S_{\boldsymbol{\lambda}}^{\rhd}(\mathbb{K}) : D_{\boldsymbol{\nu}}^{\rhd}(\mathbb{K})]_q \qquad (5\text{B}.1)$$

for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$, $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$. The *graded decomposition matrices* of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$ are the matrices

$$\mathsf{D}_n^{\mathbb{K}\lhd} = \left(\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) \right) \qquad \text{and} \qquad \mathsf{D}_n^{\mathbb{K}\rhd} = \left(\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q) \right),$$

The most important result that we need about the decomposition matrices of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$ is the following.

Theorem 5B.2. Suppose that $\mathbb{K}$ is a field and that $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$.
- (a) If $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ then $\mathsf{d}_{\boldsymbol{\mu}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) = 1$ and $\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) \neq 0$ only if $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ and $\alpha_{\boldsymbol{\lambda}} = \alpha_{\boldsymbol{\mu}}$.
- (b) If $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$ then $\mathsf{d}_{\boldsymbol{\nu}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q) = 1$ and $\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q) \neq 0$ only if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\nu}$ and $\alpha_{\boldsymbol{\lambda}} = \alpha_{\boldsymbol{\nu}}$.

Proof. Let $\triangle \in \{\lhd, \rhd\}$, $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ and $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangle}$. The theory of graded cellular algebras, via Theorem 4B.6, shows that the decomposition matrix $\mathsf{D}_n^{\mathbb{K}\triangle}$ is unitriangular when the rows and columns are ordered with respect to any total order that refines $\triangle$-dominance. Hence, $\mathsf{d}_{\boldsymbol{\mu}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q) = 1$ and $\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q) \neq 0$ only if $\boldsymbol{\lambda}\triangle\boldsymbol{\mu}$. The remaining claim follows because the cellular bases of Theorem 4F.1 give the decomposition $\mathsf{R}_n^{\Lambda}(\mathbb{K}) = \bigoplus_{\alpha \in Q_n^+} \mathsf{R}_{\alpha}^{\Lambda}(\mathbb{K}[\underline{x}])$ of $\mathsf{R}_n^{\Lambda}(\mathbb{K})$ into a direct sum of two-sided ideals. $\qquad \square$

For $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ let $Y_{\boldsymbol{\mu}}^{\lhd}$ be the *projective cover* of $D_{\boldsymbol{\mu}}^{\lhd}$ as an $\mathsf{R}_n^{\Lambda}(\mathbb{K})$-module. Similarly, let $Y_{\boldsymbol{\nu}}^{\rhd}$ be the projective cover of $D_{\boldsymbol{\nu}}^{\rhd}$ as an $\mathsf{R}_n^{\Lambda}(\mathbb{K})$-module, for $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$.

Proposition 5B.3. Let $\mathbb{K}$ be a field.
- (a) Let $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$. Then $Y_{\boldsymbol{\mu}}^{\lhd}$ has a filtration $Y_{\boldsymbol{\mu}}^{\lhd} = Y_{\boldsymbol{\mu},1}^{\lhd} \supset Y_{\boldsymbol{\mu},2}^{\lhd} \supset \cdots \supset Y_{\boldsymbol{\mu},z}^{\lhd}$ such that there exist ℓ-partitions $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_z \in \mathcal{P}_n^{\ell}$ with $Y_{\boldsymbol{\mu},k}^{\lhd}/Y_{\boldsymbol{\mu},k+1}^{\lhd} \cong \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\lambda}_k}^{\mathbb{K}\lhd}(q) S_{\boldsymbol{\lambda}_k}^{\lhd}$ and $k > l$ whenever $\boldsymbol{\lambda}_k \lhd \boldsymbol{\lambda}_l$.
- (b) Let $\boldsymbol{\mu} \in \mathcal{K}_n^{\rhd}$. Then $Y_{\boldsymbol{\nu}}^{\rhd}$ has a filtration $Y_{\boldsymbol{\nu}}^{\rhd} = Y_{\boldsymbol{\mu},1}^{\rhd} \supset Y_{\boldsymbol{\mu},2}^{\rhd} \supset \cdots \supset Y_{\boldsymbol{\mu},z}^{\rhd}$ such that there exist ℓ-partitions $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_z \in \mathcal{P}_n^{\ell}$ with $Y_{\boldsymbol{\mu},k}^{\rhd}/Y_{\boldsymbol{\mu},k+1}^{\rhd} \cong \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\lambda}_k}^{\mathbb{K}\rhd}(q) S_{\boldsymbol{\lambda}_k}^{\rhd}$ and $k > l$ whenever $\boldsymbol{\lambda}_k \rhd \boldsymbol{\lambda}_l$.

Proof. This comes from the general theory of (graded) cellular algebras; see [21, Theorem 3.7] or [24, Lemma 2.25]. □

Define *graded Cartan matrices* $\mathsf{C}_n^{\mathbb{K}\lhd} = (\mathsf{c}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q))$ and $\mathsf{C}_n^{\mathbb{K}\rhd} = (\mathsf{c}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q))$ by

$$\mathsf{c}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) = \left[Y_{\boldsymbol{\mu}}^{\lhd} : D_{\boldsymbol{\nu}}^{\lhd} \right] \qquad \text{and} \qquad \mathsf{c}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q) = [Y_{\boldsymbol{\nu}}^{\rhd} : D_{\boldsymbol{\nu}}^{\rhd}].$$

If M is matrix let M^T be its transpose.

Standard arguments now show that the $\mathbb{K}[\underline{x}]$-cellular algebra $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$ enjoys the following graded analogue of Brauer-Humphreys reciprocity; compare [24, Theorem 2.17].

Corollary 5B.4. Suppose that $\mathbb{K}$ is a field. Then $\mathsf{C}_n^{\mathbb{K}\lhd} = (\mathsf{D}_n^{\mathbb{K}\lhd})^T \mathsf{D}_n^{\mathbb{K}\lhd}$ and $\mathsf{C}_n^{\mathbb{K}\rhd} = (\mathsf{D}_n^{\mathbb{K}\rhd})^T \mathsf{D}_n^{\mathbb{K}\rhd}$.

5C. **Adjustment matrices.** Following Lemma 3A.3, in this subsection we assume that $\Bbbk = \mathbb{Z}$, so the content system (c, r) is defined over $\mathbb{Z}[\underline{x}]$. By assumption, $\mathbb{K}$ is a field that is a $\Bbbk$-algebra, which means that we are assuming that $\mathbb{K}$ is a field. Then the algebra $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}]) \cong \mathbb{K}[\underline{x}] \otimes_{\mathbb{Z}[\underline{x}]} \mathsf{R}_n^{\Lambda}(\mathbb{Z}[\underline{x}])$ is a graded $\mathbb{K}[\underline{x}]$-cellular algebra by Theorem 4F.1. The main result of this subsection compares the decomposition matrices of the two algebras $\mathsf{R}_n^{\Lambda}(\mathbb{Q}[\underline{x}])$ and $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$.

Let $\mathcal{A}[I^n]$ be the free $\mathcal{A}$-module generated by I^n. The *q-character* of a finite dimensional $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$-module M is

$$\mathrm{ch}\, M = \sum_{\mathbf{i} \in I^n} (\dim_q M_{\mathbf{i}})\, \mathbf{i} \in \mathcal{A}[I^n],$$

where $M_{\mathbf{i}} = 1_{\mathbf{i}} M$, for $\mathbf{i} \in I^n$. For example, $\mathrm{ch}\, S_{\boldsymbol{\lambda}}^{\triangle}(\mathbb{K}[\underline{x}]) = \sum_{\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})} q^{\deg^{\triangle}(\mathsf{t})} \mathsf{r}(\mathsf{t})$.

The *bar involution* is the $\mathbb{Z}$-linear involution on $\mathcal{A}$ given by setting $\overline{f(q)} = f(q^{-1})$, for $f(q) \in \mathbb{Z}$. Extend the bar involution to an automorphism of $\mathcal{A}[I^n]$ by declaring that $\overline{\mathbf{i}} = \mathbf{i}$, for $\mathbf{i} \in I^n$. It is easy to see that $\mathrm{ch}(M^{\circledast}) = \overline{\mathrm{ch}\, M}$.

The following result is well-known and is easily proved by induction of the height of $\alpha \in Q^+$. This result is stated as [36, Theorem 3.17], with the reader being invited to repeat the proof of [39, Theorem 3.3.1].

Theorem 5C.1. Let $\mathbb{K}$ be a field. Then the character map $\mathrm{ch}\colon [\mathrm{Rep}\,\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])] \longrightarrow \mathcal{A}[I^n]$ is injective.

The definition of the modules $D_{\boldsymbol{\mu}}^{\lhd}(L)$ and $D_{\boldsymbol{\nu}}^{\rhd}(L)$, and the radicals of the Specht modules, makes sense for any $\mathbb{Z}[\underline{x}]$-module L. For $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangle}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$ define

$$E_{\boldsymbol{\mu}}^{\lhd}(L) = L \otimes_{\mathbb{Z}[\underline{x}]} D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{Z}[\underline{x}]) \qquad \text{and} \qquad E_{\boldsymbol{\nu}}^{\rhd}(L) = L \otimes_{\mathbb{Z}[\underline{x}]} E_{\boldsymbol{\nu}}^{\rhd}(\mathbb{Z}[\underline{x}])..$$

For $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$, let

$$G_{\boldsymbol{\lambda}}^{\triangle} = \left(\langle \psi_{\mathsf{s}}^{\triangle}, \psi_{\mathsf{t}}^{\triangle} \rangle_{\boldsymbol{\lambda}}^{\triangle} \right)_{\mathsf{s},\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})}$$

be the Gram matrix of the bilinear form (5A.1) on the Specht module $S_{\boldsymbol{\lambda}}^{\triangle}$. By considering the Smith normal form of $G_{\boldsymbol{\lambda}}^{\triangle}$, it is straightforward to prove the following. (Compare with [52, Theorem 3.7.4].)

Lemma 5C.2. Let $\boldsymbol{\mu} \in \mathcal{P}_n^{\ell}$ and $\triangle \in \{\lhd, \rhd\}$. Then $E_{\boldsymbol{\mu}}^{\triangle}(\mathbb{Z}[\underline{x}])$ is a $\mathbb{Z}[\underline{x}]$-free $\mathsf{R}_n^{\Lambda}(\mathbb{Z}[\underline{x}])$-module. Moreover, $D_{\boldsymbol{\mu}}^{\triangle}(\mathbb{Q}) \cong E_{\boldsymbol{\mu}}^{\triangle}(\mathbb{Q})$.

The following polynomials define a map between the Grothendieck groups of $\mathsf{R}_n^{\Lambda}(\mathbb{Q}[\underline{x}])$ and $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$.

Definition 5C.3. Let $\mathbb{K}$ be a field, $\triangle \in \{\lhd, \rhd\}$ and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{K}_n^\triangle$. Define Laurent polynomials $\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q)$ by

$$\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q) = \sum_{q \in \mathbb{Z}} \left[E_{\boldsymbol{\nu}}^\triangle(\mathbb{K}) : q^d D_{\boldsymbol{\mu}}^\triangle(\mathbb{K}) \right] q^d \in \mathbb{N}\left[q, q^{-1} \right].$$

The matrix $\mathsf{A}_n^{\mathbb{K}\triangle} = (\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q))$ is the *graded adjustment matrix* of $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$.

Theorem 5C.4. Suppose that $\mathbb{K}$ is a field and let $\triangle \in \{\lhd, \rhd\}$.

(a) If $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{K}_n^\triangle$ then $\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q) \neq 0$ only if $\boldsymbol{\mu}\underline{\triangle}\boldsymbol{\nu}$ and $\alpha_{\boldsymbol{\mu}} = \alpha_{\boldsymbol{\nu}}$. Moreover, $\overline{\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q)} = \mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q)$.

(b) As matrices, $\mathsf{D}_n^{\mathbb{K}\triangle} = \mathsf{D}_n^{\mathbb{Q}\triangle}\mathsf{A}_n^{\mathbb{K}\triangle}$. That is, if $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $\boldsymbol{\mu} \in \mathcal{K}_n^\triangle$ then

$$\mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}[\underline{x}]\triangle}(q) = \sum_{\boldsymbol{\nu} \in \mathcal{K}_n^\triangle} \mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{Q}\triangle}(q)\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q).$$

Proof. Every composition factor of $E_{\boldsymbol{\mu}}^\triangle(\mathbb{K})$ is a composition factor of $S_{\boldsymbol{\mu}}^\triangle(\mathbb{K})$, so the first statement in (a) follows from Theorem 5B.2. By Lemma 5C.2, the adjustment matrix induces a well-defined map of Grothendieck groups $\mathsf{A}_n^{\mathbb{K}\triangle} : [\operatorname{Rep}\mathsf{R}_n^\Lambda(\mathbb{Q}[\underline{x}])] \longrightarrow [\operatorname{Rep}\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])]$ given by

$$\mathsf{A}_n^{\mathbb{K}\triangle}\left(\left[D_{\boldsymbol{\nu}}^\triangle(\mathbb{Q}) \right] \right) = \left[E_{\boldsymbol{\nu}}^\triangle(\mathbb{K}) \right] = \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\triangle} \mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q) \left[D_{\boldsymbol{\mu}}^\triangle(\mathbb{K}) \right].$$

Taking q-characters, $\operatorname{ch} D_{\boldsymbol{\mu}}^\triangle(\mathbb{Q}) = \sum_{\boldsymbol{\nu}} \mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q) \operatorname{ch} D_{\boldsymbol{\mu}}^\triangle(\mathbb{K})$. Applying $\circledast$ to both sides, the self-duality of the simple modules now implies that $\overline{\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q)} = \mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q)$, which completes the proof of part (a). To prove (b), observe that

$$\sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\triangle} \mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\triangle}(q) \operatorname{ch} D_{\boldsymbol{\mu}}^\triangle(\mathbb{K}) = \operatorname{ch} S_{\boldsymbol{\lambda}}^\triangle(\mathbb{K}) = \operatorname{ch} S_{\boldsymbol{\lambda}}^\triangle(\mathbb{Q})$$

$$= \sum_{\boldsymbol{\nu} \in \mathcal{K}_n^\triangle} \mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{Q}\triangle}(q) \operatorname{ch} D_{\boldsymbol{\nu}}^\triangle(\mathbb{Q})$$

$$= \sum_{\boldsymbol{\nu} \in \mathcal{K}_n^\triangle} \mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{Q}\triangle}(q) \operatorname{ch} E_{\boldsymbol{\nu}}^\triangle(\mathbb{K})$$

$$= \sum_{\boldsymbol{\nu} \in \mathcal{K}_n^\triangle} \mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{Q}\triangle}(q) \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\triangle} \mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\triangle}(q) \operatorname{ch} D_{\boldsymbol{\mu}}^\triangle(\mathbb{K}).$$

Comparing the coefficient of $\operatorname{ch} D_{\boldsymbol{\mu}}^\triangle(\mathbb{K})$ on both sides using Theorem 5C.1 proves part (b). $\square$

We prove in Theorem 6F.14 below that $\mathcal{K}_n^\triangle(\mathbb{K}) = \mathcal{K}_n^\triangle(\mathbb{Q})$ for any field $\mathbb{K}$, which implies that $\mathsf{A}_n^{\mathbb{K}\triangle}$ is a square unitriangular matrix.

5D. A Mullineux-like involution. Theorem 5A.3 gives two descriptions of the simple $\mathsf{R}_n^\Lambda(\mathbb{K})$-modules $\{q^z D_{\boldsymbol{\mu}}^\lhd(\mathbb{K})\}$ and $\{q^z D_{\boldsymbol{\nu}}^\rhd(\mathbb{K})\}$. The aim of this subsection is set up the machinery for comparing these different constructions of the simple $\mathsf{R}_n^\Lambda(\mathbb{K})$-modules. We start with a definition.

Definition 5D.1. Let $\mathsf{m} \colon \mathcal{K}_n^\lhd \longrightarrow \mathcal{K}_n^\rhd$ be the unique bijection such that $D_{\boldsymbol{\mu}}^\lhd(\mathbb{K}) \cong D_{\mathsf{m}(\boldsymbol{\mu})}^\rhd(\mathbb{K})$, for $\boldsymbol{\mu} \in \mathcal{K}_n^\lhd$.

If $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$ then, by Theorem 5A.3, the modules $q^z D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{K})$ and $q^y D_{\boldsymbol{\nu}}^{\rhd}(\mathbb{K})$ are self-dual if and only if $z = 0$ and $y = 0$, respectively. Hence, the map m of Definition 5D.1 is well-defined.

Like the sets $\mathcal{K}_n^{\lhd}$ and $\mathcal{K}_n^{\rhd}$, priori, the map m depends on Λ, $\boldsymbol{\rho}$, and the field $\mathbb{K}$. We give an explicit description of m in Corollary 6F.15 below, which shows that m is independent of $\mathbb{K}$. In the next subsection we show that m is closely related to the sign isomorphism. In particular, in the special case of the symmetric groups, the map $\boldsymbol{\mu} \mapsto \mathsf{m}(\boldsymbol{\mu})'$ is the Mullineux map [59].

Recall from subsection 5B that $Y_{\boldsymbol{\mu}}^{\triangle}$ is the projective cover of $D_{\boldsymbol{\mu}}^{\triangle}$, for $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangle}$. Hence, we have:

Lemma 5D.2. Let $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$. Then $Y_{\boldsymbol{\mu}}^{\lhd} \cong Y_{\mathsf{m}(\boldsymbol{\mu})}^{\rhd}$.

Using m we can give the precise relationship between the graded decomposition numbers $\mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\lhd}(q)$ and $\mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{K}\rhd}(q)$. In particular, this shows that the graded decomposition matrices $\mathsf{D}_n^{\mathbb{K}\lhd}$ and $\mathsf{D}_n^{\mathbb{K}\rhd}$ encode equivalent information.

Recall from the last subsection that the *bar involution* is the $\mathbb{Z}$-linear automorphism of $\mathcal{A}$ given by $\overline{f(q)} = f(q^{-1})$.

Proposition 5D.3. Suppose that $\mathbb{K}$ is a field.

(a) If $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ and $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ then $\mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\lhd}(q) = q^{\operatorname{def} \boldsymbol{\lambda}} \overline{\mathsf{d}_{\boldsymbol{\lambda}\mathsf{m}(\boldsymbol{\mu})}^{\mathbb{K}\rhd}(q)}$.
(b) If $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ and $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ then $\mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\lhd}(q) \neq 0$ only if $\mathsf{m}(\boldsymbol{\mu}) \unlhd \boldsymbol{\lambda} \unlhd \boldsymbol{\mu}$.
(c) If $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$ then $\mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{K}\rhd}(q) \neq 0$ only if $\mathsf{m}^{-1}(\boldsymbol{\nu}) \unrhd \boldsymbol{\lambda} \unrhd \boldsymbol{\nu}$.

Proof. Using formal characters and Proposition 4F.9, we have

$$\sum_{\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}} \mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\lhd}(q) \operatorname{ch} D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{K}) = \operatorname{ch} S_{\boldsymbol{\lambda}}^{\lhd}(\mathbb{K}) = q^{\operatorname{def}(\boldsymbol{\lambda})} \operatorname{ch} S_{\boldsymbol{\lambda}}^{\rhd}(\mathbb{K})^{\circledast} = q^{\operatorname{def}(\boldsymbol{\lambda})} \overline{\operatorname{ch} S_{\boldsymbol{\lambda}}^{\rhd}(\mathbb{K})}$$

$$= q^{\operatorname{def}(\boldsymbol{\lambda})} \overline{\sum_{\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}} \mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{K}\rhd}(q) \operatorname{ch} D_{\boldsymbol{\nu}}^{\rhd}(\mathbb{K})}$$

$$= q^{\operatorname{def}(\boldsymbol{\lambda})} \sum_{\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}} \overline{\mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{K}\rhd}(q)} \operatorname{ch} D_{\boldsymbol{\nu}}^{\rhd}(\mathbb{K})$$

$$= q^{\operatorname{def}(\boldsymbol{\lambda})} \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}} \overline{\mathsf{d}_{\boldsymbol{\lambda}\mathsf{m}(\boldsymbol{\mu})}^{\mathbb{K}\rhd}(q)} \operatorname{ch} D_{\mathsf{m}(\boldsymbol{\mu})}^{\rhd}(\mathbb{K})$$

where the second last equality follows because $D_{\boldsymbol{\nu}}^{\rhd}(\mathbb{K})$ is self-dual by Theorem 5A.3. Part (a) follows by comparing the coefficient of $\operatorname{ch} D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{K})$ on both sides using Theorem 6F.8.

For (b), if $\mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\lhd}(q) \neq 0$ then $\boldsymbol{\lambda} \unlhd \boldsymbol{\mu}$ by Theorem 5B.2. Moreover, $\mathsf{d}_{\boldsymbol{\lambda}\mathsf{m}(\boldsymbol{\mu})}^{\mathbb{K}\rhd}(q) \neq 0$ by (a), so $\boldsymbol{\lambda} \unrhd \mathsf{m}(\boldsymbol{\mu})$ by Theorem 5B.2. The proof of (c) is similar. $\square$

Recalling the adjustment matrices of subsection 5C, we obtain:

Corollary 5D.4. Let $\mathbb{K}$ be a field and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{K}_n^{\lhd}$. Then $\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\lhd}(q) = \overline{\mathsf{a}_{\mathsf{m}(\boldsymbol{\nu})\mathsf{m}(\boldsymbol{\mu})}^{\mathbb{K}\rhd}(q)}$.

Proof. Using Theorem 5C.4(b), twice, and Proposition 4F.9,

$$\sum_{\boldsymbol{\nu},\boldsymbol{\mu}\in\mathcal{K}_n^{\lhd}} \mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{Q}\lhd}(q)\mathsf{a}_{\boldsymbol{\nu\mu}}^{\mathbb{K}\lhd}(q)\operatorname{ch} D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{K}) = \operatorname{ch} S_{\boldsymbol{\lambda}}^{\lhd}(\mathbb{K}) = q^{\operatorname{def}(\boldsymbol{\lambda})}\overline{\operatorname{ch} S_{\boldsymbol{\lambda}}^{\rhd}(\mathbb{K})}$$

$$= q^{\operatorname{def}(\boldsymbol{\lambda})}\sum_{\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{K}_n^{\rhd}} \overline{\mathsf{d}_{\boldsymbol{\lambda\sigma}}^{\mathbb{Q}\rhd}(q)}\,\overline{\mathsf{a}_{\boldsymbol{\sigma\tau}}^{\mathbb{K}\rhd}(q)}\operatorname{ch} D_{\boldsymbol{\tau}}^{\rhd}(\mathbb{K})$$

$$= \sum_{\boldsymbol{\mu}\in\mathcal{K}_n^{\lhd}}\sum_{\boldsymbol{\nu}\in\mathcal{K}_n^{\lhd}} \mathsf{d}_{\boldsymbol{\lambda\nu}}^{\mathbb{Q}\lhd}(q)\,\overline{\mathsf{a}_{\mathsf{m}(\boldsymbol{\nu})\mathsf{m}(\boldsymbol{\mu})}^{\mathbb{K}\rhd}(q)}\operatorname{ch} D_{\boldsymbol{\mu}}^{\lhd}(\mathbb{K}),$$

where the last equality uses Proposition 5D.3(a), where we set $\boldsymbol{\sigma} = \mathsf{m}(\boldsymbol{\nu})$ and $\boldsymbol{\tau} = \mathsf{m}(\boldsymbol{\mu})$. The result follows by Theorem 5C.1. $\square$

Part (a) and Theorem 5B.2 imply that if $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ then $\mathsf{d}_{\mathsf{m}(\boldsymbol{\mu})\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) = q^{\operatorname{def}(\boldsymbol{\mu})} = \mathsf{d}_{\boldsymbol{\mu}\mathsf{m}(\boldsymbol{\mu})}^{\mathbb{K}\rhd}(q)$.

Example 5D.5. Suppose that Γ is a quiver of type $C_2^{(1)}$, $\Lambda = \Lambda_0$ and $n = 6$. Direct calculation shows that the graded decomposition numbers of $\mathsf{R}_6^{\Lambda_0}(\mathbb{K}[\underline{x}])$ are:

	(6)	(5,1)	(4,2)	(4,1²)	(3,2,1)
(6)	1				
(5,1)	q	1			
(4,2)	q	q^2	1		
(4,1²)	.	.	.	1	
(3²)	q^2	.	q	.	
(3,2,1)	.	.	.	.	1
(3,1³)	.	.	.	q	.
(2³)	q	.	q^2	.	.
(2²,1²)	q^2	q	q^3	.	.
(2,1⁴)	q^2	q^3	.	.	.
(1⁶)	q^3	.	.	.	.

	(1⁶)	(2,1⁴)	(2²,1²)	(3,1³)	(3,2,1)
(1⁶)	1				
(2,1⁴)	q	1			
(2²,1²)	q	q^2	1		
(2³)	q^2	.	q		
(3,1³)	.	.	.	1	
(3,2,1)	.	.	.	.	1
(3²)	q	.	q^2	.	.
(4,1²)	.	.	.	q	.
(4,2)	q^2	q	q^3	.	.
(5,1)	q^2	q^3	.	.	.
(6)	q^3	.	.	.	.

Graded decomposition matrix $\mathsf{D}_6^{\mathbb{K}[x]\lhd}$ Graded decomposition matrix $\mathsf{D}_6^{\mathbb{K}[x]\rhd}$

In particular, these decomposition matrices are independent of the characteristic and, in this example, the map m sends a partition to its conjugate, as defined in subsection 4A.

Remark 5D.6. If $\mathbb{K}$ is a field of characteristic zero, and if $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$ is an algebra of type $A_{e-1}^{(1)}$, then Proposition 5D.3 implies that if $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$ then $0 < \deg \mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\Delta}(q) \leq \operatorname{def}(\boldsymbol{\mu})$, with equality if and only if $\boldsymbol{\lambda} = \mathsf{m}(\boldsymbol{\mu})$; see [52, Corollary 3.6.7]. This result follows because in this case $\mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\Delta}(q) \in \delta_{\boldsymbol{\lambda\mu}} + q\mathbb{N}[q]$ by Corollary 6E.17 below. In positive characteristic, and in type $C_{e-1}^{(1)}$, this is no longer true. Even in type $A_{e-1}^{(1)}$, combining Proposition 5D.3 and [20, Corollary 5] (and [52, Example 3.7.13]), shows that the degrees of the graded decomposition numbers are not bounded by the defect in positive characteristic.

5E. **The sign isomorphism.** A *sign isomorphism* of the KLR algebras of type $A_{e-1}^{(1)}$ was introduced in [40, (3.14)]. This subsection generalises this map to include the quivers of type $C_{e-1}^{(1)}$ and it describes its effect on the Specht modules and simple modules of R_n^{Λ}. In type $A_{e-1}^{(1)}$, many of the results in this subsection are graded analogues of results in [27, §3].

Definition 5E.1. The *sign automorphism* of Γ is quiver automorphism $\varepsilon \colon \Gamma \longrightarrow \Gamma$ given by

$$\varepsilon(i) = \begin{cases} e - i \pmod{e} & \text{for type } A_{e-1}^{(1)}, \\ e - 1 - i & \text{for type } C_{e-1}^{(1)}, \end{cases}$$

for $i \in I$. If $\mathbf{i} = (i_1, \ldots, i_n) \in I^n$ let $\mathbf{i}^\varepsilon = (\varepsilon(i_1), \ldots, \varepsilon(i_n)) \in I^n$.

It is straightforward to check that $\mathsf{c}_{ij} = \mathsf{c}_{\varepsilon(i)\varepsilon(j)}$, for all $i, j \in I$, showing that ε is a quiver automorphism of Γ. The sign automorphism of Γ induces automorphisms of the lattices P^+ and Q^+, given by $\Lambda \mapsto \Lambda^\varepsilon$ and $\alpha \mapsto \alpha^\varepsilon$, that are uniquely determined by

$$\left(\alpha_i^\vee \mid \Lambda^\varepsilon\right) = \left(\alpha_{\varepsilon(i)}^\vee \mid \Lambda\right) \quad \text{and} \quad \left(\alpha_j^\vee \mid \alpha^\varepsilon\right) = \left(\alpha_{\varepsilon(j)}^\vee \mid \alpha\right), \qquad \text{for } i, j \in I,$$

respectively.

By definition, the algebra $\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ depends on the families polynomials $\mathbf{W}_I^x$ and $\mathbf{Q}_I^x$ from Notation 2C.7. Define polynomials $\mathbf{W}_I^{x,\varepsilon} = (W_i^{x,\varepsilon}(u))_{i \in I}$ and $\mathbf{Q}_I^{x,\varepsilon} = (Q_{ij}^{x,\varepsilon}(u,v))_{i,j \in I}$ by

$$W_i^{x,\varepsilon}(u) = W_{\varepsilon(i)}^x(-u) \quad \text{and} \quad Q_{ij}^{x,\varepsilon}(u,v) = Q_{\varepsilon(i)\varepsilon(j)}^x(-u,-v), \qquad \text{for } i,j \in I. \qquad (5E.2)$$

Set $^\varepsilon\mathsf{R}_\alpha^\Lambda = \mathsf{R}_{\alpha^\varepsilon}^{\Lambda^\varepsilon}(\mathbf{Q}_I^\varepsilon, \mathbf{W}_I^\varepsilon)$. If (c, r) is a (graded) content system for R_n^Λ then $(-\mathsf{c}, \varepsilon \circ \mathsf{r})$ is a graded content system with values in $\Bbbk[\underline{x}]$ for $^\varepsilon\mathsf{R}_\alpha^\Lambda$.

If $\boldsymbol{\rho} = (\kappa_1, \ldots, \kappa_\ell)$ is an ℓ-charge for Λ then $\boldsymbol{\rho}^\varepsilon = (-\kappa_\ell, \ldots, -\kappa_1)$ is the corresponding *signed charge*.

Proposition 5E.3. Let $\Lambda \in P^+$ and $\alpha \in Q^+$. Then there is a unique graded algebra isomorphism $\varepsilon \colon \mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}]) \longrightarrow {}^\varepsilon\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ such that

$$\varepsilon(1_\mathbf{i}) = 1_{\mathbf{i}^\varepsilon}, \qquad \varepsilon(\psi_k) = -\psi_k \qquad \text{and} \qquad \varepsilon(y_m) = -y_m,$$

for $\mathbf{i} \in I^n$, $1 \le k < n$ and $1 \le m \le n$.

Proof. Checking the relations in Definition 2C.2 shows that there is a well-defined surjective homomorphism isomorphism $\varepsilon \colon \mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}]) \longrightarrow {}^\varepsilon\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ of graded algebras. By symmetry, there is also a well-defined surjective graded algebra homomorphism $\varepsilon' \colon {}^\varepsilon\mathsf{R}_\alpha^\Lambda \longrightarrow \mathsf{R}_\alpha^\Lambda$. By definition, $\varepsilon \circ \varepsilon'$ and $\varepsilon' \circ \varepsilon$ are identity maps, so the result follows. (Hereafter, we abuse notation and use ε for both of these isomorphisms.) $\qquad\square$

The isomorphism $\varepsilon \colon \mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}]) \longrightarrow {}^\varepsilon\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ of Proposition 5E.3 is the *sign isomorphism*. This generalises the sign automorphism of the group algebra of the symmetric group, which corresponds to the special case when $\Lambda = \Lambda_0$ in type $A_{e-1}^{(1)}$ for $\mathscr{R}_n^\Lambda(\mathbb{K})$, when $\mathbb{K}$ is a field. By base change, Proposition 5E.3 induces isomorphisms $\mathsf{R}_\alpha^\Lambda(L) \xrightarrow{\sim} {}^\varepsilon\mathsf{R}_\alpha^\Lambda(L)$ for any $\Bbbk[\underline{x}]$-algebra L. Setting $\underline{x} = 0$ we obtain an analogous isomorphism $\varepsilon \colon \mathscr{R}_\alpha^\Lambda(\Bbbk) \longrightarrow {}^\varepsilon\mathscr{R}_\alpha^\Lambda(\Bbbk)$.

If M is an $^\varepsilon\mathsf{R}_\alpha^\Lambda$-module let M^ε be the ε-*twisted* $\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$-module that is equal to M as a $\Bbbk[\underline{x}]$-module and where the $\mathsf{R}_\alpha^\Lambda$-action is twisted by ε, so that $a \cdot m = \varepsilon(a)m$, for $a \in \mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ and $m \in M$. By Proposition 5E.3, this induces an equivalence of categories $\mathrm{Rep}\,{}^\varepsilon\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}]) \to \mathrm{Rep}\,\mathsf{R}_\alpha^\Lambda(\Bbbk[\underline{x}])$ given by $M \mapsto M^\varepsilon$. In the special case of the symmetric groups, this is the equivalence of categories induced by tensoring with the sign representation. This follows because if $\mathbb{K}$ is a field then there is an isomorphism $\mathscr{R}_n^{\Lambda_0}(\mathbb{K}) \cong \mathbb{K}\mathfrak{S}_n$ by the main result of [10] and in this case ε induces an auto-equivalence of $\mathrm{Rep}\,\mathscr{R}_n^{\Lambda_0}(\mathbb{K})$. More generally, ε induces an auto-equivalence of $\mathrm{Rep}\,\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}])$ whenever $\Lambda = \Lambda^\varepsilon$.

Most of our notation so far implicitly depends on Λ and sometimes α and $\boldsymbol{\rho}$. To avoid ambiguity, we decorate our notation with ε whenever it is applied to objects associated with the algebra ${}^{\varepsilon}\mathsf{R}_{\alpha}^{\Lambda}(\Bbbk[\underline{x}])$, and we continue to use our existing notation for the algebras $\mathsf{R}_{\alpha}^{\Lambda}(\Bbbk[\underline{x}])$. In particular, $S_{\boldsymbol{\lambda}}^{\triangle\varepsilon}$ and $D_{\boldsymbol{\mu}}^{\triangle\varepsilon}$ are the graded Specht and simple ${}^{\varepsilon}\mathsf{R}_{\alpha}^{\Lambda}(\Bbbk[\underline{x}])$–modules. The main results of this subsection explore the twisted modules $(S_{\boldsymbol{\lambda}}^{\triangle\varepsilon})^{\varepsilon}$ and $(D_{\boldsymbol{\mu}}^{\triangle\varepsilon})^{\varepsilon}$, for $\boldsymbol{\lambda} \in \mathcal{P}_{\alpha}^{\ell}$ and $\boldsymbol{\mu} \in \mathcal{K}_{\alpha}^{\triangle}$.

We need "sign adapted" combinatorics for the KLR algebras. As suggested by the terminology, in the representation theory of the symmetric groups this is given by conjugate partitions and tableaux, as defined in subsection 4A.

Extending the definition of the conjugate of an L-partition from subsection 4A, the *conjugate* of the node $A = (m, r, c)$ is the node $A' = (\ell - m + 1, c, r)$. In particular, if $\boldsymbol{\lambda} \in \mathcal{P}_{n}^{\ell}$ then its conjugate is $\boldsymbol{\lambda}' = \{A' \,|\, A \in \boldsymbol{\lambda}\}$ and the conjugate of $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ is the tableau $\mathsf{t}' \in \mathrm{Std}(\boldsymbol{\lambda}')$ given by $\mathsf{t}'(A) = \mathsf{t}(A')$, for $A \in \boldsymbol{\lambda}'$. If A is a node then $(A')' = A$, so conjugation is an involution on the sets of ℓ-partitions and standard tableaux.

A straightforward walk through the definitions reveals that the following identities hold.

Lemma 5E.4. Let $\boldsymbol{\lambda} \in \mathcal{P}_{\alpha}^{\ell}$, for $\alpha \in Q^{+}$. If $A \in \boldsymbol{\lambda}'$ then

$$d_{A}^{\triangleleft\varepsilon}(\boldsymbol{\lambda}') = d_{A'}^{\triangleright}(\boldsymbol{\lambda}), \qquad d_{A}^{\triangleright\varepsilon}(\boldsymbol{\lambda}') = d_{A'}^{\triangleleft}(\boldsymbol{\lambda}),$$

$$d_{i}^{\varepsilon}(\boldsymbol{\lambda}') = d_{\varepsilon(i)}(\boldsymbol{\lambda}) \qquad \text{and} \qquad \mathrm{def}^{\varepsilon}(\boldsymbol{\lambda}') = \mathrm{def}(\boldsymbol{\lambda}).$$

Moreover, if $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $\mathsf{r}(\mathsf{s}') = \mathsf{r}(\mathsf{s})^{\varepsilon}$, $\deg_{\varepsilon}^{\triangleleft}(\mathsf{s}') = \deg^{\triangleright}(\mathsf{s})$ and $\deg_{\varepsilon}^{\triangleright}(\mathsf{s}') = \deg^{\triangleleft}(\mathsf{s})$.

Proposition 5E.5. Suppose that $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_{\alpha}^{\ell}$. Then

$$\varepsilon(\psi_{\mathsf{st}}^{\triangleleft}) = \pm\psi_{\mathsf{s}'\mathsf{t}'}^{\triangleright\varepsilon} \qquad \text{and} \qquad \varepsilon(\psi_{\mathsf{st}}^{\triangleright}) = \pm\psi_{\mathsf{s}'\mathsf{t}'}^{\triangleleft\varepsilon}.$$

Proof. This is a straightforward exercise in the definitions. Observe that $\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleleft} = \mathsf{t}_{\boldsymbol{\lambda}'}^{\triangleright\varepsilon}$ and $\mathsf{t}_{\boldsymbol{\lambda}}^{\triangleright} = \mathsf{t}_{\boldsymbol{\lambda}'}^{\triangleleft\varepsilon}$. Consequently, if $\mathsf{u} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $d_{\mathsf{u}}^{\triangleleft\varepsilon} = d_{\mathsf{u}'}^{\triangleright}$ and $d_{\mathsf{u}}^{\triangleright\varepsilon} = d_{\mathsf{u}'}^{\triangleleft}$. By Definition 4A.5 and (5E.2), $y_{\boldsymbol{\lambda}'}^{\triangleleft\varepsilon} = \pm y_{\boldsymbol{\lambda}}^{\triangleright}$ and $y_{\boldsymbol{\lambda}'}^{\triangleright\varepsilon} = \pm y_{\boldsymbol{\lambda}}^{\triangleleft}$, implying the result. $\square$

For the Specht modules of the symmetric groups, James [29, Theorem 8.15] proved the famous result that $S^{\lambda'} \cong \mathrm{sgn} \otimes S^{\lambda}$, where S^{λ} is a Specht module for the symmetric group $\mathfrak{S}_{n}$ and sgn is its sign representation. This next result generalises James' theorem.

Corollary 5E.6. Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_{\alpha}^{\ell}$, for $\alpha \in Q^{+}$. Then $S_{\boldsymbol{\lambda}}^{\triangleleft} \cong (S_{\boldsymbol{\lambda}'}^{\triangleright\varepsilon})^{\varepsilon}$ and $S_{\boldsymbol{\lambda}}^{\triangleright} \cong (S_{\boldsymbol{\lambda}'}^{\triangleleft\varepsilon})^{\varepsilon}$.

Proof. By Proposition 5E.5, $(\mathsf{R}_{\alpha}^{\Lambda})^{\triangleleft\boldsymbol{\lambda}} \cong (({}^{\varepsilon}\mathsf{R}_{\alpha}^{\Lambda})^{\triangleright\boldsymbol{\lambda}'})^{\varepsilon}$ and $(\mathsf{R}_{\alpha}^{\Lambda})^{\triangleright\boldsymbol{\lambda}} \cong (({}^{\varepsilon}\mathsf{R}_{\alpha}^{\Lambda})^{\triangleleft\boldsymbol{\lambda}'})^{\varepsilon}$, implying the result. $\square$

This allows us to identify the twisted simple ${}^{\varepsilon}\mathsf{R}_{\alpha}^{\Lambda}$-modules as $\mathsf{R}_{\alpha}^{\Lambda}$-modules. The result says that these modules are isomorphic once you conjugate the ℓ-partitions and interchange the $\triangleleft$-simple modules and the $\triangleright$-simple modules. The simple modules are defined over the field $\Bbbk$.

Corollary 5E.7. Let $\boldsymbol{\mu} \in \mathcal{K}_{\alpha}^{\triangleleft}$ and $\boldsymbol{\nu} \in \mathcal{K}_{\alpha}^{\triangleright}$. Then $D_{\boldsymbol{\mu}}^{\triangleleft} \cong (D_{\boldsymbol{\mu}'}^{\triangleright\varepsilon})^{\varepsilon}$ and $D_{\boldsymbol{\nu}}^{\triangleright} \cong (D_{\boldsymbol{\nu}'}^{\triangleleft\varepsilon})^{\varepsilon}$.

Proof. Let $\mathrm{head}(M)$ be the head of M, which is its maximal semisimple quotient. Then, using Corollary 5E.6, $D_{\boldsymbol{\mu}}^{\triangleleft} \cong \mathrm{head}(S_{\boldsymbol{\mu}}^{\triangleleft}) \cong (\mathrm{head}\, S_{\boldsymbol{\mu}'}^{\triangleright\varepsilon})^{\varepsilon} \cong (D_{\boldsymbol{\mu}'}^{\triangleright\varepsilon})^{\varepsilon}$. The second isomorphism is proved in exactly the same way. $\square$

Recall from Definition 5D.1 that $\mathsf{m}\colon \mathcal{K}_{n}^{\triangleleft} \longrightarrow \mathcal{K}_{n}^{\triangleright}$ is the map given by $D_{\boldsymbol{\mu}}^{\triangleleft} \cong D_{\mathsf{m}(\boldsymbol{\mu})}^{\triangleright}$, for $\boldsymbol{\mu} \in \mathcal{K}_{n}^{\triangleleft}$. In the special case of the symmetric groups the next result says that the map $\boldsymbol{\mu} \mapsto \mathsf{m}(\boldsymbol{\mu})'$ is the Mullineux map.

Corollary 5E.8. Let $\boldsymbol{\mu} \in \mathcal{K}_\alpha^\lhd$. Then

$$D_{\boldsymbol{\mu}}^\lhd \cong (D_{\mathsf{m}(\boldsymbol{\mu})'}^{\lhd\varepsilon})^\varepsilon, \quad D_{\mathsf{m}(\boldsymbol{\mu})}^\rhd \cong (D_{\boldsymbol{\mu}'}^{\rhd\varepsilon})^\varepsilon, \quad Y_{\boldsymbol{\mu}}^\lhd \cong (Y_{\mathsf{m}(\boldsymbol{\mu})'}^{\lhd\varepsilon})^\varepsilon \quad \text{and} \quad Y_{\mathsf{m}(\boldsymbol{\mu})}^\rhd \cong (Y_{\boldsymbol{\mu}'}^{\rhd\varepsilon})^\varepsilon,$$

In particular, $\{D_{\boldsymbol{\mu}}^{\lhd\varepsilon} \mid \boldsymbol{\mu}' \in \mathcal{K}_n^\rhd\}$ and $\{D_{\boldsymbol{\nu}}^{\rhd\varepsilon} \mid \boldsymbol{\nu}' \in \mathcal{K}_n^\lhd\}$ are both complete sets of pairwise non-isomorphic self-dual irreducible graded ${}^\varepsilon\mathsf{R}_\alpha^\Lambda$-modules.

Proof. Using Corollary 5E.7, $D_{\boldsymbol{\mu}}^\lhd \cong D_{\mathsf{m}(\boldsymbol{\mu})}^\rhd \cong (D_{\mathsf{m}(\boldsymbol{\mu})'}^{\lhd\varepsilon})^\varepsilon$. The proof of the second isomorphism is similar and the remaining isomorphisms follow by the uniqueness of projective covers. $\qquad\square$

If M is an R_n^Λ-module then its *socle*, $\operatorname{soc} M$, is its maximal semisimple submodule. Dually, the *head* of M, $\operatorname{head} M$, is the maximal semisimple subquotient of M.

Corollary 5E.9. Let $\boldsymbol{\mu} \in \mathcal{K}_\alpha^\lhd$ and $\boldsymbol{\nu} \in \mathcal{K}_\alpha^\rhd$. Then

$$\operatorname{soc} S_{\boldsymbol{\mu}}^\lhd \cong q^{\operatorname{def}(\boldsymbol{\mu})} \left(D_{\mathsf{m}(\boldsymbol{\mu})'}^{\lhd\varepsilon}\right)^\varepsilon \qquad \text{and} \qquad \operatorname{soc} S_{\boldsymbol{\nu}}^\rhd \cong q^{\operatorname{def}(\boldsymbol{\nu})} \left(D_{\mathsf{m}(\boldsymbol{\nu})'}^{\rhd\varepsilon}\right)^\varepsilon.$$

Proof. Using Proposition 4F.9,

$$\operatorname{soc} S_{\boldsymbol{\mu}}^\lhd \cong \operatorname{soc}\left(q^{\operatorname{def}(\boldsymbol{\mu})} S_{\boldsymbol{\mu}}^{\rhd\circledast}\right) \cong q^{\operatorname{def}(\boldsymbol{\mu})} \operatorname{head}\left(S_{\boldsymbol{\mu}}^\rhd\right)^{\circledast} \cong q^{\operatorname{def}(\boldsymbol{\mu})} D_{\boldsymbol{\mu}}^\rhd \cong q^{\operatorname{def}(\boldsymbol{\mu})} \left(D_{\mathsf{m}(\boldsymbol{\mu})'}^{\lhd\varepsilon}\right)^\varepsilon.$$

where the last isomorphism follows from Corollary 5E.8. The second isomorphism is similar. $\qquad\square$

The last result in this subsection can be viewed as a generalisation of [43, Theorem 7.2].

Corollary 5E.10. Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $\boldsymbol{\mu} \in \mathcal{K}_n^\lhd$ and $\boldsymbol{\nu} \in \mathcal{K}_n^\rhd$. Then

$$\left[S_{\boldsymbol{\lambda}}^{\lhd\varepsilon} : D_{\mathsf{m}(\boldsymbol{\mu})'}^{\lhd\varepsilon}\right]_q = q^{\operatorname{def}\boldsymbol{\lambda}'} \overline{\left[S_{\boldsymbol{\lambda}'}^\lhd : D_{\boldsymbol{\mu}}^\lhd\right]}_q \qquad \text{and} \qquad \left[S_{\boldsymbol{\lambda}}^{\rhd\varepsilon} : D_{\boldsymbol{\mu}'}^{\rhd\varepsilon}\right]_q = q^{\operatorname{def}\boldsymbol{\lambda}'} \overline{\left[S_{\boldsymbol{\lambda}'}^\rhd : D_{\mathsf{m}(\boldsymbol{\mu})}^\rhd\right]}_q.$$

Proof. We prove only the second identity. Using Corollary 5E.6 and Corollary 5E.7,

$$\left[S_{\boldsymbol{\lambda}}^{\rhd\varepsilon} : D_{\boldsymbol{\mu}'}^{\rhd\varepsilon}\right]_q = \left[(S_{\boldsymbol{\lambda}}^{\rhd\varepsilon})^\varepsilon : (D_{\boldsymbol{\mu}'}^{\rhd\varepsilon})^\varepsilon\right]_q = \left[S_{\boldsymbol{\lambda}'}^\lhd : D_{\boldsymbol{\mu}}^\lhd\right]_q = q^{\operatorname{def}\boldsymbol{\lambda}'} \overline{\left[S_{\boldsymbol{\lambda}'}^\rhd : D_{\mathsf{m}(\boldsymbol{\mu})}^\rhd\right]}_q$$

where the last equality follows from Proposition 5D.3(a) and Lemma 5E.4. $\qquad\square$

6. Categorification

This section brings together all of our previous work to prove that the algebras $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ categorify the integrable highest weight modules of the corresponding Kac–Moody algebras, which is Theorem **B** from the introduction. As applications, we classify the simple $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules (Theorem **C**), and prove their modular branching rules (Theorem **D**). To do this we first use the algebras $\mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm])$ to prove the branching rules for the graded Specht modules of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$, which leads almost directly to our categorification theorem. We then use the representation theory of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ to describe the canonical bases of the highest weight modules, which gives us a way of studying the simple modules of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$.

Throughout this section we continue to assume that (c, r) is a (graded) content system with values in $\mathbb{k}[x]$ for a cyclotomic KLR algebra $\mathsf{R}_n^\Lambda(\mathbb{k}[x])$, and $\mathbb{K}$ is a field that is a $\mathbb{k}$-algebra so that $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ is a graded $\mathbb{K}[x]$-cellular algebra by Corollary 4F.3. In particular, as discussed in the last section, Corollary 4F.4 implies that the results in this section apply to the standard cyclotomic KLR algebras of types $A_{e-1}^{(1)}$, A_∞, $C_{e-1}^{(1)}$ and C_∞.

6A. **Branching rules.** This subsection proves analogues of the classical branching rules of the symmetric groups for the R_n^Λ-Specht modules. That is, we describe the modules obtained by inducing and restricting the graded Specht modules. The strategy is to first prove the branching rules for the algebras $R_n^\Lambda(\Bbbk[x^\pm])$ and then to use this result to prove the branching rules for $R_n^\Lambda(\Bbbk[x])$, after which the branching rules for R_n^Λ and $\mathscr{R}_n^\Lambda$ follow by specialisation. In the next subsection we use these results to show that R_n^Λ categorifies the integral highest weight modules of $U_q(\mathfrak{g}_\Gamma)$.

Before we can begin, we need to define the categories that we are going to work in. Fix $\alpha \in Q_n^+$. Let $\operatorname{Rep} R_\alpha^\Lambda(\Bbbk[x])$ be the category of finitely generated graded $R_\alpha^\Lambda(\Bbbk[x])$-modules, and similarly define $\operatorname{Rep} R_\alpha^\Lambda(\mathbb{K}[x])$. Let $\operatorname{Rep}_\mathbb{K} R_\alpha^\Lambda(\mathbb{K}[x])$ be the full subcategory of $\operatorname{Rep} R_\alpha^\Lambda(\mathbb{K}[x])$ consisting of graded $R_\alpha^\Lambda(\mathbb{K}[x])$-modules that are *finite dimensional* as $\mathbb{K}$-vector spaces. Let $\operatorname{Proj} R_\alpha^\Lambda(\Bbbk[x])$ and $\operatorname{Proj}_\mathbb{K} R_\alpha^\Lambda(\Bbbk[x])$ be the additive subcategories of graded projective modules in $\operatorname{Rep} R_\alpha^\Lambda(\Bbbk[x])$ and $\operatorname{Rep}_\mathbb{K} R_\alpha^\Lambda(\Bbbk[x])$, respectively. Similarly, let $\operatorname{Rep} \mathscr{R}_\alpha^\Lambda(\Bbbk)$ and $\operatorname{Proj} \mathscr{R}_\alpha^\Lambda(\Bbbk)$ $\operatorname{Rep} R_\alpha^\Lambda(\mathbb{K})$ and $\operatorname{Proj} R_\alpha^\Lambda(\mathbb{K})$ be the corresponding subcategories of graded $\mathscr{R}_\alpha^\Lambda(\Bbbk)$-module. and graded $R_\alpha^\Lambda(\mathbb{K})$-modules, respectively.

Set $\operatorname{Rep} R_n^\Lambda(\Bbbk[x]) = \bigoplus_{\alpha \in Q_n^+} \operatorname{Rep} R_\alpha^\Lambda(\Bbbk[x])$, and similarly for the other categories defined above.

Ultimately, we are most interested in the category $\operatorname{Rep}_\mathbb{K} R_n^\Lambda(\mathbb{K}[x])$, which is quite different to $\operatorname{Rep} R_n^\Lambda(\mathbb{K}[x])$. For example, the graded Specht module $S_{\boldsymbol{\lambda}}^\triangle(\mathbb{K}[x])$ does not belong to $\operatorname{Rep} R_n^\Lambda(\mathbb{K}[x])$ but it does belong to $\operatorname{Rep} R_n^\Lambda(\mathbb{K}[x])$. The categories $\operatorname{Rep}_\mathbb{K} R_n^\Lambda(\mathbb{K}[x])$ and $\operatorname{Rep} \mathscr{R}_n^\Lambda(\mathbb{K})$ are also not equivalent but they have isomorphic Grothendieck groups by the remarks after Theorem 5A.3.

Let $i \in I$ and $\alpha \in Q_n^+$. Set $\mathbf{1}_{\alpha,i} = \sum_{\mathbf{j} \in I^\alpha} \mathbf{1}_{\mathbf{j}i}$. Define *i-restriction* and *i-induction* functors:

$$E_i^\Lambda \colon \ \operatorname{Rep} R_{\alpha+\alpha_i}^\Lambda \Bbbk[x]) \longrightarrow \operatorname{Rep} R_\alpha^\Lambda(\Bbbk[x]); M \mapsto \mathbf{1}_{\alpha,i} R_{\alpha+\alpha_i}^\Lambda(\Bbbk[x]) \otimes_{R_{\alpha+\alpha_i}^\Lambda} M,$$

$$F_i^\Lambda \colon \ \operatorname{Rep} R_\alpha^\Lambda(\Bbbk[x]) \longrightarrow \operatorname{Rep} R_{\alpha+\alpha_i}^\Lambda(\Bbbk[x]); M \mapsto R_{\alpha+\alpha_i}^\Lambda(\Bbbk[x]) \mathbf{1}_{\alpha,i} \otimes_{R_\alpha^\Lambda(\Bbbk[x])} M.$$

Abusing notation, we also write $E_i^\Lambda \colon \operatorname{Rep} R_{n+1}^\Lambda \longrightarrow \operatorname{Rep} R_n^\Lambda$ and $F_i^\Lambda \colon \operatorname{Rep} R_n^\Lambda \longrightarrow \operatorname{Rep} R_{n+1}^\Lambda$ for the corresponding induced functors on these module categories. These functors can be defined as the direct sum of the functors defined above or they can be defined directly by replacing each occurrence of $\mathbf{1}_{\alpha,i}$ in the definitions above with $\mathbf{1}_{n,i} = \sum_{\alpha \in Q_n^+} \mathbf{1}_{\alpha,i}$. We further abuse notation and use E_i^Λ and F_i^Λ for the induced functors on all of the categories defined above.

Proposition 6A.1. Let $i \in I$. There is a (non-unital) embedding of graded algebras $\iota_{n,i} \colon R_n^\Lambda \hookrightarrow R_{n+1}^\Lambda$ such that

$$\mathbf{1}_{\mathbf{j}} \mapsto \mathbf{1}_{\mathbf{j}i}, \quad \psi_r \mathbf{1}_{\mathbf{j}} \mapsto \psi_r \mathbf{1}_{\mathbf{j}i} \quad \text{and} \quad y_m \mathbf{1}_{\mathbf{j}} \mapsto y_m \mathbf{1}_{\mathbf{j}i},$$

for $\mathbf{j} \in I^n$, $1 \le r < n$ and $1 \le m \le n$. Moreover, if $M \in \operatorname{Rep} R_{n+1}^\Lambda$ then $E_i^\Lambda(M) = \mathbf{1}_{n,i} M$ and if $N \in \operatorname{Rep} R_n^\Lambda$ then $F_i^\Lambda(N) = R_{n+1}^\Lambda \mathbf{1}_{n,i} N$, so E_i^Λ and F_i^Λ are exact functors.

Proof. The relations Definition 2C.2, together with Theorem 4F.1, imply that there is a unique non-unital algebra embedding $\iota_{\alpha,\alpha_i} \colon R_\alpha^\Lambda \hookrightarrow R_{\alpha+\alpha}^\Lambda$ such that

$$\mathbf{1}_{\mathbf{j}} \mapsto \mathbf{1}_{\mathbf{j}i}, \quad \psi_r \mathbf{1}_{\mathbf{j}} \mapsto \psi_r \mathbf{1}_{\mathbf{j}i} \quad \text{and} \quad y_m \mathbf{1}_{\mathbf{j}} \mapsto y_m \mathbf{1}_{\mathbf{j}i},$$

for $\mathbf{j} \in I^\alpha$, $1 \le r < n$ and $1 \le m \le n$. In particular, E_i^Λ is an exact functor. Kashiwara [35, Corollary 3.3] proves that F_i^Λ is exact. $\square$

The aim of this subsection is to describe the modules $E_i^\Lambda S_{\boldsymbol{\lambda}}^\triangleleft$ and $F_i^\Lambda S_{\boldsymbol{\lambda}}^\triangleright$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. We start with the easier case of restriction, following [53]. If $\triangle \in \{\triangleleft, \triangleright\}$ then Proposition 4A.17, $S_{\boldsymbol{\lambda}}^\triangle(\mathbb{K}[x^\pm])$ has an $f^\triangle$-basis and a $\psi^\triangle$-basis, for which the transition matrices are unitriangular. Note that $S_{\boldsymbol{\lambda}}^\triangleleft(\mathbb{K}[x^\pm]) \cong S_{\boldsymbol{\lambda}}^\triangleright(\mathbb{K}[x^\pm])$ in view of Corollary 3C.10 and Proposition 3C.2.

If $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ let $\mathsf{t}_\downarrow = \mathsf{t}_{\downarrow(n-1)}$. Let $\mathbb{K}'$ be the field of fractions of $\mathbb{k}$.

Lemma 6A.2. Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_{\alpha+\alpha_i}^\ell$. Then, as $\mathsf{R}_\alpha^\Lambda(\mathbb{K}'[x^\pm])$-modules,

$$E_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\triangleleft(\mathbb{K}'[x^\pm])\right) \cong \bigoplus_{B \in \operatorname{Rem}_i(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}-B}^\triangleleft(\mathbb{K}'[x^\pm]). \quad \text{and} \quad E_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\triangleright(\mathbb{K}'[x^\pm])\right) \cong \bigoplus_{B \in \operatorname{Rem}_i(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}-B}^\triangleright(\mathbb{K}'[x^\pm]).$$

Proof. This follows from Lemma 3E.1 but to understand how the Specht modules restrict over $\mathbb{k}[x]$ we need to describe the isomorphism explicitly. Let $\triangle \in \{\triangleleft, \triangleright\}$. By Theorem 4C.3, $E_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangle(\mathbb{K}'[x^\pm])$ has basis $\{f_{\mathsf{s}}^\triangle \mid \mathsf{s} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ and } \mathsf{r}_n(\mathsf{t}) = i\}$, which is in bijection with the set of tableaux $\bigcup_B \operatorname{Std}(\boldsymbol{\lambda}-B)$ where $B \in \operatorname{Rem}_i(\boldsymbol{\lambda})$. Define a $\mathbb{K}'[x^\pm]$-linear map

$$\theta \colon E_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\triangle(\mathbb{K}'[x^\pm])\right) \longrightarrow \bigoplus_{B \in \operatorname{Rem}_i(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}-B}^\triangle(\mathbb{K}'[x^\pm]); \quad f_{\mathsf{s}}^\triangle \mapsto f_{\mathsf{s}_\downarrow}^\triangle, \qquad \text{for } \mathsf{s} \in \operatorname{Std}(\boldsymbol{\lambda}).$$

$$(6A.3)$$

By Proposition 4A.10 this is an isomorphism of $\mathsf{R}_n^\Lambda(\mathbb{K}'[x^\pm])$-modules. □

There are no grading shifts in Lemma 6A.2 because $\mathbb{K}'[x^\pm] \cong q^d \mathbb{K}'[x^\pm]$ as a $\mathbb{Z}$-graded ring, for $d \in \mathbb{Z}$. The analogue of this result over $\mathbb{k}[x]$ requires grading shifts that are given by the integers $d_A^\triangleleft(\boldsymbol{\lambda})$ and $d_A^\triangleright(\boldsymbol{\lambda})$ from Definition 4D.3.

Proposition 6A.4. Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_{\alpha+\alpha_i}^\ell$ and let $A_1 > \cdots > A_z$ be the removable i-nodes of $\boldsymbol{\lambda}$. Then there exist $\mathsf{R}_\alpha^\Lambda(\mathbb{k}[x])$-module filtrations

$$E_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangleleft(\mathbb{k}[x])) = S_{\boldsymbol{\lambda},z}^\triangleleft(\mathbb{k}[x]) \supset S_{\boldsymbol{\lambda},z-1}^\triangleleft(\mathbb{k}[x]) \supset \cdots \supset S_{\boldsymbol{\lambda},2}^\triangleleft(\mathbb{k}[x]) \supset S_{\boldsymbol{\lambda},1}^\triangleleft(\mathbb{k}[x]) \supset 0$$

$$E_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangleright(\mathbb{k}[x])) = S_{\boldsymbol{\lambda},1}^\triangleright(\mathbb{k}[x]) \supset S_{\boldsymbol{\lambda},2}^\triangleright(\mathbb{k}[x]) \supset \cdots \supset S_{\boldsymbol{\lambda},z-1}^\triangleright(\mathbb{k}[x]) \supset S_{\boldsymbol{\lambda},z}^\triangleright(\mathbb{k}[x]) \supset 0$$

with

$$S_{\boldsymbol{\lambda},k}^\triangleleft(\mathbb{k}[x])/S_{\boldsymbol{\lambda},k-1}^\triangleleft(\mathbb{k}[x]) \cong q^{d_{A_k}^\triangleleft(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}-A_k}^\triangleleft(\mathbb{k}[x])$$

and

$$S_{\boldsymbol{\lambda},k}^\triangleright(\mathbb{k}[x])/S_{\boldsymbol{\lambda},k+1}^\triangleright(\mathbb{k}[x]) \cong q^{d_{A_k}^\triangleright(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}-A_k}^\triangleright(\mathbb{k}[x]), \quad \text{for } 1 \le k \le z.$$

Proof. Consider $E_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangleleft)$. As in Lemma 6A.2, the module $E_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangle(\mathbb{k}[x]))$ has basis

$$\{\psi_{\mathsf{s}}^\triangleleft \mid \mathsf{s} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ and } \mathsf{r}_n(\mathsf{s}) = i\} = \bigcup_{k=1}^z \{\psi_{\mathsf{s}}^\triangleleft \mid \mathsf{s}_\downarrow \in \operatorname{Std}(\boldsymbol{\lambda} - A_k)\}.$$

For $1 \le k \le z$, define $S_{\boldsymbol{\lambda},k}^\triangleleft(\mathbb{k}[x]) = \langle \psi_{\mathsf{s}}^\triangleleft \mid \mathsf{s}_\downarrow \in \operatorname{Std}(\boldsymbol{\lambda} - A_s) \text{ for } 1 \le s \le k \rangle$. Then $E_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangleleft(\mathbb{k}[x])) = S_{\boldsymbol{\lambda},z}^\triangleleft(\mathbb{k}[x]) \supset \cdots \supset S_{\boldsymbol{\lambda},1}^\triangleleft(\mathbb{k}[x]) \supset 0$ is an $\mathsf{R}_\alpha^\Lambda(\mathbb{k}[x])$-module filtration of $E_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangleleft(\mathbb{k}[x]))$ by Proposition 4C.5 and Corollary 4F.10. In view of Proposition 4A.17, it follows easily by induction on dominance that the $\mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm])$-module isomorphism θ defined in (6A.3) induces $\mathsf{R}_n^\Lambda(\mathbb{k}[x])$-module isomorphisms

$$\theta_k \colon S_{\boldsymbol{\lambda},k}^\triangleleft(\mathbb{k}[x])/S_{\boldsymbol{\lambda},k-1}^\triangleleft(\mathbb{k}[x]) \longrightarrow q^{d_A^\triangleleft(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}-A_k}^\triangleleft(\mathbb{k}[x]); \psi_{\mathsf{s}}^\triangleleft \mapsto \psi_{\mathsf{s}_\downarrow}^\triangleleft.$$

This completes the proof for $E_i^\Lambda(S_{\boldsymbol\lambda}^\triangleleft(\Bbbk[x]))$. The filtration of $E_i^\Lambda(S_{\boldsymbol\lambda}^\triangleright(\Bbbk[x]))$ can be constructed in exactly the same way. Alternatively, it can be deduced from the filtration of $E_i^\Lambda(S_{\boldsymbol\lambda}^\triangleleft(\Bbbk[x]))$ using Proposition 4F.9 and (4.32a). □

By base change, we obtain the corresponding result over any ring L that is a $\Bbbk[x]$-module.

Corollary 6A.5. Suppose that L is a $\Bbbk[x]$-module, $\boldsymbol\lambda \in \mathcal{P}_{\alpha+\alpha_i}^\ell$ and let $A_1 > \cdots > A_z$ be the removable i-nodes of $\boldsymbol\lambda$. Then there exist $\mathsf{R}_\alpha^\Lambda(L)$-module filtrations

$$E_i^\Lambda(S_{\boldsymbol\lambda}^\triangleleft(L)) = S_{\boldsymbol\lambda,z}^\triangleleft(L) \supset S_{\boldsymbol\lambda,z-1}^\triangleleft(L) \supset \cdots \supset S_{\boldsymbol\lambda,2}^\triangleleft(L) \supset S_{\boldsymbol\lambda,1}^\triangleleft(L) \supset 0$$
$$E_i^\Lambda(S_{\boldsymbol\lambda}^\triangleright(L)) = S_{\boldsymbol\lambda,1}^\triangleright(L) \supset S_{\boldsymbol\lambda,2}^\triangleright(L) \supset \cdots \supset S_{\boldsymbol\lambda,z-1}^\triangleright(L) \supset S_{\boldsymbol\lambda,z}^\triangleright(L) \supset 0$$

such that

$$S_{\boldsymbol\lambda,k}^\triangleleft(L)/S_{\boldsymbol\lambda,k-1}^\triangleleft(L) \cong q^{d_{A_k}^\triangleleft(\boldsymbol\lambda)} S_{\boldsymbol\lambda-A_k}^\triangleleft(L)$$

and

$$S_{\boldsymbol\lambda,k}^\triangleright(L)/S_{\boldsymbol\lambda,k+1}^\triangleright(L) \cong q^{d_{A_k}^\triangleright(\boldsymbol\lambda)} S_{\boldsymbol\lambda-A_k}^\triangleright(L), \quad for\, 1 \le k \le z.$$

In view of Proposition 2C.8, a special case of Corollary 6A.5 gives Specht filtrations of the Specht modules $S_{\boldsymbol\lambda}^\triangle(L)$ for the standard cyclotomic algebras $\mathscr{R}_n^\Lambda(L)$, for $\triangle \in \{\triangleleft, \triangleright\}$. In type $A_{e-1}^{(1)}$ this recovers [13, Theorem 4.11] when L is a field and [53, §5] for general L.

Next we consider the induced modules $F_i^\Lambda(S_{\boldsymbol\lambda}^\triangleleft)$ and $F_i^\Lambda(S_{\boldsymbol\lambda}^\triangleright)$ using ideas that go back to Ryom-Hansen [64]. First, some notation. Let $\triangle \in \{\triangleleft, \triangleright\}$ and suppose $A \in \mathrm{Add}_i(\boldsymbol\lambda)$. Let $\mathsf{t}_{\boldsymbol\lambda,A}^\triangle \in \mathrm{Std}(\boldsymbol\lambda+A)$ be the unique standard tableau such that $(\mathsf{t}_{\boldsymbol\lambda,A}^\triangle)_\downarrow = \mathsf{t}_{\boldsymbol\lambda}^\triangle$. Note that this forces $\mathsf{t}_{\boldsymbol\lambda,A}^\triangle(A) = n+1$.

The following example is suggestive of how the graded induction formulas are proved for the Specht modules are proved over $\Bbbk[x]$.

Example 6A.6. Let $\boldsymbol\lambda = (3^2, 2)$ and consider the quivers $A_2^{(1)}$ and $C_2^{(1)}$. The residues in $\boldsymbol\lambda$ are:
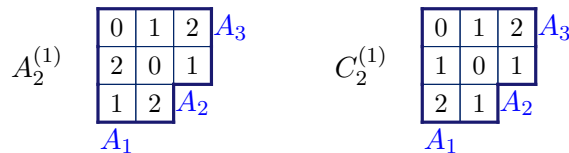
In type $A_2^{(1)}$, take $i = 0$ so that $\mathrm{Add}_i(\boldsymbol\lambda) = \{A_1, A_2, A_3\}$ where, as above, $A_1 = (4,1)$, $A_2 = (3,2)$ and $A_3 = (1,4)$. The standard tableaux $\mathsf{t}_{\boldsymbol\lambda,A_r}^\triangleleft$ and $\mathsf{t}_{\boldsymbol\lambda,A_r}^\triangleright$ are:
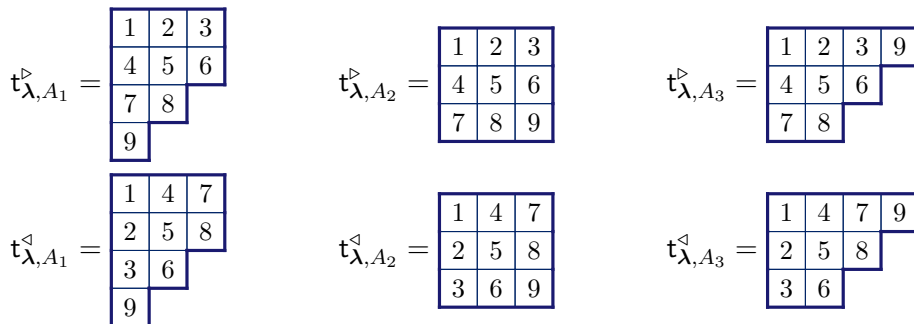
In type $C_2^{(1)}$, take $i = 1$ so that $\mathrm{Add}_i(\boldsymbol\lambda) = \{A_1, A_3\}$.

Lemma 6A.7. Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$, for $\alpha \in Q_n^+$. Then, as $\mathsf{R}_{\alpha+\alpha_i}^\Lambda(\mathbb{K}'[x^\pm])$-modules,

$$F_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\lhd(\mathbb{K}'[x^\pm])\right) \cong \bigoplus_{A \in \mathrm{Add}_i(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}+A}^\lhd(\mathbb{K}'[x^\pm]).$$

and

$$F_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\rhd(\mathbb{K}'[x^\pm])\right) \cong \bigoplus_{A \in \mathrm{Add}_i(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}+A}^\rhd(\mathbb{K}'[x^\pm]).$$

Proof. Let $\triangle \in \{\lhd, \rhd\}$. By Lemma 5A.4, $S_{\boldsymbol{\lambda}}^\triangle \cong \mathsf{R}_n^\Lambda(\mathbb{K}'[x^\pm])z_{\boldsymbol{\lambda}}^\triangle$. Hence, it is enough to describe

$$F_i^\Lambda\left(\mathsf{R}_n^\Lambda\left(\mathbb{K}'[x^\pm]\right) z_{\boldsymbol{\lambda}}^\triangle\right) = \mathsf{R}_{\alpha+\alpha_i}^\Lambda\left(\mathbb{K}'[x^\pm]\right) z_{\boldsymbol{\lambda}}^\triangle.$$

Let $\iota_{\alpha,i}\colon \mathsf{R}_\alpha^\Lambda(\mathbb{K}'[x^\pm]) \longrightarrow \mathsf{R}_{\alpha+\alpha_i}^\Lambda(\mathbb{K}'[x^\pm])$ be the embedding of Proposition 6A.1. Now $z_{\boldsymbol{\lambda}}^\triangle = \gamma_{\boldsymbol{\lambda}}^\Diamond F_{\mathsf{t}_{\boldsymbol{\lambda}}^\triangle}$ by Proposition 4E.9, so

$$
\begin{aligned}
\iota_{\alpha,i}(z_{\boldsymbol{\lambda}}^\triangle) = \gamma_{\boldsymbol{\lambda}}^\Diamond F_{\mathsf{t}_{\boldsymbol{\lambda}}^\triangle} 1_{\mathbf{i}_{\boldsymbol{\lambda}}^\triangle i} &= \gamma_{\boldsymbol{\lambda}}^\Diamond F_{\mathsf{t}_{\boldsymbol{\lambda}}^\triangle} \sum_{\mathsf{t} \in \mathrm{Std}(\mathbf{i}_{\boldsymbol{\lambda}}^\triangle i)} \frac{1}{\gamma_{\mathsf{t}}^\triangle} F_{\mathsf{t}} \\
&= \gamma_{\boldsymbol{\lambda}}^\Diamond \sum_{\substack{\mathsf{t} \in \mathrm{Std}(\mathbf{i}_{\boldsymbol{\lambda}}^\triangle i) \\ \mathsf{t}_\downarrow = \mathsf{t}_{\boldsymbol{\lambda}}^\triangle}} \frac{1}{\gamma_{\mathsf{t}}^\triangle} F_{\mathsf{t}} = \sum_{A \in \mathrm{Add}_i(\mathsf{t}_{\boldsymbol{\lambda}}^\triangle)} \frac{\gamma_{\boldsymbol{\lambda}}^\Diamond}{\gamma_{\mathsf{t}_{\boldsymbol{\lambda},A}^\triangle}^\triangle} F_{\mathsf{t}_{\boldsymbol{\lambda},A}^\triangle},
\end{aligned}
\tag{6A.8}
$$

where the second equality follows from Lemma 3B.4 and Proposition 4A.10. Note that the coefficients in the last equation are homogeneous and, hence, invertible in $\mathbb{K}'[x^\pm]$. Therefore, by Lemma 4A.7, the induced module $F_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangle(\mathbb{K}'[x^\pm]))$ is spanned by the elements $\{f_{\mathsf{st}_{\boldsymbol{\lambda},A}^\triangle}^\triangle \mid \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda}+A) \text{ and } A \in \mathrm{Add}_i(\boldsymbol{\lambda})\}$. Corollary 4A.11 now implies the result. $\qquad\square$

The second last line of the proof of Lemma 6A.7 is the reason why we are working over the polynomial rings $\mathbb{k}[x]$ and $\mathbb{K}'[x^\pm]$ in this subsection rather than over the multivariate polynomials rings $\mathbb{k}[\underline{x}]$ and $\mathbb{K}'[\underline{x}^\pm]$.

Proposition 6A.9. Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$ and let $A_1 > \cdots > A_z$ be the addable i-nodes of $\boldsymbol{\lambda}$. Then there exist $\mathsf{R}_{\alpha+\alpha_i}^\Lambda(\mathbb{k}[x])$-module filtrations

$$F_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\lhd(\mathbb{k}[x])\right) = S_{\boldsymbol{\lambda},1}^\lhd(\mathbb{k}[x]) \supset S_{\boldsymbol{\lambda},2}^\lhd(\mathbb{k}[x]) \supset \cdots \supset S_{\boldsymbol{\lambda},z-1}^\lhd(\mathbb{k}[x]) \supset S_{\boldsymbol{\lambda},z}^\lhd(\mathbb{k}[x]) \supset 0$$

$$F_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\rhd(\mathbb{k}[x])\right) = S_{\boldsymbol{\lambda},z}^\rhd(\mathbb{k}[x]) \supset S_{\boldsymbol{\lambda},z-1}^\lhd(\mathbb{k}[x]) \supset \cdots \supset S_{\boldsymbol{\lambda},2}^\rhd(\mathbb{k}[x]) \supset S_{\boldsymbol{\lambda},1}^\rhd(\mathbb{k}[x]) \supset 0$$

with $S_{\boldsymbol{\lambda},k}^\lhd(\mathbb{k}[x])/S_{\boldsymbol{\lambda},k+1}^\lhd(\mathbb{k}[x]) \cong q^{d_{A_k}^\lhd(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}+A_k}^\lhd(\mathbb{k}[x])$ and

$$S_{\boldsymbol{\lambda},k}^\rhd(\mathbb{k}[x])/S_{\boldsymbol{\lambda},k-1}^\rhd(\mathbb{k}[x]) \cong q^{d_{A_k}^\rhd(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}+A_k}^\rhd(\mathbb{k}[x]),$$

for $1 \le k \le z$.

Proof. If $\mathrm{Add}_i(\boldsymbol{\lambda}) = \emptyset$ then $F_i^\Lambda(S_{\boldsymbol{\lambda}}^\lhd(\mathbb{k}[x])) = 0$ by Lemma 6A.7, so we can assume $\mathrm{Add}_i(\boldsymbol{\lambda}) \ne \emptyset$. We only consider $F_i^\Lambda(S_{\boldsymbol{\lambda}}^\lhd(\mathbb{k}[x]))$. Set $Z_{\boldsymbol{\lambda}\uparrow}^\lhd = q^{-\mathrm{def}(\boldsymbol{\lambda})-\mathrm{deg}^\lhd(\mathsf{t}_{\boldsymbol{\lambda}}^\lhd)} \mathsf{R}_{\alpha+\alpha_i}^\Lambda \iota_{n,i}(z_{\boldsymbol{\lambda}}^\lhd)$. Then $F_i^\Lambda(S_{\boldsymbol{\lambda}}^\lhd(\mathbb{k}[x])) \cong Z_{\boldsymbol{\lambda}\uparrow}^\lhd$, by Lemma 5A.4, so, it is enough to show that $Z_{\boldsymbol{\lambda}\uparrow}^\lhd$ has the required filtration. To do this we first construct a basis for $Z_{\boldsymbol{\lambda}\uparrow}^\lhd$.

By Theorem 4F.1, $\iota_{n,i}(\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^\rhd \mathsf{t}_{\boldsymbol{\lambda}}^\lhd}^\rhd) = \sum_{(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_{n+1}^\ell)} a_{\mathsf{st}} \psi_{\mathsf{st}}^\rhd$, for $a_{\mathsf{st}} \in \mathbb{k}[x]$. Therefore, if $h \in \mathsf{R}_{n+1}^\Lambda(\mathbb{k}[x])$ then

$$\iota_{n,i}(hz_{\boldsymbol{\lambda}}^\lhd) = \sum_{(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_{n+1}^\ell)} a_{\mathsf{st}} h y_{\boldsymbol{\lambda}}^\lhd 1_{\mathbf{i}_{\boldsymbol{\lambda}}^\lhd i} \psi_{\mathsf{st}}^\rhd$$

By (6A.8), we may assume that $a_{\mathsf{st}} \neq 0$ only if $\mathsf{t} = \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k}$, for $1 \leq k \leq z$. Further, by Corollary 4F.10, if $\mathsf{s} \neq \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k}$ then $y^{\lhd}_{\boldsymbol{\lambda}} 1_{\mathbf{i}^{\lhd}_{\boldsymbol{\lambda}} i} \psi^{\rhd}_{\mathsf{st}}$ can be written as a linear combination of more dominant terms, so we can assume that $\mathsf{s} = \mathsf{t}$. That is,

$$\iota_{n,i}(h z^{\lhd}_{\boldsymbol{\lambda}}) = \sum_{k=1}^{z} a_k h y^{\lhd}_{\boldsymbol{\lambda}} 1_{\mathbf{i}^{\lhd}_{\boldsymbol{\lambda}} i} \psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k}}, \qquad \text{for } a_k \in \Bbbk[x].$$

By Corollary 4E.7, the product $\psi^{\lhd}_{\mathsf{uv}} \psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k}} \neq 0$ only if $\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k} \unrhd \mathsf{v}$. Since we also need $\mathsf{r}(\mathsf{v}) = \mathsf{r}(\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k})$, the term $\psi^{\lhd}_{\mathsf{uv}} \psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k}}$ is nonzero only if $\mathsf{v} = \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_l}$ for $1 \leq l \leq k$.

For $1 \leq k \leq z$ let $n_k = \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}+A_k}(A_k) \in \{1, \dots, n\}$, $\psi_{n..n_k} = \psi_n \dots \psi_{n_k}$ if $n_k < n+1$ and set $\psi_{n..n_k} = 1$ if $n_k = n+1$. Observe that $\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k} = \psi_{n..n_k} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}+A_k}$. Therefore, in $\mathsf{R}^{\Lambda}_{n+1}(\Bbbk[x])$,

$$y_{n+1}^{d^{\lhd}_{A_k}(\boldsymbol{\lambda})} \psi_{n..n_k} \iota_{n,i}\left(\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}}}\right) = y_{n+1}^{d^{\lhd}_{A_k}(\boldsymbol{\lambda})} \psi_{n..n_k} y^{\lhd}_{\boldsymbol{\lambda}} 1_{\mathbf{i}^{\lhd}_{\boldsymbol{\lambda}} i} = y^{\lhd}_{\boldsymbol{\lambda}+A_k} 1_{\mathbf{i}^{\lhd}_{\boldsymbol{\lambda}} i} \psi_{n..n_k} = \psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}+A_k} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k}}.$$

For $\mathsf{s} \in \mathrm{Std}(\mathcal{P}^{\ell}_{\boldsymbol{\lambda}+A_k})$ set $z^{\lhd}_{\mathsf{s}\uparrow} = \psi^{\lhd}_{\mathsf{st}^{\lhd}_{\boldsymbol{\lambda}, A_l}} \psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k}}$. Then we have shown that

$$y_{n+1}^{d^{\lhd}_{A_k}(\boldsymbol{\lambda})} \psi_{n..n_k} \iota_{n,i}\left(\psi^{\lhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}}}\right) \psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_l}} = \sum_{l=k}^{z} a_l \psi^{\lhd}_{\mathsf{st}^{\lhd}_{\boldsymbol{\lambda}, A_l}} \psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_l}} = a_k z^{\lhd}_{\mathsf{s}\uparrow},$$

where the equality follows from Corollary 4E.7. In particular, $a_k z^{\lhd}_{\mathsf{s}\uparrow} \in F^{\Lambda}_i S^{\lhd}_{\boldsymbol{\lambda}}$, whenever $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda} + A_k)$ and $1 \leq k \leq z$.

Let M be the free $\Bbbk[x]$-module spanned by $\{z^{\lhd}_{\mathsf{s}\uparrow} \mid \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda} + A_k) \text{ and } 1 \leq k \leq z\}$. We claim that $M = Z^{\lhd}_{\boldsymbol{\lambda}\uparrow} = F^{\Lambda}_i S^{\lhd}_{\boldsymbol{\lambda}}(\Bbbk)$, which is equivalent to claiming that $a_k \in \Bbbk^{\times}$, for $1 \leq k \leq z$. If x divides some a_k then the $\Bbbk'$-dimension of $Z^{\lhd}_{\boldsymbol{\lambda}\uparrow} \otimes_{\Bbbk[x]} \Bbbk'$ is strictly smaller than the $\Bbbk'[x^{\pm}]$-rank of $F^{\Lambda}_i S^{\lhd}_{\boldsymbol{\lambda}}(\Bbbk'[x^{\pm}])$ by Lemma 6A.7, which is a contradiction. Therefore, $a_k \in \Bbbk$ for $1 \leq k \leq z$. An easy argument using Nakayama's lemma (cf. [25, Proposition 4.6]), now shows that $M = Z^{\lhd}_{\boldsymbol{\lambda}\uparrow}$. In particular, this shows that $\{z^{\lhd}_{\mathsf{s}\uparrow} \mid \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda} + A_k) \text{ and } 1 \leq k \leq z\}$ is a basis of $Z^{\lhd}_{\boldsymbol{\lambda}\uparrow}$.

We can construct the promised filtration of $Z^{\lhd}_{\boldsymbol{\lambda}\uparrow}$. Define

$$S^{\lhd}_{\boldsymbol{\lambda}, k}(\Bbbk[x]) = \left\langle z^{\lhd}_{\mathsf{s}\uparrow} \,\middle|\, \mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda}+A_m) \text{ for } 1 \leq m \leq k \right\rangle, \qquad \text{for } 0 \leq k \leq z.$$

Then $Z^{\lhd}_{\boldsymbol{\lambda}\uparrow} = S^{\lhd}_{\boldsymbol{\lambda}, 1}(\Bbbk[x]) \supset S^{\lhd}_{\boldsymbol{\lambda}, 2}(\Bbbk[x]) \supset \cdots \supset S^{\lhd}_{\boldsymbol{\lambda}, z-1}(\Bbbk[x]) \supset S^{\lhd}_{\boldsymbol{\lambda}, z}(\Bbbk[x]) \supset 0$ and each $S^{\lhd}_{\boldsymbol{\lambda}, k}(\Bbbk[x])$ is an $\mathsf{R}^{\Lambda}_n(\Bbbk[x])$-submodule of $Z^{\lhd}_{\boldsymbol{\lambda}\uparrow}$ by Theorem 4F.1. By Corollary 4E.7, for $1 \leq k \leq z$ define homogeneous $\mathsf{R}^{\Lambda}_n(\Bbbk[x])$-module homomorphisms

$$\pi_k \colon q^{d^{\lhd}_{A_k}(\boldsymbol{\lambda})} S^{\lhd}_{\boldsymbol{\lambda}+A_k}(\Bbbk[x]) \longrightarrow S^{\lhd}_{\boldsymbol{\lambda}, k}(\Bbbk[x])/S^{\lhd}_{\boldsymbol{\lambda}, k-1}(\Bbbk[x])$$

by

$$\pi_k\left(\psi^{\lhd}_{\mathsf{st}^{\lhd}_{\boldsymbol{\lambda}, A_k}} + \left(\mathsf{R}^{\Lambda}_n(\Bbbk[\underline{x}])\right)^{\lhd(\boldsymbol{\lambda}+A_k)}\right) = \psi^{\lhd}_{\mathsf{st}^{\lhd}_{\boldsymbol{\lambda}, A_k}} \psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}+A_z} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_z}} + S^{\lhd}_{\boldsymbol{\lambda}, k-1}(\Bbbk[x]) = z^{\lhd}_{\mathsf{s}\uparrow} + S^{\lhd}_{\boldsymbol{\lambda}, k-1}(\Bbbk[x]),$$

for $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda}+A_k)$. By construction, these maps are surjective and hence bijective in view of Lemma 6A.7. To complete the proof we need to check that the map π_k is homogeneous of degree 0. Now, $\deg^{\lhd}(\mathsf{t}^{\rhd}_{\boldsymbol{\lambda}, A}) = \deg^{\lhd}(\mathsf{t}^{\rhd}_{\boldsymbol{\lambda}}) + d^{\rhd}_A(\boldsymbol{\lambda})$ and $\deg^{\rhd}(\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A}) = \deg^{\lhd}(\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}}) + d^{\lhd}_A(\boldsymbol{\lambda})$. Recalling the degree shifts in the definition of $Z^{\lhd}_{\boldsymbol{\lambda}\uparrow}$,

$$\deg \pi_k = \deg\left(\psi^{\rhd}_{\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}+A_z} \mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_z}}\right) + \deg^{\lhd}\left(\mathsf{t}^{\lhd}_{\boldsymbol{\lambda}, A_k}\right) - (\mathrm{def}(\boldsymbol{\lambda}) + \deg^{\rhd}(\mathsf{t}^{\rhd}_{\boldsymbol{\lambda}})) - d^{\lhd}_{A_k}(\boldsymbol{\lambda}) = 0,$$

where we have once again used Corollary 4D.6. $\qquad\square$

Corollary 6A.10. Suppose that L is a $\Bbbk[x]$-module, $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$ and let $A_1 > \cdots > A_z$ be the addable i-nodes of $\boldsymbol{\lambda}$. Then there exist $\mathsf{R}_{\alpha+\alpha_i}^\Lambda(L)$-module filtrations

$$F_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\lhd(L)\right) = S_{\boldsymbol{\lambda},1}^\lhd(L) \supset S_{\boldsymbol{\lambda},2}^\lhd(L) \supset \cdots \supset S_{\boldsymbol{\lambda},z-1}^\lhd(L) \supset S_{\boldsymbol{\lambda},z}^\lhd(L) \supset 0$$

$$F_i^\Lambda\left(S_{\boldsymbol{\lambda}}^\rhd(L)\right) = S_{\boldsymbol{\lambda},z}^\rhd(L) \supset S_{\boldsymbol{\lambda},z-1}^\lhd(L) \supset \cdots \supset S_{\boldsymbol{\lambda},2}^\rhd(L) \supset S_{\boldsymbol{\lambda},1}^\rhd(L) \supset 0$$

such that

$$S_{\boldsymbol{\lambda},k}^\lhd(L)/S_{\boldsymbol{\lambda},k+1}^\lhd(L) \cong q^{d_{A_k}^\lhd(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}+A_k}^\lhd(L)$$

and

$$S_{\boldsymbol{\lambda},k}^\rhd(L)/S_{\boldsymbol{\lambda},k-1}^\rhd(L) \cong q^{d_{A_k}^\rhd(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}+A_k}^\rhd(L), \quad \text{for } 1 \leq k \leq z.$$

In particular, this result includes filtrations of the induced Specht modules for the cyclotomic KLR algebras $\mathscr{R}_n^\Lambda(\Bbbk)$. In type $A_{e-1}^{(1)}$, this includes the main theorem of [25, Theorem 4.11], which describes Specht filtrations of the $\mathscr{R}_n^{\alpha+\alpha_i}(L)$-modules $F_i^\Lambda(S_{\boldsymbol{\lambda}}^\triangle(L))$ for $\triangle \in \{\lhd, \rhd\}$.

Finally, we note that we obtain the graded branching rules for the Specht modules of $\mathsf{R}_n^\Lambda(\Bbbk[x])$ by taking $L = \Bbbk$, or $L = \Bbbk[x]$, in Corollary 6A.5 and Corollary 6A.10.

6B. **Two dualities.** As in subsection 6A, we continue to assume that (c,r) is a content system with values in $\Bbbk[x]$ and let $\Bbbk$ be a field that is a $\Bbbk$-algebra. In this subsection we work in the categories $\operatorname{Rep}_\Bbbk \mathsf{R}_n^\Lambda(\Bbbk[x])$ and $\operatorname{Proj}_\Bbbk \mathsf{R}_n^\Lambda(\Bbbk[x])$ of graded $\mathsf{R}_n^\Lambda(\Bbbk[x])$-modules that are finite dimensional as $\Bbbk$-vector spaces.

Recall from (2C.4) that $\circledast$ defines a graded duality on $\mathsf{R}_n^\Lambda(\Bbbk[x])$-modules. Similarly, define $\#$ to be the graded functor given by

$$M^\# = \operatorname{HOM}_{\mathsf{R}_n^\Lambda(\Bbbk[x])}\left(M, \mathsf{R}_n^\Lambda(\Bbbk[x])\right), \qquad \text{for } M \in \operatorname{Rep}_\Bbbk \mathsf{R}_n^\Lambda(\Bbbk[x]), \tag{6B.1}$$

with the natural action of $\mathsf{R}_n^\Lambda(\Bbbk[x])$ on $M^\#$. Consider $\circledast$ and $\#$ as endofunctors of $\operatorname{Rep}_\Bbbk \mathsf{R}_n^\Lambda(\Bbbk[x])$ and $\operatorname{Proj}_\Bbbk \mathsf{R}_n^\Lambda(\Bbbk[x])$. As noted in [11, Remark 4.7], Theorem 4E.10 implies that these two functors agree up to shift.

Lemma 6B.2. Let $\alpha \in Q^+$. Then $\# \cong q^{2\operatorname{def}(\alpha)} \circ \circledast$ as endofunctors of $\operatorname{Rep}_\Bbbk \mathsf{R}_\alpha^\Lambda(\Bbbk[x])$.

Proof. By Theorem 4E.10, $\mathsf{R}_\alpha^\Lambda(\Bbbk[x]) \cong q^{2\operatorname{def}(\alpha)}(\mathsf{R}_\alpha^\Lambda(\Bbbk[x]))^\circledast$. If $M \in \operatorname{Rep}_\Bbbk \mathsf{R}_\alpha^\Lambda(\Bbbk[x])$ then

$$M^\# = \operatorname{HOM}_{\mathsf{R}_\alpha^\Lambda(\Bbbk[x])}\left(M, \mathsf{R}_\alpha^\Lambda(\Bbbk[x])\right) = \operatorname{HOM}_{\mathsf{R}_\alpha^\Lambda}\left(M, q^{2\operatorname{def}(\alpha)}\left(\mathsf{R}_\alpha^\Lambda(\Bbbk[x])\right)^\circledast\right)$$

$$\cong \operatorname{HOM}_{\mathsf{R}_\alpha^\Lambda}\left(M, q^{2\operatorname{def}(\alpha)} \operatorname{HOM}_{\Bbbk[x]}\left(\mathsf{R}_\alpha^\Lambda(\Bbbk[x]), \Bbbk[x]\right)\right)$$

$$\cong q^{2\operatorname{def}(\alpha)} \operatorname{HOM}_{\Bbbk[x]}\left(M \otimes_{\mathsf{R}_\alpha^\Lambda(\Bbbk[x])} \mathsf{R}_\alpha^\Lambda(\Bbbk[x]), \Bbbk[x]\right)$$

$$\cong q^{2\operatorname{def}(\alpha)} M^\circledast,$$

where the third isomorphism is the standard hom-tensor adjointness. All of these isomorphisms are functorial, so the lemma follows. $\square$

As M is a finite dimensional $\Bbbk$-vector space, $(M^\circledast)^\circledast \cong M$ for all $M \in \operatorname{Rep}_\Bbbk \mathsf{R}_n^\Lambda(\Bbbk[x])$. Hence, $(M^\#)^\# \cong M$ by Lemma 6B.2. Therefore, $\circledast$ and $\#$ define self-dual equivalences on the module categories $\operatorname{Rep}_\Bbbk \mathsf{R}_n^\Lambda(\Bbbk[x])$ and $\operatorname{Proj}_\Bbbk \mathsf{R}_n^\Lambda(\Bbbk[x])$.

Proposition 6B.3. Suppose that $i \in I$. Then there are functorial isomorphisms

$$\circledast \circ E_i^\Lambda \cong E_i^\Lambda \circ \circledast \colon \ \mathrm{Rep}_\mathbb{K} \, \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x]) \longrightarrow \mathrm{Rep}_\mathbb{K} \, \mathsf{R}_n^\Lambda(\mathbb{K}[x]),$$

$$\# \circ F_i^\Lambda \cong F_i^\Lambda \circ \# \colon \ \mathrm{Proj}_\mathbb{K} \, \mathsf{R}_n^\Lambda(\mathbb{K}[x]) \longrightarrow \mathrm{Proj}_\mathbb{K} \, \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x]).$$

Proof. The isomorphism $\circledast \circ E_i^\Lambda \cong E_i^\Lambda \circ \circledast$ is immediate from the definitions. For the second isomorphism, recall that if $P \in \mathrm{Proj}_\mathbb{K} \, \mathsf{R}_n^\Lambda(\mathbb{K}[x])$ then

$$\mathrm{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}(P, M) \cong \mathrm{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}\left(M, \mathsf{R}_n^\Lambda(\mathbb{K}[x])\right) \otimes_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])} M,$$

for any $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-module M. Now,

$$\left(\mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\mathbf{1}_{n,i}\right)^\# = \mathrm{HOM}_{\mathsf{R}_{n+1}^\Lambda \mathbb{K}[x]}\left(\mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\mathbf{1}_{n,1}, \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\right) \cong \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\mathbf{1}_{n,i},$$

where the last isomorphism follows because $\mathbf{1}_{n,i}^* = \mathbf{1}_{n,i}$. Therefore,

$$
\begin{aligned}
F_i^\Lambda\left(P^\#\right) &= \mathrm{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}\left(P, \mathsf{R}_n^\Lambda(\mathbb{K}[x])\right) \otimes_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])} \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\mathbf{1}_{n,i} \\
&\cong \mathrm{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}\left(P, \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\mathbf{1}_{n,i}\right) \\
&\cong \mathrm{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}\left(P, \mathrm{HOM}_{\mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])}\left(\mathbf{1}_{n,i}\mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x]), \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\right)\right) \\
&\cong \mathrm{HOM}_{\mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])}\left(P \otimes_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])} \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\mathbf{1}_{n,i}, \mathsf{R}_{n+1}^\Lambda(\mathbb{K}[x])\right) \\
&\cong \left(F_i^\Lambda P\right)^\#,
\end{aligned}
$$

where the second last isomorphism is the usual tensor-hom adjointness. $\qquad \square$

It follows from Proposition 6B.3 and Lemma 6B.2 that the functors $\circledast$ and F_i^Λ, and $\#$ and E_i^Λ, commute up to shift.

6C. **Grothendieck groups and the Cartan pairing.** We are now ready to prove the categorification theorems from the introduction, which will allow us to classify the simple $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules and prove our modular branching rules. As in the last two subsections we continue to assume that $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ is defined using a graded content system with values in $\Bbbk[x]$, and where the field $\mathbb{K}$ is a $\Bbbk$-algebra. In particular, this means that the graded branching rules for the Specht modules for $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ are given by the results in subsection 6A.

Recall that q is an indeterminate over $\mathbb{Z}$ and that $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Let $[\mathrm{Rep}_\mathbb{K} \, \mathsf{R}_n^\Lambda(\mathbb{K}[x])]$, $[\mathrm{Proj}_\mathbb{K} \, \mathsf{R}_n^\Lambda(\mathbb{K}[x])]$, be the Grothendieck groups of the corresponding categories of graded $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules, which are categories of finite dimensional $\mathbb{K}$-vector spaces. We consider each of these Grothendieck groups as $\mathcal{A}$-modules, where q acts by grading shift. If M is a module in one of these categories, let $[M]$ be its image in the corresponding Grothendieck group. Since q is the grading shift functor, which is exact, $[qM] = q[M]$.

Rather than considering the Grothendieck groups in isolation it is advantageous to consider all of them together. Define

$$\left[\mathrm{Rep}_\mathbb{K} \, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right] = \bigoplus_{n \geq 0} \left[\mathrm{Rep}_\mathbb{K} \, \mathsf{R}_n^\Lambda(\mathbb{K}[x])\right]$$

and

$$\left[\mathrm{Proj}_\mathbb{K} \, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right] = \bigoplus_{n \geq 0} \left[\mathrm{Proj}_\mathbb{K} \, \mathsf{R}_n^\Lambda(\mathbb{K}[x])\right].$$

These Grothendieck groups are independent of the choice of cellular basis in Theorem 4F.1, however, we give parallel categorification results for the two ψ-bases of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$.

By Proposition 6A.1, the induction and restriction functors F_i^Λ and E_i^Λ are exact and send projectives to projectives. Therefore they induce $\mathcal{A}$-linear automorphisms of the Grothendieck groups $[\operatorname{Rep}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ and $[\operatorname{Proj}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$, which are given by

$$F_i^\Lambda[M] = \left[F_i^\Lambda M\right] \qquad \text{and} \qquad E_i^\Lambda[M] = \left[E_i^\Lambda M\right]$$

for all modules M and $i \in I$.

Let M and N be free $\mathcal{A}$-modules. A *semilinear* map of $\mathcal{A}$-modules is a $\mathbb{Z}$-linear map $\theta \colon M \longrightarrow N$ such that $\theta(q^d m) = q^{-d}\theta(m) = \overline{q^d}\theta(m)$, for all $d \in \mathbb{Z}$ and $m \in M$. A *sesquilinear* map $f \colon M \times N \longrightarrow \mathcal{A}$ is a function that is semilinear in the first variable and linear in the second.

Let $\langle\ ,\ \rangle \colon [\operatorname{Proj}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])] \times [\operatorname{Rep}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])] \longrightarrow \mathcal{A}$ be the *Cartan pairing*, which is determined by

$$\langle[P], [M]\rangle = \delta_{mn} \dim_q \operatorname{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}(P, M), \tag{6C.1}$$

for $P \in \operatorname{Proj}_\mathbb{K} \mathsf{R}_m^\Lambda(\mathbb{K}[x])$ and $M \in \operatorname{Rep}_\mathbb{K} \mathsf{R}_n^\Lambda(\mathbb{K}[x])$. The Cartan pairing is sesquilinear because

$$\operatorname{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}\left(q^{-k}P, M\right) \cong \operatorname{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}\left(P, q^k M\right) \cong q^k \operatorname{HOM}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}(P, M),$$

for any $k \in \mathbb{Z}$. The Cartan pairing is characterised by either of the two properties:

$$\langle[Y_{\boldsymbol{\lambda}}^\lhd], [D_{\boldsymbol{\mu}}^\lhd]\rangle = \delta_{\boldsymbol{\lambda\mu}} \qquad \text{or} \qquad \langle[Y_{\boldsymbol{\nu}}^\rhd], [D_{\boldsymbol{\sigma}}^\rhd]\rangle = \delta_{\boldsymbol{\nu\sigma}} \tag{6C.2}$$

for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_n^\lhd$ or $\boldsymbol{\nu}, \boldsymbol{\sigma} \in \mathcal{K}_n^\rhd$, respectively.

Remark 6C.3. By the remarks after Theorem 5A.3, as abelian groups,

$$\left[\operatorname{Rep}_\mathbb{K} \mathsf{R}_n^\Lambda(\mathbb{K}[x])\right] \cong \left[\operatorname{Rep} \mathscr{R}_n^\Lambda(\mathbb{K})\right] \qquad \text{and} \qquad \left[\operatorname{Proj}_\mathbb{K} \mathsf{R}_n^\Lambda(\mathbb{K}[x])\right] \cong \left[\operatorname{Proj} \mathscr{R}_n^\Lambda(\mathbb{K})\right].$$

In what follows, we could work with the Grothendieck groups

$$\left[\operatorname{Rep} \mathscr{R}_n^\Lambda(\mathbb{K})\right] \qquad \text{and} \qquad \left[\operatorname{Proj} \mathscr{R}_n^\Lambda(\mathbb{K})\right].$$

6D. Fock spaces. This subsection proves that $[\operatorname{Proj}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ and $[\operatorname{Rep}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ categorify the integral form and its dual, respectively, of an irreducible integrable highest weight module of the quantised Kac–Moody algebra $U_q(\mathfrak{g}_\Gamma)$. We start by recalling the results and definitions that we need from the Kac–Moody universe. The arguments in this subsection are mostly standard, and follow (and correct) [52]. Our approach is similar to [11] except that we use the representation theory of the KLR algebras to construct the canonical bases, rather than vice versa. What is non-standard is that these arguments apply simultaneously in types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$.

Recall $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Set $\mathbb{A} = \mathbb{Q}(q)$. For $i \in I$ and $k \in \mathbb{Z}$ let $[k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1})$, where $q_i = q^{\mathsf{d}_i}$. If $k > 0$ set $[k]_i! = [1]_i[2]_i \ldots [k]_i$. For non-commuting indeterminates u and v and $i \in I$ set

$$\left(\operatorname{ad}_{q^i} u\right)^c (v) = \sum_{d=0}^c (-1)^d \frac{[c]_i!}{[c-d]_i![k]_i!} u^{c-d} v u^d.$$

Definition 6D.1. The *quantum group* $U_q(\mathfrak{g}_\Gamma)$ is the $\mathbb{A}$-algebra with generators E_i, F_i, $K_i^\pm$, for $i \in I$, and relations:

$$K_i K_j = K_j K_i, K_i K_i^{-1} = 1, [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$K_i E_j K_i^{-1} = q^{\mathsf{c}_{ij}} E_j, K_i F_j K_i^{-1} = q^{-\mathsf{c}_{ij}} F_j,$$

$$(\mathrm{ad}_{q^i} E_i)^{1-\mathsf{c}_{ij}}(E_j) = 0 = (\mathrm{ad}_{q^i} F_i)^{1-\mathsf{c}_{ij}}(F_j), \qquad \text{for } i \neq j.$$

The quantum group $U_q(\mathfrak{g}_\Gamma)$ is a Hopf algebra with coproduct determined by

$$\Delta(K_i) = K_i \otimes K_i, \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \quad \text{and} \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

for $i \in I$.

We will only need basic facts about highest weight theory and canonical bases for $U_q(\mathfrak{g}_\Gamma)$. Detailed accounts of the representation theory of $\mathfrak{g}_\Gamma$ and $U_q(\mathfrak{g}_\Gamma)$ can be found in [3, 30, 47].

Definition 6D.2. Let $\Lambda \in P^+$. The *combinatorial Fock spaces* $\mathscr{F}_{\mathcal{A}}^{\Lambda \lhd}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda \rhd}$ are the free $\mathcal{A}$-modules with basis the sets of symbols $\{\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} \mid \boldsymbol{\lambda} \in \mathcal{P}_{\bullet}^{\ell}\}$ and $\{\mathsf{s}_{\boldsymbol{\lambda}}^{\rhd} \mid \boldsymbol{\lambda} \in \mathcal{P}_{\bullet}^{\ell}\}$, respectively.
   Set $\mathscr{F}_{\mathbb{A}}^{\Lambda \lhd} = \mathbb{A} \otimes_{\mathcal{A}} \mathscr{F}_{\mathcal{A}}^{\Lambda \lhd}$ and $\mathscr{F}_{\mathbb{A}}^{\Lambda \rhd} = \mathbb{A} \otimes_{\mathcal{A}} \mathscr{F}_{\mathcal{A}}^{\Lambda \rhd}$.

By definition, $\mathscr{F}_{\mathbb{A}}^{\Lambda \lhd}$ and $\mathscr{F}_{\mathbb{A}}^{\Lambda \rhd}$ are infinite dimensional $\mathbb{A}$-vector spaces. For $\triangle \in \{\lhd, \rhd\}$, identify $\mathsf{s}_{\boldsymbol{\lambda}}^{\triangle}$ with $1_{\mathbb{A}} \otimes_{\mathcal{A}} \mathsf{s}_{\boldsymbol{\lambda}}^{\triangle}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$. Then $\{\mathsf{s}_{\boldsymbol{\lambda}}^{\triangle} \mid \boldsymbol{\lambda} \in \mathcal{P}_{\bullet}^{\ell}\}$ is an $\mathbb{A}$-basis $\mathscr{F}_{\mathcal{A}}^{\Lambda \triangle}$.
   Let $\mathbf{0}_\ell = (0 | \ldots | 0) \in \mathcal{P}_{\bullet}^{\ell}$ be the empty ℓ-partition. Recall the integers $d_A^{\lhd}(\boldsymbol{\lambda})$, $d_A^{\rhd}(\boldsymbol{\lambda})$, and $d_i(\boldsymbol{\lambda})$ from Definition 4D.3. Note that these definitions depend on $(\Lambda, \boldsymbol{\rho})$.

Theorem 6D.3 (Hayashi [23], Misra-Miwa [58], Premat [61]). Let $\Lambda \in P^+$.

(a) The Fock space $\mathscr{F}_{\mathcal{A}}^{\Lambda \lhd}$ is an integrable $U_q(\mathfrak{g}_\Gamma)$-module with $U_q(\mathfrak{g}_\Gamma)$-action determined by

$$E_i \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \sum_{B \in \mathrm{Rem}_i(\boldsymbol{\lambda})} q^{-d_B^{\rhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda} - B}^{\lhd}, \quad F_i \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \sum_{A \in \mathrm{Add}_i(\boldsymbol{\lambda})} q^{d_A^{\lhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda} + A}^{\lhd},$$

and $K_i \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = q^{-d_i(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}$, for $i \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$.

(b) The Fock space $\mathscr{F}_{\mathcal{A}}^{\Lambda \rhd}$ is an integrable $U_q(\mathfrak{g}_\Gamma)$-module with $U_q(\mathfrak{g}_\Gamma)$-action determined by

$$E_i \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\rhd} = \sum_{B \in \mathrm{Rem}_i(\boldsymbol{\lambda})} q^{-d_B^{\lhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda} - B}^{\rhd}, \quad F_i \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\rhd} = \sum_{A \in \mathrm{Add}_i(\boldsymbol{\lambda})} q^{d_A^{\rhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda} + A}^{\rhd},$$

and $K_i \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\rhd} = q^{-d_i(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}}^{\rhd}$, for $i \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$.

Proof. To prove (a) and (b) it is enough to verify that these actions respect the relations of $U_q(\mathfrak{g}_\Gamma)$. Recall the sign automorphism of subsection 5E. In particular, by Lemma 5E.4, $d_A^{\lhd}(\boldsymbol{\lambda}) = d_{A'}^{\rhd \varepsilon}(\boldsymbol{\lambda}')$, where if $A \in \mathrm{Add}(\boldsymbol{\lambda}) \cup \mathrm{Rem}(\boldsymbol{\lambda})$ then $d_A^{\lhd}(\boldsymbol{\lambda})$ is computed with respect to $(\Lambda, \boldsymbol{\rho})$ and $d_{A'}^{\rhd \varepsilon}(\boldsymbol{\lambda})$ is computed with respect to $(\Lambda^\varepsilon, \boldsymbol{\rho}^\varepsilon)$. Hence, parts (a) and (b) are equivalent and it suffices to prove (b).

If Γ is a quiver of type $A_{e-1}^{(1)}$ then (b) is due to Hayashi [23] in level 1, with the result in higher levels following by applying the coproduct, as was observed by Misra and Miwa [58]. For quivers of type $C_{e-1}^{(1)}$, this was proved by Premat [61, Theorem 3.1] in level 1 (see also Kim and Shin [37]), with the result in higher levels again following by applying the coproduct, as noted already in [8, §1]. $\qquad \square$

Theorem 6D.3 does not give the $U_q(\mathfrak{g}_\Gamma)$-actions on the Fock spaces that we want because this action does not commute with the bar involution on $L(\Lambda)$, which is introduced in subsection 6E below. Let $\tau \colon U_q(\mathfrak{g}_\Gamma) \longrightarrow U_q(\mathfrak{g}_\Gamma)$ be anti-linear anti-automorphism given by

$$\tau(K_i) = K_i^{-1}, \quad \tau(E_i) = q^{\mathsf{d}_i} F_i K_i^{-1} \quad \text{and} \quad \tau(F_i) = q^{-\mathsf{d}_i} K_i E_i \qquad \text{for } i \in I.$$

This map is not an involution but it is invertible. Twisting the $U_q(\mathfrak{g}_\Gamma)$-action from Theorem 6D.3 by τ gives the $U_q(\mathfrak{g}_\Gamma)$-action on the Fock space that we need.

Corollary 6D.4. Suppose that $\Lambda \in P^+$.
  (a) The Fock space $\mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ is an integrable $U_q(\mathfrak{g}_\Gamma)$-module with $U_q(\mathfrak{g}_\Gamma)$-action determined by

$$E_i \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \sum_{B \in \operatorname{Rem}_i(\boldsymbol{\lambda})} q^{d_B^{\lhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}-B}^{\lhd}, \quad F_i \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \sum_{A \in \operatorname{Add}_i(\boldsymbol{\lambda})} q^{-d_A^{\rhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}+A}^{\lhd},$$

  and $K_i \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = q^{d_i(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}$, for $i \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$.
  (b) The Fock space $\mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$ is an integrable $U_q(\mathfrak{g}_\Gamma)$-module with $U_q(\mathfrak{g}_\Gamma)$-action determined by

$$E_i \mathsf{s}_{\boldsymbol{\lambda}}^{\rhd} = \sum_{B \in \operatorname{Rem}_i(\boldsymbol{\lambda})} q^{d_B^{\rhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}-B}^{\rhd}, \quad F_i \mathsf{s}_{\boldsymbol{\lambda}}^{\rhd} = \sum_{A \in \operatorname{Add}_i(\boldsymbol{\lambda})} q^{-d_A^{\lhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}+A}^{\rhd},$$

  and $K_i \mathsf{s}_{\boldsymbol{\lambda}}^{\rhd} = q^{d_i(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}}^{\rhd}$, for $i \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$.

Proof. We consider only (a) and leave part (b) to the reader since this is similar. Using Theorem 6D.3, and the fact that τ is an anti-isomorphism of $U_q(\mathfrak{g}_\Gamma)$, we can define a new action of $U_q(\mathfrak{g}_\Gamma)$ on $\mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ by $E_i \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \tau(F_i) \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}$, $F_i \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \tau(E_i) \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}$ and $K_i \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \tau(K_i) \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}$, for $i \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Therefore,

$$E_i \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \tau(F_i) \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = q^{-\mathsf{d}_i} K_i E_i \cdot \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} = \sum_{B \in \operatorname{Rem}_i(\boldsymbol{\lambda})} q^{\mathsf{d}_i + d_i(\boldsymbol{\lambda}) - d_B^{\rhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}-B}^{\lhd}$$

$$= \sum_{B \in \operatorname{Rem}_i(\boldsymbol{\lambda})} q^{d_B^{\lhd}(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}-B}^{\lhd},$$

where the last equality follows from (4.32a). The other identities are similar. $\square$

In what follows we always use the $U_q(\mathfrak{g}_\Gamma)$-action on the Fock spaces $\mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$ from Corollary 6D.4. We work with both Fock spaces because they are closely intertwined and by using both Fock spaces we will be able to determine the labelling of the simple $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules and the map m from Definition 5D.1. As our notation suggests, the Fock spaces $\mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$ can be naturally associated with the $\psi^\lhd$ and $\psi^\rhd$-bases of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$, respectively. To make this connection precise we need a little more notation.

A vector v in a $U_q(\mathfrak{g}_\Gamma)$-module has *weight* $\operatorname{wt}(v) = \theta$ if $K_i v = q^{(\theta|\alpha_i)} v$, for all $i \in I$. Corollary 6D.4, and (4.32b), imply that if $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$ then

$$\operatorname{wt}(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}) = \Lambda - \alpha = \operatorname{wt}(\mathsf{s}_{\boldsymbol{\lambda}}^{\rhd}), \qquad \text{for all } \boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell. \tag{6D.5}$$

In particular, $\mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$ are both integrable highest weight modules for $U_q(\mathfrak{g}_\Gamma)$ and $\mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\lhd}$ and $\mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\rhd}$ are highest weight vectors of weight Λ.

Let $L(\Lambda)_{\mathbb{A}}$ be the irreducible integrable highest weight module for $U_q(\mathfrak{g}_\Gamma)$ with highest weight Λ. Then $L(\Lambda)_{\mathbb{A}} = U_q(\mathfrak{g}_\Gamma) v_\Lambda$, where v_Λ is a highest weight vector of weight Λ.

Corollary 6D.6. Let $\Lambda \in P^+$. Then $U_q(\mathfrak{g}_\Gamma) \mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\lhd} \cong L(\Lambda)_{\mathbb{A}} \cong U_q(\mathfrak{g}_\Gamma) \mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\rhd}$ as $U_q(\mathfrak{g}_\Gamma)$-modules.

Proof. By Corollary 6D.4 and (6D.5), the vectors $\mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\lhd} \in \mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ and $\mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\rhd} \in \mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$ are both highest weight vectors of weight Λ. Therefore, $U_q(\mathfrak{g}_\Gamma)\mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\lhd} \cong L(\Lambda)_{\mathbb{A}} \cong U_q(\mathfrak{g}_\Gamma)\mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\rhd}$ required. □

To make use of this result, recall from subsection 6C that $\left[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ and $\left[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ are the direct sums of Grothendieck groups of graded $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules and graded projective $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules, respectively, for $n \geq 0$. In particular, $\left[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ and $\left[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ are free an $\mathcal{A}$-modules.

Let $\mathcal{P}_\bullet^\ell = \bigcup_{n \geq 0} \mathcal{P}_n^\ell$, $\mathcal{K}_\bullet^\lhd = \bigcup_{n \geq 0} \mathcal{K}_n^\lhd$ and $\mathcal{K}_\bullet^\rhd = \bigcup_{n \geq 0} \mathcal{K}_n^\rhd$. By Theorem 5A.3 and Theorem 5B.2, $\left[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ comes equipped with four distinguished bases:

$$\{[D_{\boldsymbol{\mu}}^\lhd] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^\lhd\}, \quad \{[S_{\boldsymbol{\mu}}^\lhd] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^\lhd\} \quad \{[D_{\boldsymbol{\nu}}^\rhd] \mid \boldsymbol{\nu} \in \mathcal{K}_\bullet^\rhd\}, \quad \text{and} \quad \{[S_{\boldsymbol{\nu}}^\rhd] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^\rhd\} \quad (6\mathrm{D}.7)$$

Here, $D_{\boldsymbol{\mu}}^\lhd = D_{\boldsymbol{\mu}}^\lhd(\mathbb{K})$, $S_{\boldsymbol{\lambda}}^\lhd = S_{\boldsymbol{\lambda}}^\lhd(\mathbb{K})$, $D_{\boldsymbol{\nu}}^\rhd = D_{\boldsymbol{\nu}}^\rhd(\mathbb{K})$ and $S_{\boldsymbol{\nu}}^\rhd = S_{\boldsymbol{\nu}}^\rhd(\mathbb{K})$ are finite dimensional $\mathbb{K}$-modules. In contrast, the projective Grothendieck group $\left[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ has only two natural bases:

$$\{[Y_{\boldsymbol{\mu}}^\lhd] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^\lhd\} \qquad \text{and} \qquad \{[Y_{\boldsymbol{\nu}}^\rhd] \mid \boldsymbol{\nu} \in \mathcal{K}_\bullet^\rhd\}, \qquad (6\mathrm{D}.8)$$

where, as in subsection 5B, $Y_{\boldsymbol{\mu}}^\lhd = Y_{\boldsymbol{\mu}}^\lhd(\mathbb{K})$ and $Y_{\boldsymbol{\nu}}^\rhd = Y_{\boldsymbol{\nu}}^\rhd(\mathbb{K})$ are the projective covers of $D_{\boldsymbol{\mu}}^\lhd$ and $D_{\boldsymbol{\nu}}^\rhd$, respectively. Define elements $\{\mathsf{y}_{\boldsymbol{\mu}}^\lhd \mid \boldsymbol{\mu} \in \mathcal{K}_n^\lhd\}$ and $\{\mathsf{y}_{\boldsymbol{\mu}}^\rhd \mid \boldsymbol{\nu} \in \mathcal{K}_n^\rhd\}$ of $\mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$, respectively, by setting

$$\mathsf{y}_{\boldsymbol{\mu}}^\lhd = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n^\ell} \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q)\mathsf{s}_{\boldsymbol{\lambda}}^\lhd \qquad \text{and} \qquad \mathsf{y}_{\boldsymbol{\nu}}^\rhd = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n^\ell} \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q)\mathsf{s}_{\boldsymbol{\lambda}}^\rhd. \qquad (6\mathrm{D}.9)$$

Set

$$\left[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]_{\mathbb{A}} = \mathbb{A} \otimes_{\mathcal{A}} \left[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$$

and

$$\left[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]_{\mathbb{A}} = \mathbb{A} \otimes_{\mathcal{A}} \left[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right].$$

Proposition 6D.10. Suppose that $\Lambda \in P^+$. Identify E_i and E_i^Λ, and F_i and $F_i^\Lambda \circ q^{\mathsf{d}_i} K_i^{-1}$, for $i \in I$. Then there are $U_q(\mathfrak{g}_\Gamma)$-module embeddings

$$\mathsf{d}_T^\lhd \colon \left[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]_{\mathbb{A}} \longrightarrow \mathscr{F}_{\mathbb{A}}^{\Lambda\lhd}; [Y_{\boldsymbol{\mu}}^\lhd] \mapsto \mathsf{y}_{\boldsymbol{\mu}}^\lhd \quad \mathsf{d}_T^\rhd \colon \left[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]_{\mathbb{A}} \longrightarrow \mathscr{F}_{\mathbb{A}}^{\Lambda\rhd}; [Y_{\boldsymbol{\nu}}^\rhd] \mapsto \mathsf{y}_{\boldsymbol{\nu}}^\rhd$$

and $U_q(\mathfrak{g}_\Gamma)$-module surjections

$$\mathsf{d}^\lhd \colon \mathscr{F}_{\mathbb{A}}^{\Lambda\lhd} \longrightarrow \left[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]_{\mathbb{A}}; \mathsf{s}_{\boldsymbol{\lambda}}^\lhd \mapsto [S_{\boldsymbol{\lambda}}^\lhd] \qquad \mathsf{d}^\rhd \colon \mathscr{F}_{\mathbb{A}}^{\Lambda\rhd} \longrightarrow \left[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]_{\mathbb{A}}; \mathsf{s}_{\boldsymbol{\lambda}}^\rhd \mapsto [S_{\boldsymbol{\lambda}}^\rhd]$$

Consequently, $[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]_{\mathbb{A}} \cong L(\Lambda) \cong [\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]_{\mathbb{A}}$ as $U_q(\mathfrak{g}_\Gamma)$-modules.

Proof. Let $\{\triangle, \triangledown\} = \{\lhd, \rhd\}$. By Theorem 5B.2 and Proposition 5B.3, there are well-defined $\mathbb{A}$-linear maps $\mathsf{d}_T^\triangle$ and $\mathsf{d}^\triangle$, with $\mathsf{d}_T^\triangle$ injective and $\mathsf{d}^\triangle$ surjective. It remains to check that these maps are homomorphisms of $U_q(\mathfrak{g}_\Gamma)$-modules.

Let $i \in I$. By Proposition 6A.1, the functors E_i^Λ and F_i^Λ are exact, and send projective modules to projective modules, so they both induce $\mathbb{A}$-linear endomorphisms of

the Grothendieck groups $\left[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ and $\left[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$. Taking $L = \mathbb{K}$ in Corollary 6A.5 and Corollary 6A.10,

$$E_i\left[S_{\boldsymbol{\lambda}}^\triangle\right] = \left[E_i^\Lambda S_{\boldsymbol{\lambda}}^\triangle\right] = \sum_{B\,\in\,\mathrm{Rem}_i(\boldsymbol{\lambda})} q^{d_B^\triangle(\boldsymbol{\lambda})}\left[S_{\boldsymbol{\lambda}-B}^\triangle\right],$$

$$F_i\left[S_{\boldsymbol{\lambda}}^\triangle\right] = \left[F_i^\Lambda \circ q^{\mathsf{d}_i} K_i^{-1} S_{\boldsymbol{\lambda}}^\triangle\right] = \sum_{A\,\in\,\mathrm{Add}_i(\boldsymbol{\lambda})} q^{d_A^\triangle(\boldsymbol{\lambda})+\mathsf{d}_i-d_i(\boldsymbol{\lambda})}\left[S_{\boldsymbol{\lambda}+A}^\triangle\right]$$

$$= \sum_{A\,\in\,\mathrm{Add}_i(\boldsymbol{\lambda})} q^{-d_A^\triangledown(\boldsymbol{\lambda})}\left[S_{\boldsymbol{\lambda}+A}^\triangle\right],$$

where the last equality uses (4.32a). Therefore, by identifying E_i with the functor E_i^Λ, and F_i with the functor $F_i^\Lambda \circ q^{\mathsf{d}_i} K_i^{-1}$, the linear maps $\mathsf{d}_T^\triangleleft$ and $\mathsf{d}^\triangleleft$ become well-defined $U_q(\mathfrak{g}_\Gamma)$-module homomorphisms by Corollary 6D.4. As $U_q(\mathfrak{g}_\Gamma)$-modules, $\left[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ and $\left[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ are both cyclic because they are both generated by $[Y_{\underline{\mathbf{0}}_\ell}^\triangle] = [S_{\underline{\mathbf{0}}_\ell}^\triangle] = [D_{\underline{\mathbf{0}}_\ell}^\triangle]$. By definition, $\mathsf{d}_T^\triangleleft([Y_{\underline{\mathbf{0}}_\ell}^\triangleleft]) = \mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangleleft$ and $\mathsf{d}^\triangleleft(\mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangleleft) = [S_{\underline{\mathbf{0}}_\ell}^\triangleleft]$, so the proposition follows since $U_q(\mathfrak{g}_\Gamma)\mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangleleft \cong L(\Lambda) \cong U_q(\mathfrak{g}_\Gamma)\mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangleright$ is an irreducible $U_q(\mathfrak{g}_\Gamma)$-module. $\qquad\square$

Since $K_i \mathsf{s}_{\boldsymbol{\lambda}}^\triangle = q^{d_i(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\lambda}}^\triangle$, for $\boldsymbol{\lambda} \in \mathcal{P}_\bullet^\ell$, we view K_i as a grading shift functor on $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_n^\Lambda(\mathbb{K}[x])$, for $i \in I$. Hereafter, for $i \in I$ we identify E_i and E_i^Λ, and F_i and $F_i^\Lambda \circ q^{\mathsf{d}_i} K_i^{-1}$, as functors on $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])$ and $\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])$.

Remark 6D.11. Let $\triangle \in \{\triangleleft, \triangleright\}$. Then Proposition 6D.10 can be interpreted as saying that there is a commutative diagram of $U_q(\mathfrak{g}_\Gamma)$-modules:

$$
\begin{array}{ccc}
\left[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]_{\mathbb{A}} & \xrightarrow{\ \mathsf{d}_T^\triangle\ } & \mathscr{F}_{\mathbb{A}}^{\Lambda\triangle} \\
& \searrow^{\mathsf{c}^\triangle} & \downarrow^{\mathsf{d}^\triangle} \\
& & \left[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]_{\mathbb{A}}
\end{array}
$$

The map $\mathsf{c}^\triangle : [\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]_{\mathbb{A}} \longrightarrow [\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]_{\mathbb{A}}$ is given by the Cartan matrix, which is the natural embedding of $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]_{\mathbb{A}}$ into $[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]_{\mathbb{A}}$. Of course, $\mathsf{d}^\triangle$ is the decomposition map and $\mathsf{d}_T^\triangle$ is its transpose. Hence, Corollary 5B.4 categorifies Proposition 6D.10.

Remark 6D.12. Let ε be the sign automorphism of Γ from Definition 5E.1. Abusing notation slightly, the quiver automorphism ε induces a unique automorphism of $U_q(\mathfrak{g}_\Gamma)$ such that

$$\varepsilon(E_i) = E_{\varepsilon(i)}, \qquad \varepsilon(F_i) = F_{\varepsilon(i)} \qquad \text{and} \qquad \varepsilon(K_i) = K_{\varepsilon(i)}, \qquad \text{for all } i \in I$$

Let $\mathscr{F}_{\mathcal{A}}^{\Lambda^\varepsilon\triangleleft} = \langle \mathsf{s}_{\boldsymbol{\lambda}}^{\triangleleft\varepsilon} | \boldsymbol{\lambda} \in \mathcal{P}_\bullet^\ell \rangle_{\mathcal{A}}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda^\varepsilon\triangleright} = \langle \mathsf{s}_{\boldsymbol{\lambda}}^{\triangleright\varepsilon} | \boldsymbol{\lambda} \in \mathcal{P}_\bullet^\ell \rangle_{\mathcal{A}}$ be the Fock spaces with $U_{\mathcal{A}}(\mathfrak{g}_\Gamma)$-action defined using the functions $d_A^{\triangleleft\varepsilon}(\boldsymbol{\lambda})$ and $d_a^{\triangleright\varepsilon}(\boldsymbol{\lambda})$ from subsection 5E. Then Lemma 5E.4 implies that there are $U_q(\mathfrak{g}_\Gamma)$-module isomorphisms $\mathsf{t}_\triangleleft^\varepsilon : \mathscr{F}_{\mathcal{A}}^{\Lambda\triangleleft} \cong \mathscr{F}_{\mathcal{A}}^{\Lambda^\varepsilon\triangleright}$ and $\mathsf{t}_\triangleright^\varepsilon : \mathscr{F}_{\mathcal{A}}^{\Lambda\triangleright} \cong \mathscr{F}_{\mathcal{A}}^{\Lambda^\varepsilon\triangleleft}$ given by $\mathsf{t}_\triangleleft^\varepsilon(\mathsf{s}_{\boldsymbol{\lambda}}^\triangleleft) = s_{\boldsymbol{\lambda}'}^{\triangleright\varepsilon}$ and $\mathsf{t}_\triangleright^\varepsilon(\mathsf{s}_{\boldsymbol{\lambda}}^\triangleright) = s_{\boldsymbol{\lambda}'}^{\triangleleft\varepsilon}$, for $\boldsymbol{\lambda} \in \mathcal{P}_\bullet^\ell$. Equivalently, there are $U_q(\mathfrak{g}_\Gamma)$-module isomorphisms $\mathscr{F}_{\mathcal{A}}^{\Lambda\triangleleft} \cong (\mathscr{F}_{\mathcal{A}}^{\Lambda^\varepsilon\triangleright})^\varepsilon$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda\triangleright} \cong (\mathscr{F}_{\mathcal{A}}^{\Lambda^\varepsilon\triangleleft})^\varepsilon$, where the $U_q(\mathfrak{g}_\Gamma)$ actions on $\mathscr{F}_{\mathcal{A}}^{\Lambda^\varepsilon\triangleleft}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda^\varepsilon\triangleright}$ are twisted by ε. These results should be compared with Corollary 5E.6.

We need to prove an "integral" version of the $U_q(\mathfrak{g}_\Gamma)$-module isomorphisms in Proposition 6D.10 over $\mathcal{A}$. To do this recall that Lusztig's $\mathcal{A}$-form of $U_q(\mathfrak{g}_\Gamma)$ is the $\mathcal{A}$-subalgebra $U_\mathcal{A}(\mathfrak{g}_\Gamma)$ of $U_q(\mathfrak{g}_\Gamma)$ that generated by the quantised divided powers $E_i^{(k)} = E_i^k/[k]!$ and $F_i^{(k)} = F_i^k/[k]!$, for $i \in I$ and $k \geq 0$. For any $\mathcal{A}$-module A set $U_A(\mathfrak{g}_\Gamma) = A \otimes_\mathcal{A} U_\mathcal{A}(\mathfrak{g}_\Gamma)$.

Corollary 6D.4 implies that $U_\mathcal{A}(\mathfrak{g}_\Gamma)$ acts on the $\mathcal{A}$-submodule $\mathscr{F}_\mathcal{A}^{\Lambda\triangle}$ of $\mathscr{F}_\mathbb{A}^{\Lambda\triangle}$; compare with [49, Lemma 6.15] and [43, Lemma 6.2]. Set

$$\mathscr{L}_\mathcal{A}^\triangleleft(\Lambda) = U_\mathcal{A}(\mathfrak{g}_\Gamma)\mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangleleft \qquad \text{and} \qquad \mathscr{L}_\mathcal{A}^\triangleright(\Lambda) = U_\mathcal{A}(\mathfrak{g}_\Gamma)\mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangleright. \qquad (6D.13)$$

Then Proposition 6D.10 implies that $\mathbb{A} \otimes_\mathcal{A} \mathscr{L}_\mathcal{A}^\triangleleft(\Lambda) \cong L(\Lambda) \cong \mathbb{A} \otimes_\mathcal{A} \mathscr{L}_\mathcal{A}^\triangleright(\Lambda)$, as $U_q(\mathfrak{g}_\Gamma)$-modules, and that:

Corollary 6D.14. Suppose that $\Lambda \in P^+$. Then $\mathscr{L}_\mathcal{A}^\triangleleft(\Lambda) \cong \left[\text{Proj}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right] \cong \mathscr{L}_\mathcal{A}^\triangleright(\Lambda)$ as $U_\mathcal{A}(\mathfrak{g}_\Gamma)$-modules.

The analogue of this result for $\left[\text{Rep}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ requires some Lie theory. Define symmetric bilinear forms $(\ ,\ )^\triangleleft \colon \mathscr{F}_\mathcal{A}^{\Lambda\triangleleft} \times \mathscr{F}_\mathcal{A}^{\Lambda\triangleleft} \longrightarrow \mathcal{A}$ and $(\ ,\ )^\triangleright \colon \mathscr{F}_\mathcal{A}^{\Lambda\triangleright} \times \mathscr{F}_\mathcal{A}^{\Lambda\triangleright} \longrightarrow \mathcal{A}$ by

$$\left(\mathsf{s}_{\boldsymbol{\lambda}}^\triangleleft, \mathsf{s}_{\boldsymbol{\mu}}^\triangleleft\right)^\triangleleft = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} q^{\text{def }\boldsymbol{\lambda}} \qquad \text{and} \qquad \left(\mathsf{s}_{\boldsymbol{\lambda}}^\triangleright, \mathsf{s}_{\boldsymbol{\mu}}^\triangleright\right)^\triangleright = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} q^{\text{def }\boldsymbol{\lambda}} \qquad \text{for } \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_\bullet^\ell, \qquad (6D.15)$$

and extending linearly. By definition, both of these bilinear forms are non-degenerate. By restriction, we consider $(\ ,\ )^\triangleleft$ and $(\ ,\ )^\triangleright$ as (possibly degenerate) bilinear forms on $\mathscr{L}_\mathcal{A}^\triangleleft(\Lambda)$ and $\mathscr{L}_\mathcal{A}^\triangleright(\Lambda)$, respectively.

Lemma 6D.16. Let $\triangle \in \{\triangleleft, \triangleright\}$. The bilinear form $(\ ,\ )^\triangle$ on $\mathscr{L}_\mathcal{A}^\triangle(\Lambda)$ is characterised by the properties:

$$(\mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangle, \mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangle)^\triangle = 1, \quad \left(E_i u, v\right)^\triangle = \left(u, F_i v\right)^\triangle \quad \text{and} \quad \left(F_i u, v\right)^\triangle = \left(u, E_i v\right)^\triangle,$$

for all $i \in I$ and $u, v \in \mathscr{L}_\mathcal{A}^\triangle(\Lambda)$.

Proof. By definition, $(\mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangle, \mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangle) = 1$. Let $i \in I$. To show that E_i and F_i are biadjoint with respect to $(\ ,\ )$ it is enough to consider the cases when $u = \mathsf{s}_{\boldsymbol{\mu}}^\triangle$ and $v = \mathsf{s}_{\boldsymbol{\lambda}}^\triangle$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_\bullet^\ell$. By Corollary 6D.4, $(F_i \mathsf{s}_{\boldsymbol{\mu}}^\triangle, \mathsf{s}_{\boldsymbol{\lambda}}^\triangle)^\triangle = 0 = (\mathsf{s}_{\boldsymbol{\mu}}^\triangle, E_i \mathsf{s}_{\boldsymbol{\lambda}}^\triangle)^\triangle$ unless $\boldsymbol{\lambda} = \boldsymbol{\mu} + A$ for some $A \in \text{Add}_i(\boldsymbol{\lambda})$. Moreover, if $A \in \text{Add}_i(\boldsymbol{\mu})$ and $\boldsymbol{\lambda} = \boldsymbol{\mu} + A$ then using Corollary 6D.4 and Lemma 4D.4,

$$\left(F_i \mathsf{s}_{\boldsymbol{\mu}}^\triangle, \mathsf{s}_{\boldsymbol{\lambda}}^\triangle\right)^\triangle = q^{\text{def}(\boldsymbol{\lambda}) - d_A^\triangledown(\boldsymbol{\mu})} = q^{\text{def}(\boldsymbol{\lambda}) - d_i(\boldsymbol{\mu}) + \mathsf{d}_i + d_A^\triangle(\boldsymbol{\mu})} = q^{\text{def}(\boldsymbol{\mu}) + d_A^\triangle(\boldsymbol{\mu})} = \left(\mathsf{s}_{\boldsymbol{\mu}}^\triangle, E_i \mathsf{s}_{\boldsymbol{\lambda}}^\triangle\right)^\triangle.$$

Similarly, $(E_i \mathsf{s}_{\boldsymbol{\lambda}}^\triangle, \mathsf{s}_{\boldsymbol{\mu}}^\triangle)^\triangle = (\mathsf{s}_{\boldsymbol{\lambda}}^\triangle, F_i \mathsf{s}_{\boldsymbol{\mu}}^\triangle)^\triangle$, for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$. As $\mathsf{s}_{\underline{\mathbf{0}}_\ell}^\triangle$ is the highest weight vector of weight Λ in the irreducible module $\mathbb{A} \otimes_\mathcal{A} \mathscr{L}_\mathcal{A}^\triangle(\Lambda)$, it follows by induction on weight that these three properties uniquely determine the bilinear form $(\ ,\ )^\triangle$ on $\mathscr{L}_\mathcal{A}^\triangle(\Lambda)$. $\square$

As the next result shows, the pairings $(\ ,\ )^\triangleleft$ and $(\ ,\ )^\triangleright$ are closely related to the Cartan pairing defined in (6C.1). Recall the functor # from (6B.1).

Lemma 6D.17. Suppose that $u \in \left[\text{Proj}_\mathbb{K} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right]$ and $v \in \mathscr{F}_\mathcal{A}^{\Lambda\triangleleft}$ with $\text{wt}(v) = \beta$. Then

$$\left(\mathsf{d}_T^\triangleleft\left(u^\#\right), v\right)^\triangleleft = q^{\text{def}(\beta)} \langle u, \mathsf{d}^\triangleleft(v) \rangle \qquad \text{and} \qquad \left(\mathsf{d}_T^\triangleright\left(u^\#\right), v\right)^\triangleright = q^{\text{def}(\beta)} \langle u, \mathsf{d}^\triangleright(v) \rangle$$

Proof. Let $\triangle \in \{\triangleleft, \triangleright\}$. It is enough to check this when $x = q^a[Y_{\boldsymbol{\mu}}^{\triangle}]$ and $v = \mathsf{s}_{\boldsymbol{\lambda}}^{\triangle}$, for $a \in \mathbb{Z}$, $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathcal{K}_{\bullet}^{\triangle}$ and $\boldsymbol{\lambda} \in \mathcal{P}_{\bullet}^{\ell}$. As $\langle \ , \ \rangle$ is sesquilinear, and $(\ , \ )^{\triangle}$ is bilinear,

$$q^{\mathrm{def}(\boldsymbol{\lambda})} \left\langle q^a \left[Y_{\boldsymbol{\mu}}^{\triangle} \right], \mathsf{d}^{\triangle} \left(\mathsf{s}_{\boldsymbol{\lambda}}^{\triangle} \right) \right\rangle = q^{\mathrm{def}(\boldsymbol{\lambda})-a} \sum_{\boldsymbol{\nu} \in \mathcal{K}_{\bullet}^{\triangle}} \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\triangle}(q) \left\langle \left[Y_{\boldsymbol{\mu}}^{\triangle} \right], \left[D_{\boldsymbol{\nu}}^{\triangle} \right] \right\rangle$$

$$= q^{\mathrm{def}(\boldsymbol{\lambda})-a} \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q) = q^{-a} \sum_{\boldsymbol{\nu} \in \mathcal{K}_{\bullet}^{\triangle}} \mathsf{d}_{\boldsymbol{\nu}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q) \left(\mathsf{s}_{\boldsymbol{\nu}}^{\triangle}, \mathsf{s}_{\boldsymbol{\lambda}}^{\triangle} \right)^{\triangle}$$

$$= q^{-a} \left(\mathsf{d}_T^{\triangle} \left(\left[Y_{\boldsymbol{\mu}}^{\triangle} \right] \right), \mathsf{s}_{\boldsymbol{\lambda}}^{\triangle} \right)^{\triangle} = \left(\mathsf{d}_T^{\triangle} \left(\left[q^a Y_{\boldsymbol{\mu}}^{\triangle} \right]^{\#} \right), \mathsf{s}_{\boldsymbol{\lambda}}^{\triangle} \right)^{\triangle}.$$

The last equality follows because $[q^a Y_{\boldsymbol{\mu}}^{\triangle}]^{\#} = q^{-a}[Y_{\boldsymbol{\mu}}^{\triangle}]$, by (6B.1), since $Y_{\boldsymbol{\mu}}^{\triangle}$ is projective. $\qquad \square$

We can now show that the Cartan pairing is biadjoint with respect to F_i^{Λ} and E_i^{Λ}, for $i \in I$.

Theorem 6D.18. Let $u \in [\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$, $v \in [\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$, and $i \in I$. Then

$$\left\langle F_i^{\Lambda} u, v \right\rangle = \left\langle u, E_i^{\Lambda} v \right\rangle \qquad \text{and} \qquad \left\langle E_i^{\Lambda} u, v \right\rangle = \left\langle u, F_i^{\Lambda} v \right\rangle.$$

Proof. Let $\triangle \in \{\triangleleft, \triangleright\}$. Since $\mathsf{D}_n^{\mathbb{K}\triangle}$ is surjective, we can write $v = \mathsf{d}^{\triangle}(\dot{v})$ where $\dot{v} \in L_{\mathcal{A}}(\Lambda)$ and $\mathrm{wt}(\dot{v}) = \Lambda - \alpha$. Then $\left\langle E_i^{\Lambda} u, v \right\rangle = 0$ unless $\mathrm{wt}(u) = \Lambda - \alpha + \alpha_i$, in which case we compute

$$\left\langle E_i^{\Lambda} u, v \right\rangle = \left\langle E_i u, \mathsf{d}^{\triangle}(\dot{v}) \right\rangle$$

$$= q^{-\mathrm{def}(\alpha)} \left(\mathsf{d}_T^{\triangle} \left((E_i u)^{\#} \right), \dot{v} \right)^{\triangle} \qquad \text{by Lemma 6D.17,}$$

$$= q^{\mathrm{def}(\alpha)} \left(E_i \mathsf{d}_T^{\triangle}(u^{\circledast}), \dot{v} \right)^{\triangle}, \qquad \text{by Lemma 6B.2 and Proposition 6B.3,}$$

$$= q^{\mathrm{def}(\alpha)} \left(\mathsf{d}_T^{\triangle}(u^{\circledast}), F_i \dot{v} \right)^{\triangle}, \qquad \text{by Lemma 6D.16,}$$

$$= q^{-\mathrm{def}(\alpha)-2\mathrm{def}(\alpha-\alpha_i)} \left(\mathsf{d}_T^{\triangle}(u^{\#}), F_i \dot{v} \right)^{\triangle}, \quad \text{by Lemma 6B.2,}$$

$$= q^{-\mathrm{def}(\alpha-\alpha_i)} \left\langle u, F_i v \right\rangle, \qquad \text{by Lemma 6D.17,}$$

$$= \left\langle u, F_i^{\Lambda} v \right\rangle,$$

where the last equality uses (4.32c) and the identifications of F_i and $F_i^{\Lambda} \circ q^{-\mathsf{d}_i} K_i^{-1}$ from Proposition 6D.10. A similar calculation shows that $\langle u, E_i^{\Lambda} v \rangle = \langle F_i^{\Lambda} u, v \rangle$. $\qquad \square$

Remark 6D.19. Working over a positively graded ring, Kashiwara [35, Theorem 3.5] shows that $(E_i^{\Lambda}, F_i^{\Lambda})$ is a biadjoint pair, which implies Theorem 6D.18. Lemma 6D.17 can be interpreted as saying that the Cartan pairing categorifies the Shapovalov form; compare [11, Lemma 3.1 and Theorem 4.18 (4)].

The modules $\mathscr{L}_{\mathcal{A}}^{\triangleleft}(\Lambda)$ and $\mathscr{L}_{\mathcal{A}}^{\triangleright}(\Lambda)$ are *standard $\mathcal{A}$-forms* of the irreducible $U_q(\mathfrak{g}_{\Gamma})$-module $L(\Lambda)$. The corresponding *costandard $\mathcal{A}$-forms* of $L(\Lambda)$ are the dual lattices:

$$\mathscr{L}_{\mathcal{A}}^{\triangleleft}(\Lambda)^* = \{v \in \mathscr{L}_{\mathcal{A}}^{\triangleleft}(\Lambda) \mid \langle u, v \rangle \in \mathcal{A} \text{ for all } u \in \mathscr{L}_{\mathcal{A}}^{\triangleleft}(\Lambda)\}$$

$$\mathscr{L}_{\mathcal{A}}^{\triangleright}(\Lambda)^* = \{v \in \mathscr{L}_{\mathcal{A}}^{\triangleright}(\Lambda) \mid \langle u, v \rangle \in \mathcal{A} \text{ for all } v \in \mathscr{L}_{\mathcal{A}}^{\triangleright}(\Lambda)\}$$

By Lemma 6D.17, $\mathscr{L}_{\mathcal{A}}^{\triangle}(\Lambda)^* = \{v \in \mathbb{A} \otimes_{\mathcal{A}} \mathscr{L}_{\mathcal{A}}^{\triangle}(\Lambda) \mid (u, v)^{\triangle} \in \mathcal{A} \text{ for all } u \in \mathscr{L}_{\mathcal{A}}^{\triangle}(\Lambda)\}$.

We can now prove the main result of this subsection. Categorical analogues of this result have been obtained by Brundan and Kleshchev [11, Theorem 4.18] in type $A_{e-1}^{(1)}$ and Kang and Kashiwara [31, Theorem 6.2] for all symmetrisable Kac–Moody algebras. The following theorem provides an explicit bridge between the graded representation theory of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ and the representation theory of $U_{\mathcal{A}}(\mathfrak{g}_\Gamma)$, which will be exploited in the following subsections.

Theorem 6D.20 (Cyclotomic categorification)**.** Suppose that $\Lambda \in Q^+$. Then, as $U_{\mathcal{A}}(\mathfrak{g}_\Gamma)$-modules,

$$\mathscr{L}_{\mathcal{A}}^\lhd(\Lambda) \cong \left[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right] \cong \mathscr{L}_{\mathcal{A}}^\rhd(\Lambda) \qquad \text{and} \qquad \mathscr{L}_{\mathcal{A}}^\lhd(\Lambda)^* \cong \left[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])\right] \cong \mathscr{L}_{\mathcal{A}}^\rhd(\Lambda)^*.$$

Proof. The two isomorphisms for $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ were already noted in Corollary 6D.14. Let $\triangle \in \{\lhd, \rhd\}$. Using the fact that $\mathscr{L}_{\mathcal{A}}^\triangle(\Lambda) \cong [\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$, together with (6C.1) and Theorem 6D.18, shows that $\mathscr{L}_{\mathcal{A}}^\triangle(\Lambda)^* \cong [\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ as $U_{\mathcal{A}}(\mathfrak{g}_\Gamma)$-modules. $\square$

In particular, note that Theorem 6D.20 implies that the sets $\mathcal{K}_n^\lhd$ and $\mathcal{K}_n^\rhd$ are independent of the field $\mathbb{K}$. (In fact, this already follows from Proposition 6D.10.) We will soon give recursive descriptions of these sets.

6E. **Canonical bases.** A key feature of integrable highest weight modules is that they come equipped with the closely related canonical bases and crystal bases. This subsection connects the natural bases of $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ and $[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ with canonical bases of $L_{\mathcal{A}}(\Lambda)$ and $L_{\mathcal{A}}(\Lambda)^*$.

Lemma 6E.1. Let $i \in I$. Then $E_i \circ \circledast \cong \circledast \circ E_i$ and $F_i \circ \circledast \cong \circledast \circ F_i$ as functors on $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])$.

Proof. By Proposition 6B.3, E_i^Λ commutes with $\circledast$ as functors on $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])$. Therefore, it is enough to show that $F_i \circ \circledast \cong \circledast \circ F_i$ as functors on $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\alpha^\Lambda(\mathbb{K}[x])$, for $\alpha \in Q^+$. As in Proposition 6D.10, identify F_i with the functor $F_i^\Lambda \circ q^{\mathsf{d}_i} K_i^{-1} = q^{-\mathsf{d}_i} K_i \circ F_i^\Lambda$ on $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\alpha^\Lambda(\mathbb{K}[x])$. Then there are isomorphisms

$$
\begin{aligned}
F_i \circ \circledast &\cong q^{\mathsf{d}_i} F_i^\Lambda K_i^{-1} \circ q^{-2\,\mathrm{def}\,\alpha} \# && \text{by Lemma 6B.2,} \\
&\cong q^{\mathsf{d}_i - d_i(\alpha) - 2\,\mathrm{def}\,\alpha} F_i^\Lambda \circ \# && \text{where } d_i(\alpha) = (\Lambda - \alpha | \alpha_i), \\
&\cong q^{\mathsf{d}_i - d_i(\alpha) - 2\,\mathrm{def}\,\alpha} \# \circ F_i^\Lambda && \text{by Proposition 6B.3,} \\
&\cong q^{-2\,\mathrm{def}(\alpha+\alpha_i)} \# \circ q^{d_i(\alpha) - \mathsf{d}_i} \circ F_i^\Lambda && \text{by Lemma 4D.4,} \\
&\cong \circledast \circ q^{-\mathsf{d}_i} K_i F_i^\Lambda \cong \circledast \circ F_i, && \text{by Lemma 6B.2.}
\end{aligned}
$$

So, E_i and F_i commute with $\circledast$ when acting on $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])$ (and as functors on $\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])$). $\square$

In contrast, E_i and F_i do *not* commute with $\#$ — and nor do the functors F_i^Λ and $\circledast$.

The functors $\#$ and $\circledast$ of Equations (6B.1) and (2C.4), respectively, induce semilinear automorphisms of $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ and $[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$, which are given by:

$$[P]^\# = [P^\#], \qquad \text{and} \qquad [M]^\circledast = [M^\circledast]$$

for $M \in \mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_n^\Lambda(\mathbb{K}[x])$ and $P \in \mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_n^\Lambda(\mathbb{K}[x])$. Lemma 6B.2 shows that these automorphisms are closely related. By restriction, we consider $\circledast$ as a semilinear automorphism of $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$.

The *bar involution* on $: U_q(\mathfrak{g}_\Gamma) \longrightarrow U_q(\mathfrak{g}_\Gamma)$ is the unique semilinear involution such that

$$\overline{E_i} = E_i, \qquad \overline{F_i} = F_i \qquad \text{and} \qquad \overline{K_i} = K_i^{-1}, \qquad \text{for all } i \in I.$$

Recall that $\Lambda \in P^+$ is a dominant weight and that $L(\Lambda) = U_q(\mathfrak{g}_\Gamma)v_\Lambda$ is an integrable highest weight module, where v_Λ a highest weight vector of weight Λ. The bar involution of $U_q(\mathfrak{g}_\Gamma)$ induces a unique semilinear bar involution $\overline{}$ on $L(\Lambda)$ such that $\overline{v_\Lambda} = v_\Lambda$ and $\overline{av} = \overline{a}\,\overline{v}$, for all $a \in U_q(\mathfrak{g}_\Gamma)$ and $v \in L(\Lambda)$.

Corollary 6E.2. Let $u \in \mathscr{L}_{\mathcal{A}}^{\lhd}(\Lambda)$, $v \in \mathscr{L}_{\mathcal{A}}^{\rhd}(\Lambda)$ and $p \in [\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$. Then

$$\mathsf{d}^{\lhd}(u)^{\circledast} = \mathsf{d}^{\lhd}(\overline{u}), \quad \mathsf{d}^{\rhd}(v)^{\circledast} = \mathsf{d}^{\rhd}(\overline{v}), \quad \mathsf{d}_T^{\lhd}(p^{\#}) = q^{2\,\mathrm{def}(\alpha)}\overline{\mathsf{d}_T^{\lhd}(p)}, \quad \mathsf{d}_T^{\rhd}(p^{\#}) = q^{2\,\mathrm{def}(\alpha)}\overline{\mathsf{d}_T^{\rhd}(p)}.$$

Proof. Let $\triangle \in \{\lhd, \rhd\}$. Since $\overline{\mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\triangle}} = \mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\triangle} = \mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\triangle\,\circledast}$ is the highest weight vector in $\mathscr{L}_{\mathcal{A}}^{\triangle}(\Lambda)$, arguing by induction on weight using Lemma 6E.1, it follows that $\mathsf{d}^{\triangle}(\overline{f}) = (\mathsf{d}^{\triangle}(f))^{\circledast}$, for all $f \in \mathscr{L}_{\mathcal{A}}^{\triangle}(\Lambda)$. As $[\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ embeds into $[\mathrm{Rep}_{\mathbb{K}}\,\mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$, $\mathsf{d}_T^{\triangle}(p^{\circledast}) = \overline{\mathsf{d}_T^{\triangle}(p)}$, for all $p \in [\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$. Hence, $\mathsf{d}_T^{\triangle}(p^{\#}) = q^{2\,\mathrm{def}\,\alpha}\overline{\mathsf{d}_T^{\triangle}(p)}$ since $\# \cong q^{2\,\mathrm{def}(\alpha)}\circ\circledast$ by Lemma 6B.2. $\square$

That is, $\circledast$ categorifies the bar involution on the Fock space.

Remark 6E.3. The Fock spaces $\mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$ are both integrable highest weight modules. Hence, both Fock spaces come equipped with bar involutions that are unique up to a choice of scalars, corresponding to the choice of highest weight vectors. Motivated by Proposition 4F.9, let $\mathsf{t}\colon \mathscr{F}_{\mathcal{A}}^{\Lambda\lhd} \longrightarrow \mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$ be the unique linear map such that $\mathsf{t}(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}) = q^{\mathrm{def}\,\boldsymbol{\lambda}}\overline{\mathsf{s}_{\boldsymbol{\lambda}}^{\rhd}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_\bullet^\ell$. Then Corollary 6D.4, Proposition 6D.10 and Lemma 4D.4 imply that t is a $U_q(\mathfrak{g}_\Gamma)$-module isomorphism and that $\mathsf{t} \circ \overline{} = \overline{} \circ \mathsf{t}$. Similarly, the map $\mathsf{t}'\colon \mathscr{F}_{\mathcal{A}}^{\Lambda\rhd} \to \mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$, which sends $\mathsf{s}_{\boldsymbol{\lambda}}^{\rhd}$ to $q^{\mathrm{def}\,\boldsymbol{\lambda}}\overline{\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}}$ for $\boldsymbol{\lambda} \in \mathcal{P}_\bullet^\ell$, is a $U_q(\mathfrak{g}_\Gamma)$-module isomorphism and $\mathsf{t}' \circ \overline{} = \overline{} \circ \mathsf{t}'$. Moreover, $\mathsf{t} \circ \mathsf{t}'$ and $\mathsf{t}' \circ \mathsf{t}$ are both identity maps. We will not use these observations in what follows, except implicitly in the sense that, as this remark suggests, working with the two Fock spaces, $\mathscr{F}_{\mathcal{A}}^{\Lambda\lhd}$ and $\mathscr{F}_{\mathcal{A}}^{\Lambda\rhd}$, serves as a replacement for giving an explicit description of the bar involution on either Fock space.

Lemma 6E.4. Suppose that $P \in \mathrm{Proj}\,\mathscr{R}_n^\Lambda(F)$ and $M \in \mathrm{Rep}\,\mathscr{R}_n^m(F)$. Then

$$\langle [P], [M]^{\circledast} \rangle = \overline{\langle [P]^{\#}, [M] \rangle}.$$

Proof. This is a standard tensor-hom adjointness argument; see, for example, [11, Lemma 2.5]. $\square$

By (6C.1), with respect to the Cartan pairing, the bases $\{[Y_{\boldsymbol{\mu}}^{\lhd}] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^{\lhd}\}$ and $\{[Y_{\boldsymbol{\nu}}^{\rhd}] \mid \boldsymbol{\nu} \in \mathcal{K}_\bullet^{\rhd}\}$ of $[\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ are dual to the bases $\{[D_{\boldsymbol{\mu}}^{\lhd}] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^{\lhd}\}$ and $\{[D_{\boldsymbol{\nu}}^{\rhd}] \mid \boldsymbol{\nu} \in \mathcal{K}_\bullet^{\rhd}\}$ of $[\mathrm{Rep}_{\mathbb{K}}\,\mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$, respectively. The projective Grothendieck group $[\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ comes equipped with only one natural basis $\{[Y_{\boldsymbol{\mu}}^{\triangle}] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^{\triangle}\}$. In contrast, the Grothendieck group $[\mathrm{Rep}_{\mathbb{K}}\,\mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ has two quite different bases, $\{[D_{\boldsymbol{\mu}}^{\triangle}] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^{\triangle}\}$ and $\{[S_{\boldsymbol{\mu}}^{\triangle}] \mid \boldsymbol{\mu} \in \mathcal{K}_\bullet^{\triangle}\}$, given by the simple modules and the Specht modules. To define a second basis of $[\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_n^\Lambda(\mathbb{K}[x])]$, which turns out to be dual to the dual Specht modules, define the *inverse graded decomposition numbers* to be the Laurent polynomials $\mathsf{e}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\lhd}(-q), \mathsf{e}_{\boldsymbol{\mu\sigma}}^{\mathbb{K}\rhd}(-q) \in \mathcal{A}$ given by

$$\left(\mathsf{e}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\lhd}(-q)\right) = \left(\mathsf{d}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\lhd}(q)\right)^{-1} \qquad \text{and} \qquad \left(\mathsf{e}_{\boldsymbol{\sigma\nu}}^{\mathbb{K}\rhd}(-q)\right) = \left(\mathsf{d}_{\boldsymbol{\sigma\nu}}^{\mathbb{K}\rhd}(q)\right)^{-1}, \qquad (6E.5)$$

where $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$, $\boldsymbol{\nu}, \boldsymbol{\sigma} \in \mathcal{K}_n^{\rhd}$ and the rows and columns of these matrices are ordered using the lexicographic orders $<_{\mathrm{lex}}$ and $>_{\mathrm{lex}}$, respectively. These polynomials are well-defined because these submatrices of the decomposition matrices of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ are lower

unitriangular square matrices by Theorem 5B.2. For $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\triangleright}$ define virtual projective modules by

$$\mathsf{X}_{\boldsymbol{\mu}}^{\triangleleft} = \sum_{\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}} \mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(-q)\, [Y_{\boldsymbol{\lambda}}^{\triangleleft}] \qquad \text{and} \qquad \mathsf{X}_{\boldsymbol{\nu}}^{\triangleright} = \sum_{\boldsymbol{\sigma} \trianglerighteq \boldsymbol{\nu}} \mathsf{e}_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\mathbb{K}\triangleright}(-q)\, [Y_{\boldsymbol{\sigma}}^{\triangleright}], \qquad (6\mathrm{E}.6)$$

where $\boldsymbol{\lambda} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\sigma} \in \mathcal{K}_n^{\triangleright}$ in the sums. As the matrices in (6E.5) are invertible, $\bigcup_{n \geq 0}\{\mathsf{X}_{\boldsymbol{\mu}}^{\triangleleft} \,|\, \boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}\}$ and $\bigcup_{n \geq 0}\{\mathsf{X}_{\boldsymbol{\nu}}^{\triangleright} \,|\, \boldsymbol{\nu} \in \mathcal{K}_n^{\triangleright}\}$ are both $\mathcal{A}$-bases of $[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$. The definition of the $\mathsf{X}^{\triangle}$-bases suggests that these elements depend on $\mathbb{K}$ but the next result shows that these elements are independent of $\mathbb{K}$.

Lemma 6E.7. Suppose that $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\nu}, \boldsymbol{\sigma} \in \mathcal{K}_n^{\triangleright}$. Then $\langle \mathsf{X}_{\boldsymbol{\mu}}^{\triangleleft}, [S_{\boldsymbol{\lambda}}^{\triangleleft}]^{\circledast} \rangle = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ and $\langle \mathsf{X}_{\boldsymbol{\nu}}^{\triangleright}, [S_{\boldsymbol{\sigma}}^{\triangleright}]^{\circledast} \rangle = \delta_{\boldsymbol{\nu}\boldsymbol{\sigma}}$.

Proof. It is enough to prove the first statement as the second follows by symmetry. By the definitions,

$$\left\langle \mathsf{X}_{\boldsymbol{\mu}}^{\triangleleft}, [S_{\boldsymbol{\sigma}}^{\triangleleft}]^{\circledast} \right\rangle = \left\langle \sum_{\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}} \mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(-q)\, [Y_{\boldsymbol{\lambda}}^{\triangleleft}], [S_{\boldsymbol{\sigma}}^{\triangleleft}]^{\circledast} \right\rangle = \sum_{\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}} \overline{\mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(-q)} \left\langle [Y_{\boldsymbol{\lambda}}^{\triangleleft}], [S_{\boldsymbol{\sigma}}^{\triangleleft}]^{\circledast} \right\rangle$$

$$= \sum_{\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}} \overline{\mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(-q)} \left\langle [Y_{\boldsymbol{\lambda}}^{\triangleleft}], \sum_{\boldsymbol{\tau} \trianglerighteq \boldsymbol{\sigma}} \overline{\mathsf{d}_{\boldsymbol{\sigma}\boldsymbol{\tau}}^{\mathbb{K}\triangleleft}(q)}[D_{\boldsymbol{\tau}}^{\triangleleft}]^{\circledast} \right\rangle$$

$$= \sum_{\substack{\boldsymbol{\tau} \trianglerighteq \boldsymbol{\sigma} \\ \boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}}} \overline{\mathsf{d}_{\boldsymbol{\sigma}\boldsymbol{\tau}}^{\mathbb{K}\triangleleft}(q)}\, \overline{\mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(-q)} \left\langle [Y_{\boldsymbol{\lambda}}^{\triangleleft}], [D_{\boldsymbol{\tau}}^{\triangleleft}] \right\rangle$$

$$= \sum_{\boldsymbol{\sigma} \trianglelefteq \boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}} \overline{\mathsf{d}_{\boldsymbol{\sigma}\boldsymbol{\lambda}}^{\mathbb{K}\triangleleft}(q)}\, \overline{\mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(-q)},$$

where the last equality follows by (6C.2). Note that in these sums, $\boldsymbol{\lambda}, \boldsymbol{\tau} \in \mathcal{K}_n^{\triangle}$. The result now follows by (6E.5). $\qquad \square$

Applying the two bar involutions $\#$ and $\circledast$ shows that if $\triangle \in \{\triangleleft, \triangleright\}$ then

$$\left[Y_{\boldsymbol{\mu}}^{\triangle}\right]^{\#} = \left[Y_{\boldsymbol{\mu}}^{\triangle}\right] \quad \text{and} \quad \left[D_{\boldsymbol{\mu}}^{\triangle}\right]^{\circledast} = \left[D_{\boldsymbol{\mu}}^{\triangle}\right], \qquad \text{for } \boldsymbol{\mu} \in \mathcal{K}_n^{\triangle}, \qquad (6\mathrm{E}.8)$$

with the $\#$-identities following because $Y_{\boldsymbol{\mu}}^{\triangleleft}$ and $Y_{\boldsymbol{\nu}}^{\triangleright}$ are projective and the $\circledast$-identities coming from Theorem 5A.3. It is less clear what these involutions do to the other bases of $[\mathrm{Proj}_{\mathbb{K}} \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$ and $[\mathrm{Rep}_{\mathbb{K}} \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$.

Lemma 6E.9. Let $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\triangleright}$. Then

$$\left(\mathsf{X}_{\boldsymbol{\mu}}^{\triangleleft}\right)^{\#} = \mathsf{X}_{\boldsymbol{\mu}}^{\triangleleft} + \sum_{\boldsymbol{\lambda} \vartriangleleft \boldsymbol{\mu}} x_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\triangleleft}(q)\mathsf{X}_{\boldsymbol{\lambda}}^{\triangleleft}, \qquad\qquad \left[S_{\boldsymbol{\mu}}^{\triangleleft}\right]^{\circledast} = \left[S_{\boldsymbol{\mu}}^{\triangleleft}\right] + \sum_{\boldsymbol{\lambda} \vartriangleleft \boldsymbol{\mu}} s_{\boldsymbol{\mu}\boldsymbol{\lambda}}^{\triangleleft}(q)\, [S_{\boldsymbol{\lambda}}^{\triangleleft}],$$

$$(\mathsf{X}_{\boldsymbol{\nu}}^{\triangleright})^{\#} = \mathsf{X}_{\boldsymbol{\nu}}^{\triangleright} + \sum_{\boldsymbol{\sigma} \vartriangleright \boldsymbol{\nu}} x_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\triangleright}(q)\mathsf{X}_{\boldsymbol{\sigma}}^{\triangleright}, \qquad\qquad [S_{\boldsymbol{\nu}}^{\triangleright}]^{\circledast} = [S_{\boldsymbol{\nu}}^{\triangleright}] + \sum_{\boldsymbol{\sigma} \vartriangleright \boldsymbol{\nu}} s_{\boldsymbol{\nu}\boldsymbol{\sigma}}^{\triangleright}(q)\, [S_{\boldsymbol{\sigma}}^{\triangleright}].$$

for Laurent polynomials $x_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\triangleleft}(q), s_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\triangleleft}(q), x_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\triangleright}(q), s_{\boldsymbol{\sigma}\boldsymbol{\mu}}^{\triangleright}(q) \in \mathcal{A}$ with $\boldsymbol{\lambda} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\sigma} \in \mathcal{K}_n^{\triangleright}$.

Proof. Let $\boldsymbol{\sigma} \in \mathcal{K}_n^{\lhd}$. Using Theorem 5B.2 and (6E.5),

$$
\begin{aligned}
\left[S_{\boldsymbol{\mu}}^{\lhd}\right]^{\circledast} &= \left(\sum_{\boldsymbol{\alpha} \unlhd \boldsymbol{\mu}} \mathsf{d}_{\boldsymbol{\mu}\boldsymbol{\alpha}}^{\mathbb{K}\lhd}(q)\,[D_{\boldsymbol{\alpha}}^{\lhd}]\right)^{\circledast} = \sum_{\boldsymbol{\alpha} \unlhd \boldsymbol{\mu}} \overline{\mathsf{d}_{\boldsymbol{\mu}\boldsymbol{\alpha}}^{\mathbb{K}\lhd}(q)}\,[D_{\boldsymbol{\alpha}}^{\lhd}] \\
&= \sum_{\boldsymbol{\alpha} \unlhd \boldsymbol{\mu}} \overline{\mathsf{d}_{\boldsymbol{\mu}\boldsymbol{\alpha}}^{\mathbb{K}\lhd}(q)} \sum_{\boldsymbol{\lambda} \unlhd \boldsymbol{\alpha}} \mathsf{e}_{\boldsymbol{\alpha}\boldsymbol{\lambda}}^{\mathbb{K}\lhd}(-q)\,[S_{\boldsymbol{\lambda}}^{\lhd}] \\
&= \left[S_{\boldsymbol{\mu}}^{\lhd}\right] + \sum_{\boldsymbol{\lambda} \lhd \boldsymbol{\mu}} \left(\sum_{\substack{\boldsymbol{\alpha} \in \mathcal{K}_n^{\lhd} \\ \boldsymbol{\lambda} \unlhd \boldsymbol{\alpha} \unlhd \boldsymbol{\mu}}} \overline{\mathsf{d}_{\boldsymbol{\mu}\boldsymbol{\alpha}}^{\mathbb{K}\lhd}(q)}\mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(-q)\,[\boldsymbol{\alpha}\boldsymbol{\lambda}]\right)\,[S_{\boldsymbol{\lambda}}^{\lhd}],
\end{aligned}
$$

where the last equality follows because $\mathsf{d}_{\boldsymbol{\mu}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) = 1 = \mathsf{e}_{\boldsymbol{\mu}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(-q)$ by Theorem 5B.2. This proves the result for $[S_{\boldsymbol{\mu}}^{\lhd}]^{\circledast}$, which this implies that $\mathsf{X}_{\boldsymbol{\mu}}^{\lhd \#}$ has the required expansion by Lemma 6E.7 and Lemma 6E.4. The remaining claims are similar. $\qquad\square$

Theorem 6E.10. *Let $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$. Then there exist bases $\{\mathbb{Y}_{\boldsymbol{\mu}}^{\lhd} \mid \boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}\}$ and $\{\mathbb{Y}_{\boldsymbol{\nu}}^{\rhd} \mid \boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}\}$ of $[\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$, and $\{\mathbb{D}_{\boldsymbol{\mu}}^{\lhd} \mid \boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}\}$ and $\{\mathbb{D}_{\boldsymbol{\nu}}^{\rhd} \mid \boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}\}$ of $[\mathrm{Rep}_{\mathbb{K}}\,\mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$, that are uniquely determined by the conditions:*

$$
(\mathbb{Y}_{\boldsymbol{\mu}}^{\lhd})^{\#} = \mathbb{Y}_{\boldsymbol{\mu}}^{\lhd} \quad \text{and} \quad \mathbb{Y}_{\boldsymbol{\mu}}^{\lhd} = \mathsf{X}_{\boldsymbol{\mu}}^{\lhd} + \sum_{\boldsymbol{\lambda} \lhd \boldsymbol{\mu}} \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd}(q)\mathsf{X}_{\boldsymbol{\lambda}}^{\lhd}
$$

$$
(\mathbb{Y}_{\boldsymbol{\nu}}^{\rhd})^{\#} = \mathbb{Y}_{\boldsymbol{\nu}}^{\rhd} \quad \text{and} \quad \mathbb{Y}_{\boldsymbol{\mu}}^{\rhd} = \mathsf{X}_{\boldsymbol{\mu}}^{\rhd} + \sum_{\boldsymbol{\lambda} \rhd \boldsymbol{\nu}} \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\rhd}(q)\mathsf{X}_{\boldsymbol{\lambda}}^{\rhd}
$$

$$
(\mathbb{D}_{\boldsymbol{\mu}}^{\lhd})^{\circledast} = \mathbb{D}_{\boldsymbol{\mu}}^{\lhd} \quad \text{and} \quad \mathbb{D}_{\boldsymbol{\mu}}^{\lhd} = [S_{\boldsymbol{\mu}}^{\lhd}] + \sum_{\boldsymbol{\lambda} \lhd \boldsymbol{\mu}} \mathsf{e}_{\boldsymbol{\mu}\boldsymbol{\lambda}}^{\lhd}(-q)[S_{\boldsymbol{\lambda}}^{\lhd}]
$$

$$
(\mathbb{D}_{\boldsymbol{\nu}}^{\rhd})^{\circledast} = \mathbb{D}_{\boldsymbol{\nu}}^{\rhd} \quad \text{and} \quad \mathbb{D}_{\boldsymbol{\nu}}^{\rhd} = [S_{\boldsymbol{\nu}}^{\rhd}] + \sum_{\boldsymbol{\lambda} \rhd \boldsymbol{\nu}} \mathsf{e}_{\boldsymbol{\nu}\boldsymbol{\lambda}}^{\rhd}(-q)[S_{\boldsymbol{\lambda}}^{\rhd}].
$$

for polynomials $\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd}(q), \mathsf{e}_{\boldsymbol{\mu}\boldsymbol{\lambda}}^{\lhd}(-q) \in \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} + q\mathbb{Z}[q]$ and $\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\rhd}(q), \mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\rhd}(-q) \in \delta_{\boldsymbol{\lambda}\boldsymbol{\nu}} + q\mathbb{Z}[q]$, for $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$. In particular, the basis elements $\mathbb{Y}_{\boldsymbol{\mu}}^{\lhd}$, $\mathbb{Y}_{\boldsymbol{\nu}}^{\rhd}$, $\mathbb{D}_{\boldsymbol{\mu}}^{\lhd}$ and $\mathbb{D}_{\boldsymbol{\nu}}^{\rhd}$, and these polynomials, are independent of the field $\mathbb{K}$.

Proof. Given Lemma 6E.9, this result is a consequence of *Lusztig's Lemma* [47, Lemma 24.2.1], which is easily proved by induction on dominance using Gaussian elimination and Lemma 6E.9. See [52, Proposition 3.5.6] for a proof that uses very similar language to that used here. $\qquad\square$

A key point in Theorem 6E.10 is that the coefficients appearing in Lemma 6E.9 belong to $\mathcal{A}$. As the notation suggests, the polynomials $\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\triangle}(q)$ are related to the decomposition matrices of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$ and the polynomials $\mathsf{e}_{\boldsymbol{\mu}\boldsymbol{\lambda}}^{\triangle}(-q)$ are related to the inverse decomposition matrices. See Theorem 6E.16 below for a precise statement.

By Theorem 6E.10, $\{\mathbb{Y}_{\boldsymbol{\mu}}^{\lhd} \mid \boldsymbol{\mu} \in \mathcal{K}_{\bullet}^{\lhd}\}$ and $\{\mathbb{Y}_{\boldsymbol{\nu}}^{\rhd} \mid \boldsymbol{\nu} \in \mathcal{K}_{\bullet}^{\rhd}\}$ are bases of $[\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$ and $\{\mathbb{D}_{\boldsymbol{\mu}}^{\lhd} \mid \boldsymbol{\mu} \in \mathcal{K}_{\bullet}^{\lhd}\}$ and $\{\mathbb{D}_{\boldsymbol{\nu}}^{\rhd} \mid \boldsymbol{\nu} \in \mathcal{K}_{\bullet}^{\rhd}\}$ are bases of $[\mathrm{Rep}_{\mathbb{K}}\,\mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$.

Definition 6E.11.
   (a) The $\circledast$-*canonical bases* of $[\mathrm{Rep}_{\mathbb{K}}\,\mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x]]$ are the two bases $\{\mathbb{D}_{\boldsymbol{\mu}}^{\lhd} \mid \boldsymbol{\mu} \in \mathcal{K}_{\bullet}^{\lhd}\}$ and $\{\mathbb{D}_{\boldsymbol{\nu}}^{\rhd} \mid \boldsymbol{\nu} \in \mathcal{K}_{\bullet}^{\rhd}\}$.
   (b) The $\#$-*canonical bases* of $[\mathrm{Proj}_{\mathbb{K}}\,\mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$ are the two bases $\{\mathbb{Y}_{\boldsymbol{\mu}}^{\lhd} \mid \boldsymbol{\mu} \in \mathcal{K}_{\bullet}^{\lhd}\}$ and $\{\mathbb{Y}_{\boldsymbol{\nu}}^{\rhd} \mid \boldsymbol{\nu} \in \mathcal{K}_{\bullet}^{\rhd}\}$.

We frequently call these four bases *canonical bases* of $[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ and $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$. In Theorem 6F.14 below we show that, up to scaling, these bases coincide with Lusztig's (dual) canonical bases [46, §14.4] and Kashiwara's (upper and lower) global bases [33] of $L(\Lambda)$.

For now we note that Theorem 6E.10 and Lemma 6B.2 imply:

Corollary 6E.12. Suppose that $\boldsymbol{\mu} \in \mathcal{K}_n^\lhd$ and $\boldsymbol{\nu} \in \mathcal{K}_n^\rhd$. Then

$$\left(\mathbb{Y}_{\boldsymbol{\mu}}^\lhd\right)^\circledast = q^{-2\,\mathrm{def}\,\boldsymbol{\mu}}\mathbb{Y}_{\boldsymbol{\mu}}^\lhd, \quad \left(\mathbb{Y}_{\boldsymbol{\nu}}^\rhd\right)^\circledast = q^{-2\,\mathrm{def}\,\boldsymbol{\nu}}\mathbb{Y}_{\boldsymbol{\nu}}^\rhd,$$

$$\left(\mathbb{D}_{\boldsymbol{\mu}}^\lhd\right)^\# = q^{2\,\mathrm{def}\,\boldsymbol{\mu}}\mathbb{D}_{\boldsymbol{\mu}}^\lhd \quad \text{and} \quad \left(\mathbb{D}_{\boldsymbol{\nu}}^\rhd\right)^\# = q^{2\,\mathrm{def}\,\boldsymbol{\nu}}\mathbb{D}_{\boldsymbol{\nu}}^\rhd.$$

The next result shows that these bases of $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_n^\Lambda(\mathbb{K}[x])]$ and $[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_n^\Lambda(\mathbb{K}[x])]$ are dual with respect to the Cartan pairing. The matrix identities in the next result should be compared with (6E.5).

Corollary 6E.13. Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_n^\lhd$ and $\boldsymbol{\nu}, \boldsymbol{\sigma} \in \mathcal{K}_n^\rhd$. Then $\langle \mathbb{Y}_{\boldsymbol{\lambda}}^\lhd, \mathbb{D}_{\boldsymbol{\mu}}^\lhd \rangle = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ and $\langle \mathbb{Y}_{\boldsymbol{\nu}}^\rhd, \mathbb{D}_{\boldsymbol{\sigma}}^\rhd \rangle = \delta_{\boldsymbol{\nu}\boldsymbol{\sigma}}$. Equivalently, the two matrix identities hold

$$\left(\mathsf{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^\lhd(-q)\right) = \left(\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^\lhd(q)\right)^{-1} \quad \text{and} \quad \left(\mathsf{e}_{\boldsymbol{\sigma}\boldsymbol{\nu}}^\rhd(-q)\right) = \left(\mathsf{d}_{\boldsymbol{\sigma}\boldsymbol{\nu}}^\rhd(q)\right)^{-1}.$$

Proof. Let $\triangle \in \{\lhd, \rhd\}$. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{K}_n^\triangle$. Direct calculation reveals that

$$\left\langle \mathbb{Y}_{\boldsymbol{\alpha}}^\triangle, \mathbb{D}_{\boldsymbol{\beta}}^\triangle \right\rangle = \left\langle \left[\mathbb{Y}_{\boldsymbol{\alpha}}^\triangle\right], \left[\mathbb{D}_{\boldsymbol{\beta}}^\triangle\right]^\circledast \right\rangle = \left\langle \sum_{\boldsymbol{\sigma} \in \mathcal{K}_n^\triangle} \mathsf{d}_{\boldsymbol{\sigma}\boldsymbol{\alpha}}^\triangle(q)\mathsf{X}_{\boldsymbol{\sigma}}^\triangle, \sum_{\boldsymbol{\tau} \in \mathcal{K}_n^\triangle} \overline{\mathsf{e}_{\boldsymbol{\beta}\boldsymbol{\tau}}^\triangle(-q)}[S_{\boldsymbol{\tau}}^\triangle] \right\rangle$$

$$= \sum_{\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{K}_n^\triangle} \overline{\mathsf{d}_{\boldsymbol{\sigma}\boldsymbol{\alpha}}^\triangle(q)}\,\overline{\mathsf{e}_{\boldsymbol{\beta}\boldsymbol{\tau}}^\triangle(-q)} \left\langle \mathsf{X}_{\boldsymbol{\sigma}}^\triangle, [S_{\boldsymbol{\tau}}^\triangle]^\circledast \right\rangle$$

$$= \sum_{\boldsymbol{\sigma} \in \mathcal{K}_n^\triangle} \overline{\mathsf{e}_{\boldsymbol{\beta}\boldsymbol{\sigma}}^\triangle(-q)\mathsf{d}_{\boldsymbol{\sigma}\boldsymbol{\alpha}}^\triangle(q)},$$

where the last equality follows by Lemma 6E.7. Therefore, $\langle \mathbb{Y}_{\boldsymbol{\alpha}}^\triangle, \mathbb{D}_{\boldsymbol{\beta}}^\triangle \rangle \in \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} + q^{-1}\mathbb{Z}[q^{-1}]$. However, by Lemma 6E.4,

$$\left\langle \mathbb{Y}_{\boldsymbol{\alpha}}^\triangle, \mathbb{D}_{\boldsymbol{\beta}}^\triangle \right\rangle = \overline{\left\langle \mathbb{Y}_{\boldsymbol{\alpha}}^\triangle{}^\#, \mathbb{D}_{\boldsymbol{\beta}}^\triangle{}^\circledast \right\rangle} = \overline{\left\langle \mathbb{Y}_{\boldsymbol{\alpha}}^\triangle, \mathbb{D}_{\boldsymbol{\beta}}^\triangle \right\rangle} \in \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} + qZ[q].$$

Hence, $\langle \mathbb{Y}_{\boldsymbol{\alpha}}^\triangle, \mathbb{D}_{\boldsymbol{\beta}}^\triangle \rangle = \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}$. The calculation in the first displayed equation shows that this is equivalent to the matrix identity in the statement of the corollary. $\square$

In particular, this shows that the #-canonical bases of $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ and the $\circledast$-canonical bases of $[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ encode equivalent information.

Lemma 6E.14. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_n^\lhd$ and $\boldsymbol{\sigma}, \boldsymbol{\nu} \in \mathcal{K}_n^\rhd$. Then

$$\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^\lhd(q) = \left\langle \mathbb{Y}_{\boldsymbol{\mu}}^\lhd, [S_{\boldsymbol{\lambda}}^\lhd] \right\rangle,\ \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^\rhd(q) = \langle \mathbb{Y}_{\boldsymbol{\nu}}^\rhd, [S_{\boldsymbol{\sigma}}^\rhd] \rangle,\ \mathsf{e}_{\boldsymbol{\mu}\boldsymbol{\lambda}}^\lhd(-q) = \langle \mathsf{X}_{\boldsymbol{\lambda}}^\lhd, \mathbb{D}_{\boldsymbol{\mu}}^\lhd \rangle,\ \mathsf{e}_{\boldsymbol{\nu}\boldsymbol{\lambda}}^\rhd(-q) = \langle \mathsf{X}_{\boldsymbol{\sigma}}^\rhd, \mathbb{D}_{\boldsymbol{\nu}}^\rhd \rangle.$$

Proof. Let $\triangle \in \{\lhd, \rhd\}$ and $\boldsymbol{\mu} \in \mathcal{K}_n^\triangle$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. Using Lemma 6D.17 and Theorem 6E.10,

$$\left\langle \mathbb{Y}_{\boldsymbol{\mu}}^\triangle, \left[S_{\boldsymbol{\lambda}}^\triangle\right] \right\rangle = \overline{\left\langle \mathbb{Y}_{\boldsymbol{\mu}}^\triangle{}^\#, \left[S_{\boldsymbol{\lambda}}^\triangle\right]^\circledast \right\rangle} = \overline{\left\langle \mathbb{Y}_{\boldsymbol{\mu}}^\triangle, \left[S_{\boldsymbol{\lambda}}^\triangle\right]^\circledast \right\rangle} = \sum_{\boldsymbol{\tau} \in \mathcal{K}_n^\triangle} \mathsf{d}_{\boldsymbol{\tau}\boldsymbol{\mu}}^\triangle(q)\overline{\left\langle \mathsf{X}_{\boldsymbol{\tau}}^\triangle, \left[S_{\boldsymbol{\lambda}}^\triangle\right]^\circledast \right\rangle} = \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^\triangle(q),$$

where the last equality comes from Lemma 6E.7. The proof of the other identities are similar. $\square$

For $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$, $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$ and $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{P}_n^\ell$ define Laurent polynomials

$$\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd}(q) = \left\langle \mathbb{Y}_{\boldsymbol{\mu}}^{\lhd}, [S_{\boldsymbol{\lambda}}^{\lhd}] \right\rangle \qquad \text{and} \qquad \mathbb{d}_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\rhd}(q) = \langle \mathbb{Y}_{\boldsymbol{\nu}}^{\rhd}, [S_{\boldsymbol{\sigma}}^{\rhd}] \rangle. \tag{6E.15}$$

By Lemma 6E.14, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_n^{\triangle}$ then $\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\triangle}(q)$ coincides with the polynomial defined in Theorem 6E.10. In particular, if $\boldsymbol{\lambda} \in \mathcal{K}_n^{\triangle}$ then $\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\triangle}(q) \in \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} + q\mathbb{Z}[q]$ by Theorem 6E.10. We will show in Corollary 6F.16 below that this is still true when $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell \setminus \mathcal{K}_n^{\triangle}$. Moreover, we show that $\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\triangle}(q) \in \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} + q\mathbb{N}[q]$ in type $A_{e-1}^{(1)}$.

Theorem 6E.16. For $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathcal{K}_n^{\lhd}$ and $\boldsymbol{\nu}, \boldsymbol{\sigma} \in \mathcal{K}_n^{\rhd}$, there exist bar invariant polynomials $\mathbb{o}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q), \mathbb{o}_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q), \mathbb{b}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q), \mathbb{b}_{\boldsymbol{\nu}\boldsymbol{\sigma}}^{\mathbb{K}\rhd}(q) \in \mathcal{A}$ such that

$$\left[Y_{\boldsymbol{\mu}}^{\lhd}\right] = \mathbb{Y}_{\boldsymbol{\mu}}^{\lhd} + \sum_{\boldsymbol{\lambda} \lhd \boldsymbol{\mu}} \mathbb{o}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q)\mathbb{Y}_{\boldsymbol{\lambda}}^{\lhd}, \qquad\qquad [Y_{\boldsymbol{\nu}}^{\rhd}] = \mathbb{Y}_{\boldsymbol{\nu}}^{\rhd} + \sum_{\boldsymbol{\sigma} \rhd \boldsymbol{\nu}} \mathbb{o}_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q)\mathbb{Y}_{\boldsymbol{\sigma}}^{\rhd},$$

$$\left[D_{\boldsymbol{\mu}}^{\lhd}\right] = \mathbb{D}_{\boldsymbol{\mu}}^{\lhd} + \sum_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} \mathbb{b}_{\boldsymbol{\mu}\boldsymbol{\lambda}}^{\mathbb{K}\lhd}(q)\mathbb{D}_{\boldsymbol{\lambda}}^{\lhd}, \qquad\qquad [D_{\boldsymbol{\nu}}^{\rhd}] = \mathbb{D}_{\boldsymbol{\nu}}^{\rhd} + \sum_{\boldsymbol{\sigma} \lhd \boldsymbol{\nu}} \mathbb{b}_{\boldsymbol{\nu}\boldsymbol{\sigma}}^{\mathbb{K}\rhd}(q)\mathbb{D}_{\boldsymbol{\sigma}}^{\rhd}.$$

Moreover, for $\boldsymbol{\sigma}, \boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, the following matrix identities hold:

$$\left(\mathbb{b}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q)\right) = \left(\mathbb{o}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q)\right)^{-1}, \qquad\qquad \left(\mathbb{b}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q)\right) = \left(\mathbb{o}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q)\right)^{-1},$$

$$\left(\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q)\right) = \left(\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd}(q)\right)\left(\mathbb{o}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q)\right), \qquad\qquad \left(\mathbb{d}_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q)\right) = (\mathbb{d}_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\rhd}(q))\left(\mathbb{o}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q)\right).$$

Proof. Let $\triangle \in \{\lhd, \rhd\}$. By (6E.8), $[Y_{\boldsymbol{\mu}}^{\triangle}]$ is a #-invariant element of $[\text{Proj}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ and $[D_{\boldsymbol{\mu}}^{\triangle}]$ is a $\circledast$-invariant element of $[\text{Rep}_{\mathbb{K}} \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$. Hence, the first four identities follow by (6E.5) and Lemma 6E.9. (These four identities describe the transition matrices between the $\{[Y_{\boldsymbol{\mu}}^{\triangle}]\}$ and $\{[\mathbb{Y}_{\boldsymbol{\mu}}^{\triangle}]\}$ bases and between the $\{[D_{\boldsymbol{\mu}}^{\triangle}]\}$ and $\{[\mathbb{D}_{\boldsymbol{\mu}}^{\triangle}]\}$ bases.) Since $\langle[Y_{\boldsymbol{\mu}}^{\triangle}], [D_{\boldsymbol{\nu}}^{\triangle}]\rangle = \delta_{\boldsymbol{\mu}\boldsymbol{\nu}}$, by (6C.1), these transition matrices are inverse to each other by Corollary 6E.13. Finally, if $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangle}$ then

$$\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q) = \left\langle \left[Y_{\boldsymbol{\mu}}^{\triangle}\right], \left[S_{\boldsymbol{\lambda}}^{\triangle}\right] \right\rangle = \left\langle \sum_{\boldsymbol{\nu} \in \mathcal{K}_n^{\triangle}} \mathbb{o}_{\boldsymbol{\nu}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q)\mathbb{Y}_{\boldsymbol{\nu}}^{\triangle}, \left[S_{\boldsymbol{\lambda}}^{\triangle}\right] \right\rangle = \sum_{\boldsymbol{\nu} \in \mathcal{K}_n^{\triangle}} \mathbb{o}_{\boldsymbol{\nu}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q) \left\langle \mathbb{Y}_{\boldsymbol{\nu}}^{\triangle}, [S_{\boldsymbol{\lambda}}^{\triangle}] \right\rangle$$

$$= \sum_{\boldsymbol{\nu} \in \mathcal{K}_n^{\triangle}} \mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\triangle}(q)\mathbb{o}_{\boldsymbol{\nu}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q),$$

where the third equality follows because $\overline{\mathbb{o}_{\boldsymbol{\nu}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q)} = \mathbb{o}_{\boldsymbol{\nu}\boldsymbol{\mu}}^{\mathbb{K}\triangle}(q)$ is bar invariant. This gives the required factorisation of the decomposition matrices $\mathsf{d}^{\triangle}$. $\qquad\square$

As a consequence, we recover the Ariki–Brundan–Kleshchev categorification theorem.

Corollary 6E.17 (Brundan and Kleshchev [11, Theorem 5.3 and Corollary 5.15])**.** Let Γ be a quiver of type $A_{e-1}^{(1)}$ and suppose that $\mathbb{K}$ is a field of characteristic 0. Then

$$[Y_{\boldsymbol{\mu}}^{\lhd}] = \mathbb{Y}_{\boldsymbol{\mu}}^{\lhd}, \qquad [Y_{\boldsymbol{\nu}}^{\rhd}] = \mathbb{Y}_{\boldsymbol{\nu}}^{\rhd}, \qquad [D_{\boldsymbol{\mu}}^{\lhd}] = \mathbb{D}_{\boldsymbol{\mu}}^{\lhd} \qquad \text{and} \qquad [D_{\boldsymbol{\nu}}^{\rhd}] = \mathbb{D}_{\boldsymbol{\nu}}^{\rhd}.$$

for all $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ and all $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$. Consequently, if $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$ then

$$\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) = \langle \mathbb{Y}_{\boldsymbol{\mu}}^{\lhd}, [S_{\boldsymbol{\lambda}}^{\lhd}] \rangle \qquad \text{and} \qquad \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q) = \langle \mathbb{Y}_{\boldsymbol{\nu}}^{\rhd}, [S_{\boldsymbol{\lambda}}^{\rhd}] \rangle.$$

In particular, $\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q) = \mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd}(q) \in \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} + q\mathbb{N}[q]$ if $\boldsymbol{\lambda} \in \mathcal{K}_n^{\lhd}$ and $\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q) = \mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\rhd}(q) \in \delta_{\boldsymbol{\lambda}\boldsymbol{\nu}} + q\mathbb{N}[q]$ if $\boldsymbol{\lambda} \in \mathcal{K}_n^{\rhd}$.

Proof. Let $\triangle \in \{\triangleleft, \triangleright\}$. The algebras $\mathscr{R}_n^\Lambda(\mathbb{K}) \cong \mathsf{R}_n^\Lambda(\mathbb{K})$ are cellular by Corollary 4F.4, so every field is a splitting field for $\mathsf{R}_n^\Lambda(\mathbb{K})$, so we can assume that $\mathbb{K} = \mathbb{C}$. In type $A_{e-1}^{(1)}$, Brundan and Kleshchev [10] proved that the cyclotomic KLR algebra $\mathscr{R}_n^\Lambda(\mathbb{C})$ is isomorphic to a (degenerate) Ariki–Koike algebra $\mathscr{H}_n^\Lambda(\mathbb{C})$. Ariki [1, Theorem 4.4(2)], and Brundan and Kleshchev [12, Theorem 3.10] in the degenerate case, proved that the dual canonical basis of $[\mathrm{Rep}_\mathbb{C}\, \mathsf{R}_n^\Lambda(\mathbb{C}[x])]$ at $q = 1$ coincides with the basis of $[\mathrm{Rep}\, \mathscr{H}_n^\bullet] = \bigoplus_{n \geq 0}[\mathrm{Rep}\, \mathscr{H}_n^\Lambda(\mathbb{C})]$ given by the images of the irreducible $\mathscr{H}_n^\Lambda$-modules. Therefore, $\mathbb{D}_{\boldsymbol{\mu}}^\triangle = [D_{\boldsymbol{\mu}}^\triangle]$, for $\boldsymbol{\mu} \in \mathcal{K}_n^\triangle$, since the simple module $D_{\boldsymbol{\mu}}^\triangle$ is self-dual by Theorem 4B.6. The remaining claims now follow in view of Theorem 6E.10 and Lemma 6E.7. □

Example 6E.18. Given Corollary 6E.17, in type $C_{e-1}^{(1)}$ it is natural to ask if the ⊛-canonical bases of $L_\mathcal{A}(\Lambda)^*$ coincide with the bases of simple modules, and the #-canonical bases with the bases of principal indecomposable $\mathsf{R}_n^\Lambda(\mathbb{K})$-modules when $\mathbb{K}$ is a field of characteristic zero. It is shown in [17] that this first fails for the principal block of $\mathsf{R}_8^{\Lambda_0}(\mathbb{C})$ when Γ is a quiver of type $C_2^{(1)}$. Several other examples are given where the canonical bases do not coincide with the natural bases of these Grothendieck groups in type C, including an example when $n = 13$ that shows that the graded decomposition numbers of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ are not necessarily polynomials, even in characteristic zero.

The transition matrices $(\mathfrak{o}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\triangleleft}(q))$, $(\mathfrak{o}_{\boldsymbol{\lambda\nu}}^{\mathbb{K}\triangleright}(q))$, $(\mathfrak{b}_{\boldsymbol{\lambda\mu}}^{\mathbb{K}\triangleleft}(q))$ and $(\mathfrak{b}_{\boldsymbol{\lambda\nu}}^{\mathbb{K}\triangleright}(q))$ in Theorem 6E.16 are analogues of the adjustment matrices of Definition 5C.3. These matrices express the decomposition matrices of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$ in terms of the canonical bases and dual canonical bases. By taking inverses, similar "adjustment matrix" identities hold for the inverse decomposition matrices.

Recall the Mullineux involution $\mathsf{m}\colon \mathcal{K}_n^\triangleleft \longrightarrow \mathcal{K}_n^\triangleright$ from Definition 5D.1. The next result should be compared with Proposition 5D.3.

Proposition 6E.19. Let $\boldsymbol{\mu} \in \mathcal{K}_n^\triangleleft$. Then $\mathbb{Y}_{\boldsymbol{\mu}}^\triangleleft = \mathbb{Y}_{\mathsf{m}(\boldsymbol{\mu})}^\triangleright$ and $\mathbb{D}_{\boldsymbol{\mu}}^\triangleleft = \mathbb{D}_{\mathsf{m}(\boldsymbol{\mu})}^\triangleright$. Moreover, if $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ then $\mathfrak{d}_{\boldsymbol{\lambda\mu}}^\triangleleft(q) = q^{\mathrm{def}\,\boldsymbol{\lambda}}\overline{\mathfrak{d}_{\boldsymbol{\lambda}\mathsf{m}(\boldsymbol{\mu})}^\triangleright(q)}$.

Proof. By Definition 5D.1, $[D_{\boldsymbol{\mu}}^\triangleleft] = [D_{\mathsf{m}(\boldsymbol{\mu})}^\triangleright]$ and $[Y_{\boldsymbol{\mu}}^\triangleleft] = [Y_{\mathsf{m}(\boldsymbol{\mu})}^\triangleright]$. Hence, $\mathbb{Y}_{\boldsymbol{\mu}}^\triangleleft = \mathbb{Y}_{\mathsf{m}(\boldsymbol{\mu})}^\triangleright$ and $\mathbb{D}_{\boldsymbol{\mu}}^\triangleleft = \mathbb{D}_{\mathsf{m}(\boldsymbol{\mu})}^\triangleright$ by Theorem 6E.16 and the uniqueness of the canonical basis elements established in Theorem 6E.10. To prove the remaining claim, if $\boldsymbol{\mu} \in \mathcal{K}_n^\triangleleft$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ then

$$\mathfrak{d}_{\boldsymbol{\lambda\mu}}^\triangleleft(q) = \left\langle \mathbb{Y}_{\boldsymbol{\mu}}^\triangleleft, [S_{\boldsymbol{\lambda}}^\triangleleft] \right\rangle = q^{\mathrm{def}\,\boldsymbol{\lambda}} \left\langle \mathbb{Y}_{\mathsf{m}(\boldsymbol{\mu})}^\triangleright, [S_{\boldsymbol{\lambda}}^\triangleright]^\circledast \right\rangle = q^{\mathrm{def}\,\boldsymbol{\lambda}} \overline{\left\langle \mathbb{Y}_{\mathsf{m}(\boldsymbol{\mu})}^\triangleright, [S_{\boldsymbol{\lambda}}^\triangleright] \right\rangle} = q^{\mathrm{def}\,\boldsymbol{\lambda}} \overline{\mathfrak{d}_{\mathsf{m}(\boldsymbol{\mu})\boldsymbol{\lambda}}^\triangleright(q)},$$

where we have used Proposition 4F.9 and Lemma 6E.4. □

Combining Theorem 6E.10 and Proposition 6E.19, we obtain.

Corollary 6E.20. Let $\boldsymbol{\mu} \in \mathcal{K}_n^\triangleleft$, $\boldsymbol{\nu} \in \mathcal{K}_n^\triangleleft$ and $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{K}_n^\triangleleft \cup \mathcal{K}_n^\triangleright$.
    (a) If $\mathfrak{d}_{\boldsymbol{\lambda\mu}}^\triangleleft(q) \neq 0$ then $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda} \trianglelefteq \mathsf{m}(\boldsymbol{\mu})$ and $\alpha_{\boldsymbol{\lambda}} = \alpha_{\boldsymbol{\mu}}$. Moreover, $\mathfrak{d}_{\boldsymbol{\mu\mu}}^\triangleleft(q) = 1$, $\mathfrak{d}_{\mathsf{m}(\boldsymbol{\mu})\boldsymbol{\mu}}^\triangleleft(q) = q^{\mathrm{def}\,\boldsymbol{\mu}}$ and if $\mathsf{m}(\boldsymbol{\mu}) \triangleleft \boldsymbol{\lambda} \triangleleft \boldsymbol{\mu}$ then $0 < \deg \mathfrak{d}_{\boldsymbol{\lambda\mu}}^\triangleleft(q) < \mathrm{def}\,\boldsymbol{\mu}$.
    (b) If $\mathfrak{d}_{\boldsymbol{\lambda\nu}}^\triangleright(q) \neq 0$ then $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda} \trianglerighteq \mathsf{m}(\boldsymbol{\mu})$ and $\alpha_{\boldsymbol{\lambda}} = \alpha_{\boldsymbol{\mu}}$. Moreover, $\mathfrak{d}_{\boldsymbol{\mu\mu}}^\triangleright(q) = 1$, $\mathfrak{d}_{\mathsf{m}^{-1}(\boldsymbol{\mu})\boldsymbol{\mu}}^\triangleright(q) = q^{\mathrm{def}\,\boldsymbol{\mu}}$ and if $\mathsf{m}^{-1}(\boldsymbol{\mu}) \triangleright \boldsymbol{\lambda} \triangleright \boldsymbol{\mu}$ then $0 < \deg \mathfrak{d}_{\boldsymbol{\lambda\nu}}^\triangleright(q) < \mathrm{def}\,\boldsymbol{\mu}$.

Proof. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_n^\triangleleft$ then $\mathfrak{d}_{\boldsymbol{\lambda\mu}}^\triangleleft(q) \in \delta_{\boldsymbol{\lambda\mu}} + q\mathbb{Z}[q]$ by Theorem 6E.10. Hence, the only claim in (a) that is not immediate from Proposition 6E.19 is that $0 < \deg \mathfrak{d}_{\boldsymbol{\lambda\mu}}^\triangleleft(q) < \mathrm{def}\,\boldsymbol{\mu}$ when $\boldsymbol{\lambda} \in \mathcal{K}_n^\triangleright$ and $\boldsymbol{\lambda} \notin \{\boldsymbol{\mu}, \mathsf{m}(\boldsymbol{\mu})\}$. In this case, $\mathfrak{d}_{\boldsymbol{\lambda}\mathsf{m}(\boldsymbol{\mu})}^\triangleright(q) \in \delta_{\boldsymbol{\lambda}\mathsf{m}(\boldsymbol{\mu})} + q\mathbb{Z}[q]$, so $0 < \deg \mathfrak{d}_{\boldsymbol{\lambda\mu}}^\triangleleft(q) < \mathrm{def}\,\boldsymbol{\mu}$ by Proposition 6E.19. This proves (a). The proof of (b) is similar. □

Later, we will show that this result is true for $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{P}_n^\ell$. There are similar identities for the polynomials $\mathbb{e}_{\boldsymbol{\mu}\boldsymbol{\lambda}}^\lhd(-q)$ and $\mathbb{e}_{\boldsymbol{\nu}\boldsymbol{\lambda}}^\rhd(-q)$, which we leave for the reader.

Corollary 6E.21. *Let $\boldsymbol{\lambda} \in \mathcal{P}_\alpha^\ell$, for $\alpha \in Q_n^+$. Then $\operatorname{def} \alpha = \operatorname{def} \boldsymbol{\lambda} \geq 0$.*

Proof. This is implicit in Corollary 6E.20 since $\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^\lhd(q)$ and $\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^\rhd(q)$ are polynomials. $\qquad\square$

6F. **Crystal bases of Fock spaces.** The categorification results of the last few sections imply that the number of self-dual graded simple modules is independent of the characteristic, but we have not yet determined the sets $\mathcal{K}_n^\lhd$ and $\mathcal{K}_n^\rhd$ that index the simple $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules. To do this we now describe the crystal graphs of $\mathscr{L}_{\mathcal{A}}^\lhd(\Lambda)$ and $\mathscr{L}_{\mathcal{A}}^\rhd(\Lambda)$. We start by recalling Kashiwara's theory of global and crystal bases and Lusztig's theory of canonical bases.

Suppose that V be an integrable highest weight module for $U_q(\mathfrak{g}_\Gamma)$. If $i \in I$ then E_i and F_i act on V as locally nilpotent linear operators. Therefore, by [47, 16.1.4], each weight vector $v \in V$ can be written uniquely in the form

$$v = \sum_{r \geq 0} F_i^{(r)} v_r$$

such that $E_i v_r = 0$ and $K_i v_r = q^{\langle \operatorname{wt}(v_r), \alpha_i \rangle + r\mathsf{d}_i} v_r$, for $r \geq 0$. For $i \in I$, the *Kashiwara operators* e_i and f_i are the linear endomorphisms of V defined by

$$e_i v = \sum_{r \geq 1} F_i^{(r-1)} v_r \qquad \text{and} \qquad f_i v = \sum_{r \geq 0} F_i^{(r+1)} v_r. \qquad (6F.1)$$

For $\mathbf{i} \in I^n$ set $e_{\mathbf{i}} = e_{i_n} \dots e_{i_2} e_{i_1}$ and $f_{\mathbf{i}} = f_{i_n} \dots f_{i_2} f_{i_1}$.

Let $\mathbb{A}_0$ be the subring of rational functions $\mathbb{A} = \mathbb{Q}(q)$ that are regular at zero and let $\mathbb{A}_\infty$ be the rational function that are regular at infinity. To allow us to work with these two rings simultaneously, if $\omega \in \{0, \infty\}$ set

$$q_\omega = \begin{cases} q & \text{if } \omega = 0, \\ q^{-1} & \text{if } \omega = \infty. \end{cases}$$

Definition 6F.2 (Kashiwara [33, Definition 2.3.1]). Let V be an integrable $U_q(\mathfrak{g}_\Gamma)$-module. Fix $\omega \in \{0, \infty\}$. A ω-*crystal base* of V is a pair $(\mathcal{L}_\omega, \mathcal{B}_\omega)$ such that:

(a) The module $\mathcal{L}_\omega$ is a free $\mathbb{A}_\omega$-submodule of V such that $V \cong \mathbb{A} \otimes_{\mathbb{A}_\omega} \mathcal{L}_\omega$ and $\mathcal{L}_\omega$ is a direct sum of $U_q(\mathfrak{g}_\Gamma)$-weight spaces and it is invariant under the actions of e_i and f_i, for $i \in I$.
(b) The set $\mathcal{B}_\omega$ is a basis of the $\mathbb{Q}$-vector space $\mathcal{L}_\omega / q_\omega \mathcal{L}_\omega = \langle \mathcal{B}_\omega \rangle_{\mathbb{Q}}$.
(c) The elements of $\mathcal{B}_\omega$ are images of weight vectors under the map $\mathcal{L}_\omega \to \mathcal{L}_\omega / q_\omega \mathcal{L}_\omega$.
(d) If $i \in I$ then $e_i \mathcal{B}_\omega \subset \mathcal{B}_\omega \cup \{0\}$ and $f_i \mathcal{B}_\omega \subset \mathcal{B}_\omega \cup \{0\}$.
(e) If $b, b' \in \mathcal{B}_\omega$ and $i \in I$ then $e_i b = b'$ if and only if $f_i b' = b$.

This section describes the 0-crystal base $(\mathcal{L}_0, \mathcal{B}_0)$ and the ∞-crystal base $(\mathcal{L}_\infty, \mathcal{B}_\infty)$ of $L(\Lambda)$.

If $V = U_q(\mathfrak{g}_\Gamma) v_\Lambda$ is an integrable highest weight module with highest weight vector v_Λ then, as in subsection 6E, the bar involution on V is defined to be the unique semilinear automorphism such that $\overline{v_\Lambda} = v_\Lambda$ and $\overline{av} = \bar{a}\,\bar{v}$, for all $v \in V$ and $a \in U_q(\mathfrak{g}_\Gamma)$.

Theorem 6F.3 (Lusztig [47, §14.4], Kashiwara [33])**.** Let V be an integrable $U_q(\mathfrak{g}_\Gamma)$-module. Fix $\omega \in \{0, \infty\}$ and suppose that $(\mathcal{L}_\omega, \mathcal{B}_\omega)$ is an ω-crystal basis for V. Then there exists a unique $\mathcal{A}$-basis $\mathcal{B}_\omega(\Lambda) = \{G_{\omega,b} \mid b \in \mathcal{B}_\omega(\Lambda)\}$ of $V_\mathcal{A}(\Lambda)$ such that $\overline{G_{\omega,b}} = G_{\omega,b}$ and $G_{\omega,b} \equiv b \pmod{q_\omega \mathcal{L}_\omega(\Lambda)}$, for $b \in \mathcal{B}_\omega(\Lambda)$.

The basis $\mathcal{B}_0(\Lambda)$ of $V(\Lambda)$ is Lusztig's *dual canonical basis*, or Kashiwara's *lower global basis* and the basis $\mathcal{B}_\infty(\Lambda)$ is Lusztig's *canonical basis*, or Kashiwara's *upper global basis*.

To apply these results to the combinatorial Fock spaces $\mathscr{L}_\mathcal{A}^\lhd(\Lambda)$ and $\mathscr{L}_\mathcal{A}^\rhd(\Lambda)$, and the Grothendieck groups $[\mathrm{Proj}_\mathbb{K} \, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$ and $[\mathrm{Rep}_\mathbb{K} \, \mathsf{R}_\bullet^\Lambda(\mathbb{K}[x])]$, we first generalise the integers $d_A^\lhd(\boldsymbol{\lambda})$ and $d_A^\rhd(\boldsymbol{\lambda})$ from Definition 4D.3. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_\bullet^\ell$ and $i \in I$ write $\boldsymbol{\lambda} \xrightarrow{i^r} \boldsymbol{\mu}$ if $|\boldsymbol{\mu}| = |\boldsymbol{\lambda}| + r$ and $\boldsymbol{\mu} = \boldsymbol{\lambda} \cup \{A_1, \ldots, A_r\}$, where $\{A_1, \ldots, A_r\} \subseteq \mathrm{Add}_i(\boldsymbol{\lambda})$, and define

$$d_{\boldsymbol{\mu}}^\lhd(\boldsymbol{\lambda}) = \mathsf{d}_i \sum_{s=1}^r \Big(\#\{B \in \mathrm{Add}_i(\boldsymbol{\mu}) \mid B < A_s\} - \#\{B \in \mathrm{Rem}_i(\boldsymbol{\lambda}) \mid B < A_s\}\Big),$$

$$d_{\boldsymbol{\mu}}^\rhd(\boldsymbol{\lambda}) = \mathsf{d}_i \sum_{s=1}^r \Big(\#\{B \in \mathrm{Add}_i(\boldsymbol{\lambda}) \mid B > A_s\} - \#\{B \in \mathrm{Rem}_i(\boldsymbol{\lambda}) \mid B > A_s\}\Big).$$

By definition, if $\boldsymbol{\mu} = \boldsymbol{\lambda} \cup \{A\}$, for $A \in \mathrm{Add}_i(\boldsymbol{\lambda})$, then $d_{\boldsymbol{\mu}}^\lhd(\boldsymbol{\lambda}) = d_A^\lhd(\boldsymbol{\lambda})$ and $d_{\boldsymbol{\mu}}^\rhd(\boldsymbol{\lambda}) = d_A^\rhd(\boldsymbol{\lambda})$.

Lemma 6F.4. Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $i \in I$. Then, for $r \geq 0$,

$$F_i^{(r)} \mathsf{s}_{\boldsymbol{\lambda}}^\lhd = \sum_{\boldsymbol{\lambda} \xrightarrow{i^k} \boldsymbol{\mu}} q^{-d_{\boldsymbol{\mu}}^\rhd(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\mu}}^\lhd \qquad \text{and} \qquad F_i^{(r)} \mathsf{s}_{\boldsymbol{\lambda}}^\rhd = \sum_{\boldsymbol{\lambda} \xrightarrow{i^k} \boldsymbol{\mu}} q^{-d_{\boldsymbol{\mu}}^\lhd(\boldsymbol{\lambda})} \mathsf{s}_{\boldsymbol{\mu}}^\rhd$$

Proof. This follows easily by induction on r using the fact that $F_i^{(r+1)} = [r+1] F_i^{(r)}$; see [49, Lemma 6.15] for a similar argument. The base case for the induction is given by Corollary 6D.4. $\qquad\square$

Definition 6F.5 (Normal and good nodes)**.** Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $i \in I$.
   (a) A removable i-node $A \in \mathrm{Rem}_i(\boldsymbol{\lambda})$ is $\lhd$-*normal* if $d_A^\lhd(\boldsymbol{\lambda}) \leq 0$ and $d_A^\lhd(\boldsymbol{\lambda}) < d_B^\lhd(\boldsymbol{\lambda})$ if $B < A$, for $B \in \mathrm{Rem}_i(\boldsymbol{\lambda})$.
   (b) A normal i-node A is $\lhd$-*good* if $A \leq B$ whenever B is a $\lhd$-normal i-node. Equivalently, A is a $\lhd$-good i-node if $d_A^\lhd(\boldsymbol{\lambda}) \leq d_B^\lhd(\boldsymbol{\lambda})$ for all $B \in \mathrm{Rem}_i(\boldsymbol{\lambda})$ with equality only if $A \leq B$.
   (c) A removable j-node $A \in \mathrm{Rem}_j(\boldsymbol{\lambda})$ is $\rhd$-*normal* if $d_A^\rhd(\boldsymbol{\lambda}) \leq 0$ and $d_A^\rhd(\boldsymbol{\lambda}) < d_B^\rhd(\boldsymbol{\lambda})$ if $B > A$, for $B \in \mathrm{Rem}_j(\boldsymbol{\lambda})$.
   (d) A normal j-node A is $\rhd$-*good* if $A \geq B$ whenever B is a $\rhd$-normal j-node. Equivalently, A is a good i-node if $d_A^\rhd(\boldsymbol{\lambda}) \leq d_B^\rhd(\boldsymbol{\lambda})$ for all $B \in \mathrm{Rem}_i(\boldsymbol{\lambda})$ with equality only if $A \geq B$.

If $\boldsymbol{\mu} = \boldsymbol{\lambda} + A$ write $\boldsymbol{\lambda} \overset{i\lhd}{\rightsquigarrow} \boldsymbol{\mu}$ if A is a $\lhd$-good i-node of $\boldsymbol{\mu}$ and write $\boldsymbol{\lambda} \overset{j\rhd}{\rightsquigarrow} \boldsymbol{\nu}$ if A is an $\rhd$-good j-node of $\boldsymbol{\nu}$. More generally, if $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}_n^\ell$ and $\mathbf{i}, \mathbf{j} \in I^n$, write $\underline{\mathbf{0}}_\ell \overset{\mathbf{i}\lhd}{\rightsquigarrow} \boldsymbol{\mu}$ and $\underline{\mathbf{0}}_\ell \overset{\mathbf{j}\rhd}{\rightsquigarrow} \boldsymbol{\nu}$ if there exist ℓ-partitions $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n = \boldsymbol{\mu}$ and $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_n = \boldsymbol{\nu}$ such that

$$\underline{\mathbf{0}}_\ell \overset{i_1\lhd}{\rightsquigarrow} \boldsymbol{\mu}_1 \overset{i_2\lhd}{\rightsquigarrow} \ldots \overset{i_n\lhd}{\rightsquigarrow} \boldsymbol{\mu}_n = \boldsymbol{\mu} \qquad \text{and} \qquad \underline{\mathbf{0}}_\ell \overset{j_1\rhd}{\rightsquigarrow} \boldsymbol{\nu}_1 \overset{j_2\rhd}{\rightsquigarrow} \ldots \overset{j_n\rhd}{\rightsquigarrow} \boldsymbol{\nu}_n = \boldsymbol{\nu},$$

respectively.

There is a dual definition for conormal and cogood nodes.

Definition 6F.6 (Conormal and cogood nodes)**.** Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $i \in I$.

(a) An addable i-node $A \in \mathrm{Add}_i(\boldsymbol{\lambda})$ is ◁-*conormal* if $d_A^\triangleleft(\boldsymbol{\lambda}) \geq 0$ and $d_A(\boldsymbol{\lambda}) > d_B(\boldsymbol{\lambda})$ if $A < B$, for $B \in \mathrm{Add}_i(\boldsymbol{\lambda})$.

(b) A normal i-node A is ◁-*cogood* if $A \geq B$ whenever B is a ◁-normal i-node.

(c) An addable j-node $A \in \mathrm{Add}_j(\boldsymbol{\lambda})$ is ▷-*conormal* if $d_j^\triangleright(\boldsymbol{\lambda}) \geq 0$ and $d_A(\boldsymbol{\lambda}) > d_B(\boldsymbol{\lambda})$ if $A > B$, for $B \in \mathrm{Add}_j(\boldsymbol{\lambda})$.

(d) A normal j-node A is ▷-*cogood* if $A \leq B$ whenever B is a ▷-normal j-node.

In particular, if $\boldsymbol{\mu} = \boldsymbol{\lambda} \cup A$ then A is a good i-node of $\boldsymbol{\mu}$ if and only if A is a cogood i-node of $\boldsymbol{\lambda}$.

Normal and conormal nodes are often defined by listing the addable and removable i-nodes for $\boldsymbol{\lambda}$ lexicographically and then recursively deleting all adjacent addable-removable pairs for ◁-normal nodes, and removable-addable pairs for ▷-normal nodes. After all such pairs have been removed, the normal nodes are the removable nodes that remain and the conormal nodes are the addable nodes. It is slightly tedious, but straightforward, to check that these descriptions of normal and conormal nodes are equivalent to the two definitions above; compare with [3, Lemma 11.2].

Example 6F.7. Consider the partition $\lambda = (4,3,1)$ for the algebra $\mathsf{R}_6^{\Lambda_0}(\mathbb{K}[x])$ of type $C_2^{(1)}$. The type $C_2^{(1)}$ residues in λ are given by the diagram:

0	1	2	1
1	0	1
2

Then

$$\underline{\mathbf{0}}_\ell \xrightarrow{0\triangleleft} (1) \xrightarrow{1\triangleleft} (2) \xrightarrow{1\triangleleft} (2,1) \xrightarrow{0\triangleleft} (2^2) \xrightarrow{2\triangleleft} (3,2) \xrightarrow{2\triangleleft} (3,2,1) \xrightarrow{1\triangleleft} (4,2,1) \xrightarrow{1\triangleleft} (4,3,1).$$

It follows from Theorem 6F.14 below that $D_{(4,3,1)}^\triangleleft \neq 0$. In contrast,

$$(3) \xrightarrow{1\triangleright} (3,1) \xrightarrow{0\triangleright} (3,2) \xrightarrow{1\triangleright} (3^2) \xrightarrow{1\triangleright} (4,3) \xrightarrow{2\triangleright} (4,3,1).$$

The partition (3) does not have any ▷-normal nodes, so $D_{(4,3,1)}^\triangleright = 0$ by Theorem 6F.14.

Analogues of the next result are well-known. Given its importance to the main results of this paper we give the proof, following [49, Theorem 6.17]. Perhaps unexpectedly, the result mixes up the dominance and reverse dominance partial orders.

Theorem 6F.8. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$ and $i \in I$.

(a) If $\boldsymbol{\lambda}$ does not have a ▷-good j-node then $e_j \mathsf{s}_{\boldsymbol{\lambda}}^\triangleleft \in q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda\triangleleft}$.

(b) If $\boldsymbol{\lambda} \xrightarrow{j\triangleright} \boldsymbol{\mu}$ then $e_j \mathsf{s}_{\boldsymbol{\mu}}^\triangleleft = \mathsf{s}_{\boldsymbol{\lambda}}^\triangleleft \pmod{q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda\triangleleft}}$ and $f_j \mathsf{s}_{\boldsymbol{\lambda}}^\triangleleft = \mathsf{s}_{\boldsymbol{\mu}}^\triangleleft \pmod{q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda\triangleleft}}$.

(c) If $\boldsymbol{\lambda}$ does not have a ▷-good i-node then $e_i \mathsf{s}_{\boldsymbol{\lambda}}^\triangleright \in q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda\triangleright}$.

(d) If $\boldsymbol{\lambda} \xrightarrow{i\triangleleft} \boldsymbol{\mu}$ then $e_i \mathsf{s}_{\boldsymbol{\mu}}^\triangleright = \mathsf{s}_{\boldsymbol{\lambda}}^\triangleright \pmod{q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda\triangleright}}$ and $f_i \mathsf{s}_{\boldsymbol{\lambda}}^\triangleright = \mathsf{s}_{\boldsymbol{\mu}}^\triangleright \pmod{q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda\triangleright}}$.

Proof. We prove only parts (a) and (b) as the proofs of (c) and (d) follow by symmetry. First suppose that $\boldsymbol{\lambda}$ does not have a ▷-good i-node. If $A \in \mathrm{Rem}_i(\boldsymbol{\lambda})$ then $d_A^\triangleright(\boldsymbol{\lambda}) > 0$, so there are at least as many addable i-nodes below A as there are removable i-nodes. Let $\check{A}$ be the highest addable i-node of $\boldsymbol{\lambda}$ such that $A < \check{A}$ and $d_{\check{A}}^\triangleleft(\boldsymbol{\lambda}) = d_A^\triangleleft(\boldsymbol{\lambda}) + 1$. As $d_A^\triangleleft(\boldsymbol{\lambda}) > 0$ the node $\check{A}$ always exists and if $A, B \in \mathrm{Rem}_i(\boldsymbol{\lambda})$ then $\check{A} = \check{B}$ if and only if $A = B$. If $M \subseteq \mathrm{Rem}_i(\boldsymbol{\lambda})$ let $\check{\boldsymbol{\lambda}}_M = \boldsymbol{\lambda} - M + \check{M}$, where $\check{M} = \{\check{A} \mid A \in M\}$. That is, $\check{\boldsymbol{\lambda}}_M$ is

the ℓ-partition obtained from $\boldsymbol{\lambda}$ by removing the i-nodes in M from $\boldsymbol{\lambda}$ and then adding on the nodes in $\check{M}$. In particular, $|\check{\boldsymbol{\lambda}}_M| = |\boldsymbol{\lambda}|$. Now set

$$\check{\Omega}_i(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}) = \sum_{M \subseteq \mathrm{Rem}_i(\boldsymbol{\lambda})} (-q)^{-|M|} \mathsf{s}_{\check{\boldsymbol{\lambda}}_M}^{\lhd} \in \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda \lhd}.$$

By Corollary 6D.4, $\mathsf{s}_{\boldsymbol{\nu}}^{\lhd}$ appears in $E_i \check{\Omega}_i(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd})$ only if $\mathrm{Rem}_i(\boldsymbol{\nu}) = \check{M} \cup N$ where $\mathrm{Rem}_i(\boldsymbol{\lambda}) = M \sqcup N \sqcup \{A\}$ (disjoint union). Now, $\mathsf{s}_{\boldsymbol{\nu}}^{\lhd}$ appears in $E_i \mathsf{s}_{\check{\boldsymbol{\lambda}}_M}^{\lhd}$ and in $E_i \mathsf{s}_{\check{\boldsymbol{\lambda}}_{M \cup \{A\}}}^{\lhd}$, and its coefficient in $E_i \Omega_i(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd})$ is

$$(-q)^{-|M| + d_A^{\lhd}(\boldsymbol{\lambda}_M)} + (-q)^{-|M| - 1 + d_A^{\lhd}(\boldsymbol{\lambda}_{M \cup \{A\}})} = 0,$$

where the last equality follows because $d_{\check{A}}^{\lhd}(\boldsymbol{\lambda}_M) = d_{\check{A}}^{\lhd}(\boldsymbol{\lambda}) = d_A^{\lhd}(\boldsymbol{\lambda}) + 1 = d_A^{\lhd}(\boldsymbol{\lambda}_M)$, which is the key identity underpinning this theorem. Hence, $E_i \check{\Omega}_i(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}) = 0$ and, consequently, $e_i \check{\Omega}_i(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}) = 0$ by (6F.1). Therefore,

$$e_i \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} \equiv e_i \check{\Omega}_i(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}) = 0 \left(\mathrm{mod}\ q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda \lhd} \right),$$

proving (a).

To prove (b) we continue to assume that $\boldsymbol{\lambda}$ has no $\rhd$-normal i-nodes and compute $f_i^r \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}$, for $r \geq 0$. Using the notation above, set

$$\mathcal{N}_i(\boldsymbol{\lambda}) = \{A \in \mathrm{Add}_i(\boldsymbol{\lambda}) \mid A \neq \check{B} \text{ for any } B \in \mathrm{Add}_i(\boldsymbol{\lambda})\} = \{A_1 > \cdots > A_z\}.$$

Observe that $z = \#\mathcal{N}_i(\boldsymbol{\lambda}) = d_i(\boldsymbol{\lambda})$ and that $s = d_{A_s}^{\rhd}(\boldsymbol{\lambda})$, for $1 \leq s \leq z$. So, $\mathcal{N}_i(\boldsymbol{\lambda})$ is the set of $\rhd$-conormal i-nodes of $\boldsymbol{\lambda}$.

For $K \subseteq \mathrm{Add}_i(\boldsymbol{\nu})$ let $\boldsymbol{\nu} + K$ be the ℓ-partition $\boldsymbol{\nu} \cup K$. Using Equation 6F.1 for the first congruence, and Lemma 6F.4 for the following equality,

$$f_i^r \mathsf{s}_{\boldsymbol{\lambda}}^{\lhd} \equiv F_i^{(r)} \check{\Omega}_i(\mathsf{s}_{\boldsymbol{\lambda}}^{\lhd}) \quad \left(\mathrm{mod}\ q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda \lhd} \right)$$

$$= \sum_{M \subseteq \mathrm{Rem}_i(\boldsymbol{\lambda})} (-q)^{-|M|} \sum_{\substack{K \subseteq \mathrm{Add}_i(\check{\boldsymbol{\lambda}}_M) \\ |K| = r}} q^{-d_{\check{\boldsymbol{\lambda}}_M + K}^{\rhd}(\boldsymbol{\lambda})(\check{\boldsymbol{\lambda}}_M)} \mathsf{s}_{\check{\boldsymbol{\lambda}}_M + K}^{\lhd}$$

$$= \sum_{M \subseteq \mathrm{Rem}_i(\boldsymbol{\lambda})} (-q)^{-|M|} \sum_{K \subseteq \mathrm{Add}_i(\boldsymbol{\lambda}) \setminus \check{M}, |K| = r} q^{-d_{\check{\boldsymbol{\lambda}}_M + K}^{\rhd}(\boldsymbol{\lambda})(\check{\boldsymbol{\lambda}}_M)} \mathsf{s}_{\check{\boldsymbol{\lambda}}_M + K}^{\lhd}$$

$$= \sum_{\substack{K \subseteq \mathrm{Add}_i(\boldsymbol{\lambda}) \\ |K| = r}} \sum_{\substack{M \subseteq \mathrm{Rem}_i(\boldsymbol{\lambda}) \\ \check{M} \cap K = \emptyset}} (-q)^{-|M| - d_{\boldsymbol{\lambda} + K}^{\rhd}(\boldsymbol{\lambda})} \mathsf{s}_{\check{\boldsymbol{\lambda}}_M + K}^{\lhd}$$

$$\equiv \begin{cases} \mathsf{s}_{\boldsymbol{\lambda} + \{A_1, \ldots, A_r\}}^{\lhd} & \text{if } 1 \leq r \leq z, \\ 0 & \text{otherwise,} \end{cases}$$

where the last equation, which is modulo $q^{-1} \mathscr{F}_{\mathbb{A}_\infty}^{\Lambda \lhd}$, follows because if $K \neq \{A_1, \ldots, A_r\}$ or $M \neq \emptyset$ then $|M| - d_{\boldsymbol{\lambda} + K}^{\rhd}(\boldsymbol{\lambda}) > 0$. To complete the proof of (b) it remains to observe that A_r is the $\rhd$-good i-node of $\boldsymbol{\lambda} + \{A_1, \ldots, A_{r-1}\}$. $\qquad \square$

Definition 6F.9. Suppose that $\Lambda \in P^+$. Define

$$\mathsf{B}^{\lhd}(\Lambda) = \left\{ \boldsymbol{\mu} \,\middle|\, \boldsymbol{\mu} \in \mathcal{P}_n^{\ell} \text{ and } \underline{\mathbf{0}}_\ell \xrightarrow{\ \mathbf{i} \lhd\ } \boldsymbol{\mu} \text{ for some } \mathbf{i} \in I^n \text{ and } n \geq 0 \right\}$$

and

$$\mathsf{B}^{\triangleright}(\Lambda) = \left\{ \boldsymbol{\nu} \,\middle|\, \boldsymbol{\nu} \in \mathcal{P}_n^{\ell} \text{ and } \underline{\mathbf{0}}_{\ell} \overset{\mathbf{j}\triangleright}{\rightsquigarrow} \boldsymbol{\nu} \text{ for some } \mathbf{j} \in I^n \text{ and } n \geq 0 \right\}$$

and set

$$\mathscr{B}_{\infty}^{\triangleleft}(\Lambda) = \left\{ \mathsf{s}_{\boldsymbol{\nu}}^{\triangleleft} + q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda) \,\middle|\, \boldsymbol{\nu} \in \mathsf{B}^{\triangleright}(\Lambda) \right\}$$

and

$$\mathscr{B}_{\infty}^{\triangleright}(\Lambda) = \left\{ \mathsf{s}_{\boldsymbol{\mu}}^{\triangleright} + q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleright}(\Lambda) \,\middle|\, \boldsymbol{\mu} \in \mathsf{B}^{\triangleleft}(\Lambda) \right\}.$$

By definition, $\mathscr{B}_{\infty}^{\triangleleft}(\Lambda)$ is contained in $\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)/q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)$ and, similarly, $\mathscr{B}_{\infty}^{\triangleright}(\Lambda)$ is contained in $\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleright}(\Lambda)/q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleright}(\Lambda)$.

Corollary 6F.10. Let $\Lambda \in P^+$. Then $(\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda), \mathscr{B}_{\infty}^{\triangleright}(\Lambda))$ and $(\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleright}(\Lambda), \mathscr{B}_{\infty}^{\triangleleft}(\Lambda))$ are ∞-crystal bases of $L(\Lambda)$.

Proof. We only prove the result for $(\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda), \mathscr{B}_{\infty}^{\triangleright}(\Lambda))$. The only condition in Definition 6F.2 that is not clear from Theorem 6F.8 is that $\mathscr{B}_{\infty}^{\triangleright}(\Lambda)$ is a $\mathbb{Q}$-basis of

$$\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)/q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda).$$

Since $\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)$ is a highest weight module,

$$\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)/q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda) = \left\langle f_{\mathbf{i}}\mathsf{s}_{\underline{\mathbf{0}}_{\ell}}^{\triangleleft} + q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda) \,\middle|\, \mathbf{i} \in I^n \text{ for } n \geq 0 \right\rangle_{\mathbb{A}_{\infty}}.$$

Hence, it is enough to show that $\{ f_{\mathbf{i}}\mathsf{s}_{\underline{\mathbf{0}}_{\ell}}^{\triangleleft} + q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda) \,|\, \mathbf{i} \in I^n \}$ is spanned by

$$\left\{ \mathsf{s}_{\boldsymbol{\mu}}^{\triangleleft} + q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda) \,\middle|\, \boldsymbol{\mu} \in \mathsf{B}^{\triangleright}(\Lambda) \cap \mathcal{P}_n^{\ell} \text{ for } n \geq 0 \right\}.$$

We argue by induction on n. If $n = 0$ there is nothing to prove since $\mathsf{s}_{\underline{\mathbf{0}}_{\ell}}^{\triangleleft}$ is a highest weight vector in $\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)$. By way of induction, suppose that the claim is true for n and consider the statement for $n+1$. Fix $\boldsymbol{\mu} \in \mathsf{B}^{\triangleright}(\Lambda)$ and $\mathbf{i} \in I^n$ such that $\underline{\mathbf{0}}_{\ell} \overset{\mathbf{i}\triangleright}{\rightsquigarrow} \boldsymbol{\mu}$. By Theorem 6F.8, $f_i\mathsf{s}_{\boldsymbol{\lambda}}^{\triangleleft} \in q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)$ if and only if $\boldsymbol{\mu}$ has no $\triangleright$-conormal i-nodes and, moreover, if A is the $\triangleright$-cogood i-node then $f_i\mathsf{s}_{\boldsymbol{\mu}}^{\triangleleft} \equiv \mathsf{s}_{\boldsymbol{\mu}+A}^{\triangleleft} \pmod{q^{-1}\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)}$. This completes the proof of the inductive step and hence proves the corollary. $\qquad\square$

For $i, j \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ define functions $\varepsilon_i^{\triangleleft}, \varphi_i^{\triangleleft} \colon \mathsf{B}^{\triangleleft}(\Lambda) \longrightarrow \mathbb{Z}$ and $\varepsilon_j^{\triangleright}, \varphi_j^{\triangleright} \colon \mathsf{B}^{\triangleright}(\Lambda) \longrightarrow \mathbb{Z}$ by

$$\begin{aligned}
\varepsilon_i^{\triangleleft}(\boldsymbol{\mu}) &= \#\{ A \in \mathrm{Add}_i(\boldsymbol{\mu}) \,|\, A \text{ is } \triangleleft\text{-normal} \} \\
\varepsilon_i^{\triangleright}(\boldsymbol{\nu}) &= \#\{ A \in \mathrm{Add}_i(\boldsymbol{\nu}) \,|\, A \text{ is } \triangleright\text{-normal} \} \\
\varphi_i^{\triangleleft}(\boldsymbol{\mu}) &= \#\{ A \in \mathrm{Rem}_i(\boldsymbol{\mu}) \,|\, A \text{ is } \triangleleft\text{-conormal} \} \\
\varphi_i^{\triangleright}(\boldsymbol{\nu}) &= \#\{ A \in \mathrm{Rem}_i(\boldsymbol{\nu}) \,|\, A \text{ is } \triangleright\text{-conormal} \}
\end{aligned} \tag{6F.11}$$

for $\boldsymbol{\mu} \in \mathsf{B}^{\triangleleft}(\Lambda)$ and $\boldsymbol{\nu} \in \mathsf{B}^{\triangleright}(\Lambda)$. Let $i, j \in I$. These definitions readily imply that if $i \in I$ then for $\boldsymbol{\mu} \in \mathsf{B}^{\triangleleft}(\Lambda)$ and $\boldsymbol{\nu} \in \mathsf{B}^{\triangleright}(\Lambda)$.

$$d_i(\boldsymbol{\mu}) = \varphi_i^{\triangleleft}(\boldsymbol{\mu}) - \varepsilon_i^{\triangleleft}(\boldsymbol{\mu}) \quad \text{and} \quad d_i(\boldsymbol{\nu}) = \varphi_i^{\triangleright}(\boldsymbol{\nu}) - \varepsilon_i^{\triangleright}(\boldsymbol{\nu}), \tag{6F.12}$$

Abusing notation, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathsf{B}^{\triangleleft}(\Lambda)$ and $\boldsymbol{\lambda} \overset{i\triangleleft}{\rightsquigarrow} \boldsymbol{\mu}$ we write $e_i\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $f_i\boldsymbol{\lambda} = \boldsymbol{\mu}$. Similarly, if $\boldsymbol{\sigma}, \boldsymbol{\nu} \in \mathsf{B}^{\triangleright}(\Lambda)$, write $e_j\boldsymbol{\nu} = \boldsymbol{\sigma}$ and $f_j\boldsymbol{\sigma} = \boldsymbol{\nu}$ if $\boldsymbol{\sigma} \overset{j\triangleright}{\rightsquigarrow} \boldsymbol{\nu}$. If $\varepsilon_i^{\triangle}(\boldsymbol{\lambda}) = 0$ set $e_i\boldsymbol{\lambda} = 0$ and if $\varphi_i^{\triangle}(\boldsymbol{\lambda}) = 0$ set $f_i\boldsymbol{\lambda} = 0$.

By Corollary 6F.10, if m is a non-negative integer and $\boldsymbol{\lambda} \in \mathsf{B}^{\triangle}(\Lambda)$ then $e_i^m \boldsymbol{\lambda} \neq 0$ if and only if $m \leq \varepsilon_i^{\triangle}(\boldsymbol{\lambda})$ and $f_i^m \boldsymbol{\lambda} \neq 0$ if and only if $m \leq \varphi_i^{\triangle}(\boldsymbol{\lambda})$. Therefore, following [34, §7.2], the datum $(\mathsf{B}^{\triangleleft}(\Lambda), e_i, f_i, \varepsilon^{\triangleleft}, \varphi^{\triangleleft}, \mathrm{wt})$ uniquely determines Kashiwara's upper crystal graph of $\mathscr{L}_{\mathcal{A}}^{\triangleright}(\Lambda)$, where wt is the weight function of (6D.5). Similarly, the datum $(\mathsf{B}^{\triangleright}(\Lambda), e_i, f_i, \varepsilon^{\triangleright}, \varphi^{\triangleright}, \mathrm{wt})$ determines the upper crystal graph of $\mathscr{L}_{\mathcal{A}}^{\triangleleft}(\Lambda)$.

Using Theorem 6F.3, the crystal bases $\mathscr{B}_{\infty}^{\triangleleft}(\Lambda)$ and $\mathscr{B}_{\infty}^{\triangleright}(\Lambda)$ lift to canonical bases

$$\left\{ \mathsf{G}_{\infty, \boldsymbol{\nu}}^{\triangleleft} \;\middle|\; \boldsymbol{\nu} \in \mathsf{B}^{\triangleright}(\Lambda) \right\} \qquad \text{and} \qquad \left\{ \mathsf{G}_{\infty, \boldsymbol{\mu}}^{\triangleright} \;\middle|\; \boldsymbol{\mu} \in \mathsf{B}^{\triangleleft}(\Lambda) \right\}$$

of $\mathscr{L}_{\mathcal{A}}^{\triangleleft}(\Lambda)^*$ and $\mathscr{L}_{\mathcal{A}}^{\triangleright}(\Lambda)^*$, respectively, that are uniquely determined by the properties:

$$\overline{\mathsf{G}_{\infty, \boldsymbol{\nu}}^{\triangleleft}} = \mathsf{G}_{\infty, \boldsymbol{\nu}}^{\triangleleft} \quad \text{and} \quad \mathsf{G}_{\infty, \boldsymbol{\nu}}^{\triangleleft} \equiv \mathsf{s}_{\boldsymbol{\nu}}^{\triangleleft} \left(\mathrm{mod} \; q^{-1} \mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda) \right)$$

$$\overline{\mathsf{G}_{\infty, \boldsymbol{\mu}}^{\triangleright}} = \mathsf{G}_{\infty, \boldsymbol{\mu}}^{\triangleright} \quad \text{and} \quad \mathsf{G}_{\infty, \boldsymbol{\mu}}^{\triangleright} \equiv \mathsf{s}_{\boldsymbol{\mu}}^{\triangleright} \left(\mathrm{mod} \; q^{-1} \mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleright}(\Lambda) \right). \tag{6F.13}$$

for $\boldsymbol{\nu} \in \mathsf{B}^{\triangleright}(\Lambda)$ and $\boldsymbol{\mu} \in \mathsf{B}^{\triangleleft}(\Lambda)$.

When combined with Theorem 5A.3, the next result proves Theorem **C** from the introduction. As remarked at the start of section 6, this result applies to all (standard) cyclotomic KLR algebras of types $A_{e-1}^{(1)}$, A_{∞}, $C_{e-1}^{(1)}$ and C_{∞}.

Theorem 6F.14. Let $\Lambda \in P^+$. Then $\mathcal{K}_n^{\triangleleft} = \mathsf{B}^{\triangleleft}(\Lambda)$ and $\mathcal{K}_n^{\triangleright} = \mathsf{B}^{\triangleright}(\Lambda)$. Moreover, if $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$ then

$$\mathsf{d}_T^{\triangleleft} \left(q^{-\operatorname{def} \boldsymbol{\mu}} \mathbb{Y}_{\boldsymbol{\mu}}^{\triangleleft} \right) = \mathsf{G}_{\infty, \mathsf{m}(\boldsymbol{\mu})}^{\triangleleft} \quad \text{and} \quad \mathsf{d}_T^{\triangleright} \left(q^{-\operatorname{def} \boldsymbol{\mu}} \mathbb{Y}_{\mathsf{m}(\boldsymbol{\mu})}^{\triangleright} \right) = \mathsf{G}_{\infty, \boldsymbol{\mu}}^{\triangleright}.$$

Proof. By working with $\mathscr{L}_{\mathcal{A}}^{\triangleleft}(\Lambda)$ we prove that $\mathsf{B}^{\triangleright}(\Lambda) = \mathcal{K}_n^{\triangleright}$ and that $\mathsf{d}_T^{\triangleleft}(q^{-\operatorname{def} \boldsymbol{\mu}} \mathbb{Y}_{\boldsymbol{\mu}}^{\triangleleft}) = \mathsf{G}_{\infty, \mathsf{m}(\boldsymbol{\mu})}^{\triangleleft}$ for $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$. The remaining results are proved in exactly the same way and are left as an exercise for the reader. By Corollary 6E.2 and Lemma 6E.1, the functor $\circledast$ categorifies the bar involution on $\mathscr{L}_{\mathcal{A}}^{\triangleleft}(\Lambda)$, so $\{q^{-\operatorname{def} \boldsymbol{\mu}} \mathbb{Y}_{\boldsymbol{\mu}}^{\triangleleft} \mid \boldsymbol{\mu} \in \mathcal{K}_{\bullet}^{\triangleleft}\}$ is the ∞-canonical basis of $[\operatorname{Proj}_{\mathbb{K}} \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$. By Theorem 6F.3, the ∞-canonical basis is uniquely determined by the choice of highest weight vector, and $\mathsf{d}_T^{\triangleleft}$ sends $\mathbb{Y}_{\underline{\mathbf{0}}_{\ell}}^{\triangleleft}$ to $\mathsf{s}_{\underline{\mathbf{0}}_{\ell}}^{\triangleleft}$. Hence, if $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$ then $\mathsf{d}_T^{\triangleleft}(q^{-\operatorname{def} \boldsymbol{\mu}} \mathbb{Y}_{\boldsymbol{\mu}}^{\triangleleft}) = \mathsf{G}_{\infty, \boldsymbol{\nu}}^{\triangleleft}$, for some $\boldsymbol{\nu} \in \mathsf{B}^{\triangleright}(\Lambda)$. To determine the ℓ-partition $\boldsymbol{\nu}$, we compute in $\mathscr{L}_{\mathbb{A}_{\infty}}^{\triangleleft}(\Lambda)$:

$$\mathsf{d}_T^{\triangleleft} \left(q^{-\operatorname{def} \boldsymbol{\mu}} \mathbb{Y}_{\boldsymbol{\mu}}^{\triangleleft} \right)$$

$$= q^{-2 \operatorname{def} \boldsymbol{\mu}} \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}} \left(\mathsf{d}_T^{\triangleleft}(\mathbb{Y}_{\boldsymbol{\mu}}^{\triangleleft}), \mathsf{s}_{\boldsymbol{\lambda}}^{\triangleleft} \right)^{\triangleleft} \mathsf{s}_{\boldsymbol{\lambda}}^{\triangleleft} \qquad \text{by (6D.15) and Proposition 6D.10,}$$

$$= q^{-\operatorname{def} \boldsymbol{\mu}} \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}} \left\langle \mathbb{Y}_{\boldsymbol{\mu}}^{\triangleleft}, [S_{\boldsymbol{\lambda}}^{\triangleleft}] \right\rangle \mathsf{s}_{\boldsymbol{\lambda}}^{\triangleleft} \qquad \text{by Lemma 6D.17,}$$

$$= q^{-\operatorname{def} \boldsymbol{\mu}} \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}} \mathsf{d}_{\boldsymbol{\lambda} \boldsymbol{\mu}}^{\triangleleft}(q) \mathsf{s}_{\boldsymbol{\lambda}}^{\triangleleft} \qquad \text{by Lemma 6E.14,}$$

$$= \mathsf{s}_{\mathsf{m}(\boldsymbol{\mu})}^{\triangleleft} + \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}} \overline{\mathsf{d}_{\boldsymbol{\lambda} \mathsf{m}(\boldsymbol{\mu})}^{\triangleleft}(q)} \mathsf{s}_{\boldsymbol{\lambda}}^{\triangleleft} \left(\mathrm{mod} \; q^{-1} \mathscr{F}_{\mathbb{A}_{\infty}}^{\Lambda \triangleleft} \right) \qquad \text{by Proposition 6E.19}$$

$$\equiv \mathsf{s}_{\mathsf{m}(\boldsymbol{\mu})}^{\triangleleft} + \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell} \setminus (\mathcal{K}_n^{\triangleleft} \cup \mathcal{K}_n^{\triangleright})} \overline{\mathsf{d}_{\boldsymbol{\lambda} \mathsf{m}(\boldsymbol{\mu})}^{\triangleleft}(q)} \mathsf{s}_{\boldsymbol{\lambda}}^{\triangleleft} \left(\mathrm{mod} \; q^{-1} \mathscr{F}_{\mathbb{A}_{\infty}}^{\Lambda \triangleleft} \right),$$

where the last equality comes from Corollary 6E.20. Therefore, Theorem 6F.8 and Equation (6F.13) force $\boldsymbol{\nu} = \mathsf{m}(\boldsymbol{\mu})$ and $\overline{\mathsf{d}_{\boldsymbol{\lambda} \mathsf{m}(\boldsymbol{\mu})}^{\triangleleft}(q)} = q^{-\operatorname{def} \boldsymbol{\mu}} \mathsf{d}_{\boldsymbol{\lambda} \boldsymbol{\mu}}^{\triangleleft}(q) \in \delta_{\boldsymbol{\lambda} \mathsf{m}(\boldsymbol{\mu})} + q^{-1} Z[q^{-1}]$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$. That is,

$$\mathsf{d}_T^{\lhd}\left(q^{-\operatorname{def}\boldsymbol{\mu}}\mathbb{Y}_{\boldsymbol{\mu}}^{\lhd}\right) = \mathsf{G}_{\infty,\mathsf{m}(\boldsymbol{\mu})}^{\lhd} \quad \text{and} \quad \boldsymbol{\nu} = \mathsf{m}(\boldsymbol{\mu}) \in \mathcal{K}_n^{\rhd}.$$

In particular, this shows that $\mathsf{B}^{\rhd}(\Lambda) = \{\mathsf{m}(\boldsymbol{\mu}) \,|\, \boldsymbol{\mu} \in \mathcal{K}_\bullet^{\lhd}\} = \mathcal{K}_\bullet^{\rhd}$, where the last equality is Definition 5D.1. This completes the proof. □

Theorem 6F.14 completes the classification of the simple $\mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$-modules from Theorem 5A.3 by giving a description of the sets $\mathcal{K}_n^{\lhd}$ and $\mathcal{K}_n^{\rhd}$. The crystal graphs of $L(\Lambda)$ allow us to strengthen this characterisation of $\mathcal{K}_n^{\lhd}$ and $\mathcal{K}_n^{\rhd}$.

Corollary 6F.15. *Let $\mathbb{K}$ be a field and suppose that $\boldsymbol{\mu} \in \mathcal{P}_n^{\ell}$.*

   (a) *The $\mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$-module $D_{\boldsymbol{\mu}}^{\lhd}(F) \neq 0$ if and only if $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$.*
   (b) *The $\mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$-module $D_{\boldsymbol{\mu}}^{\rhd}(F) \neq 0$ if and only if $\boldsymbol{\mu} \in \mathcal{K}_n^{\rhd}$.*

   (c) *The ℓ-partition $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ if and only if $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{i}\lhd} \boldsymbol{\mu}$ for some $\mathbf{i} \in I^n$.*
   (d) *The ℓ-partition $\boldsymbol{\mu} \in \mathcal{K}_n^{\rhd}$ if and only if $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{i}\rhd} \boldsymbol{\mu}$ for some $\mathbf{i} \in I^n$.*

   (e) *If $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ and $\mathbf{i} \in I^n$ then $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{i}\lhd} \boldsymbol{\mu}$ if and only if $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{i}\rhd} \mathsf{m}(\boldsymbol{\mu})$.*

Proof. By invoking Theorem 6F.14 and Theorem 5A.3, parts (a)–(d) are restatements of the identities $\mathcal{K}_n^{\lhd} = \mathsf{B}^{\lhd}(\Lambda)$ and $\mathcal{K}_n^{\rhd} = \mathsf{B}^{\rhd}(\Lambda)$. For part (e), if $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ then $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{i}\lhd} \boldsymbol{\mu}$ if and only if the sequence $\mathbf{i}$ labels a path in the crystal graph of $\mathscr{L}_{\mathcal{A}}^{\lhd}(\Lambda)$ from $\underline{\mathbf{0}}_\ell$ to $\boldsymbol{\mu}$. By Theorem 6D.20, the $U_q(\mathfrak{g}_\Gamma)$-modules $\mathscr{L}_{\mathcal{A}}^{\lhd}(\Lambda)^*$ and $\mathscr{L}_{\mathcal{A}}^{\rhd}(\Lambda)^*$ have isomorphic crystal graphs. Any crystal isomorphism preserves the labels on the paths, so $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{i}\lhd} \boldsymbol{\mu}$ is a path in the crystal graph of $\mathscr{L}_{\mathcal{A}}^{\lhd}(\Lambda)$ if and only if $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{i}\rhd} \boldsymbol{\nu}$ is a path in the crystal graph of $\mathscr{L}_{\mathcal{A}}^{\rhd}(\Lambda)$, for some $\boldsymbol{\nu} \in \mathcal{K}_n^{\rhd}$. Applying Theorem 6F.14 twice,

$$\mathsf{G}_{\infty,\mathsf{m}(\boldsymbol{\mu})}^{\lhd} = \mathsf{d}_T^{\lhd}\left(q^{-\operatorname{def}\boldsymbol{\nu}}\mathbb{Y}_{\boldsymbol{\mu}}^{\lhd}\right) \quad \text{and} \quad \mathsf{G}_{\infty,\boldsymbol{\nu}}^{\rhd} = \mathsf{d}_T^{\rhd}\left(q^{-\operatorname{def}\boldsymbol{\nu}}\mathbb{Y}_{\mathsf{m}(\boldsymbol{\nu})}^{\rhd}\right)$$

By Proposition 6E.19, $\mathbb{Y}_{\boldsymbol{\mu}}^{\lhd} = \mathbb{Y}_{\mathsf{m}(\boldsymbol{\mu})}^{\rhd}$, so the map $\mathsf{d}_T^{\rhd} \circ (\mathsf{d}_T^{\lhd})^{-1}$ induces a crystal isomorphism

$$\left(\mathscr{L}_{\mathbb{A}_\infty}^{\lhd}(\Lambda), \mathscr{B}_\infty^{\rhd}(\Lambda)\right) \to \left(\mathscr{L}_{\mathbb{A}_\infty}^{\rhd}(\Lambda), \mathscr{B}_\infty^{\lhd}(\Lambda)\right),$$

which sends $\mathsf{G}_{\infty,\mathsf{m}(\boldsymbol{\mu})}^{\lhd} + q^{-1}\mathscr{L}_{\mathbb{A}_\infty}^{\lhd}(\Lambda)$ to $\mathsf{G}_{\infty,\boldsymbol{\mu}}^{\rhd} + q^{-1}\mathscr{L}_{\mathbb{A}_\infty}^{\rhd}(\Lambda)$. Hence, part (e) follows. □

We have now proved a strong form of Theorem **C** from the introduction.

Notice that Corollary 6F.15 gives a description of the map $\boldsymbol{\mu} \mapsto \mathsf{m}(\boldsymbol{\mu})$, for $\mathsf{m} \colon \mathcal{K}_n^{\lhd} \longrightarrow \mathcal{K}_n^{\rhd}$. Explicitly, if $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$ then we can find $\mathbf{i} \in I^n$ such that $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{i}\lhd} \boldsymbol{\mu}$ is a path in the crystal graph of $\mathscr{L}_{\mathcal{A}}^{\lhd}(\Lambda)^*$ from $\mathsf{s}_{\underline{\mathbf{0}}_\ell}^{\lhd}$ to $\mathsf{s}_{\boldsymbol{\mu}}^{\lhd}$. Then $\mathsf{m}(\boldsymbol{\mu}) \in \mathcal{K}_n^{\rhd}$ is the unique ℓ-partition such that $\underline{\mathbf{0}}_\ell \xrightarrow{\mathbf{j}\rhd} \mathsf{m}(\boldsymbol{\mu})$ in the crystal graph of $\mathscr{L}_{\mathcal{A}}^{\rhd}(\Lambda)^*$. In view of Corollary 5E.7, if Γ is a quiver of type $A_{e-1}^{(1)}$ and $\Lambda = \Lambda_0$, this gives a variation on Kleshchev's description of the Mullineux map of the symmetric group, which is the function $\boldsymbol{\mu} \mapsto \mathsf{m}(\boldsymbol{\mu})'$, for $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$.

The proof of Theorem 6F.14 gives the following strengthening of Corollary 6E.20.

Corollary 6F.16. *Let $\boldsymbol{\mu} \in \mathcal{K}_n^{\lhd}$, $\boldsymbol{\nu} \in \mathcal{K}_n^{\lhd}$ and $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{P}_n^{\ell}$.*

   (a) *If $\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd}(q) \neq 0$ then $\boldsymbol{\mu} \unlhd \boldsymbol{\lambda} \unlhd \mathsf{m}(\boldsymbol{\mu})$ and $\alpha_{\boldsymbol{\lambda}} = \alpha_{\boldsymbol{\mu}}$. Moreover, $\mathbb{d}_{\boldsymbol{\mu}\boldsymbol{\mu}}^{\lhd}(q) = 1$, $\mathbb{d}_{\mathsf{m}(\boldsymbol{\mu})\boldsymbol{\mu}}^{\lhd}(q) = q^{\operatorname{def}\boldsymbol{\mu}}$ and if $\mathsf{m}(\boldsymbol{\mu}) \lhd \boldsymbol{\lambda} \lhd \boldsymbol{\mu}$ then $0 < \deg \mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd}(q) < \operatorname{def}\boldsymbol{\mu}$.*
   (b) *If $\mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\rhd}(q) \neq 0$ then $\boldsymbol{\mu} \unrhd \boldsymbol{\lambda} \unrhd \mathsf{m}(\boldsymbol{\mu})$ and $\alpha_{\boldsymbol{\lambda}} = \alpha_{\boldsymbol{\mu}}$. Moreover, $\mathbb{d}_{\boldsymbol{\mu}\boldsymbol{\mu}}^{\rhd}(q) = 1$, $\mathbb{d}_{\mathsf{m}^{-1}(\boldsymbol{\mu})\boldsymbol{\mu}}^{\rhd}(q) = q^{\operatorname{def}\boldsymbol{\mu}}$ and if $\mathsf{m}^{-1}(\boldsymbol{\mu}) \rhd \boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ then $0 < \deg \mathbb{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\rhd}(q) < \operatorname{def}\boldsymbol{\mu}$.*

By Corollary 6E.17, $\mathsf{d}^{\triangle}_{\boldsymbol{\lambda\mu}}(q) = [S^{\triangle}_{\boldsymbol{\lambda}} : D^{\triangle}_{\boldsymbol{\mu}}]_q$ in type $A^{(1)}_{e-1}$ when $\mathbb{K}$ is a field of characteristic zero, so $\mathsf{d}^{\triangle}_{\boldsymbol{\lambda\mu}}(q) \in \delta_{\boldsymbol{\lambda\mu}} + q\mathbb{N}[q]$ in this case. In type $C^{(1)}_{e-1}$, we can only say that $\mathsf{d}^{\triangle}_{\boldsymbol{\lambda\mu}}(q) \in \delta_{\boldsymbol{\lambda\mu}} + q\mathbb{Z}[q]$, and that these polynomials approximate the graded decomposition numbers in the sense of Theorem 6E.16.

The final results in this section describe the 0-canonical bases of $\mathscr{L}^{\triangleleft}_{\mathcal{A}}(\Lambda)$ and $\mathscr{L}^{\triangleright}_{\mathcal{A}}(\Lambda)$. To do this we retrace our steps and prove a variation of Theorem 6F.8.

Theorem 6F.17. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}^{\ell}_n$ and $i \in I$.

(a) If $\boldsymbol{\lambda}$ does not have a $\triangleleft$-good i-node then $e_i\mathsf{s}^{\triangleleft}_{\boldsymbol{\lambda}} \in q\mathscr{F}^{\Lambda\triangleleft}_{\mathbb{A}_0}$.

(b) If $\boldsymbol{\lambda} \xmapsto{\;i\triangleleft\;} \boldsymbol{\mu}$ then $e_i\mathsf{s}^{\triangleleft}_{\boldsymbol{\mu}} = \mathsf{s}^{\triangleleft}_{\boldsymbol{\lambda}} \pmod{q\mathscr{F}^{\Lambda\triangleleft}_{\mathbb{A}_0}}$ and $f_i\mathsf{s}^{\triangleleft}_{\boldsymbol{\lambda}} = \mathsf{s}^{\triangleleft}_{\boldsymbol{\mu}} \pmod{q\mathscr{F}^{\Lambda\triangleleft}_{\mathbb{A}_0}}$.

(c) If $\boldsymbol{\lambda}$ does not have a $\triangleright$-good j-node then $e_j\mathsf{s}^{\triangleright}_{\boldsymbol{\lambda}} \in q\mathscr{F}^{\Lambda\triangleright}_{\mathbb{A}_0}$.

(d) If $\boldsymbol{\lambda} \xmapsto{\;j\triangleright\;} \boldsymbol{\mu}$ then $e_j\mathsf{s}^{\triangleright}_{\boldsymbol{\mu}} = \mathsf{s}^{\triangleright}_{\boldsymbol{\lambda}} \pmod{q\mathscr{F}^{\Lambda\triangleright}_{\mathbb{A}_0}}$ and $f_j\mathsf{s}^{\triangleright}_{\boldsymbol{\lambda}} = \mathsf{s}^{\triangleright}_{\boldsymbol{\mu}} \pmod{q\mathscr{F}^{\Lambda\triangleright}_{\mathbb{A}_0}}$.

Proof. The proof is almost identical to the proof of Theorem 6F.8. For (a), suppose that $\boldsymbol{\lambda}$ does not have a $\triangleleft$-good i-node. For $A \in \mathrm{Rem}_i(\boldsymbol{\lambda})$ define $\widehat{A}$ to be the lowest addable i-node of $\boldsymbol{\lambda}$ such that $A > \widehat{A}$ and $d^{\triangleleft}_{\widehat{A}}(\boldsymbol{\lambda}) = d^{\triangleleft}_A(\boldsymbol{\lambda}) + 1$. If $M \subseteq \mathrm{Rem}_i(\boldsymbol{\lambda})$ set $\widehat{\boldsymbol{\lambda}}_M = \boldsymbol{\lambda} - M + \widehat{M}$, where $\widehat{M} = \{\widehat{A} \mid A \in M\}$, and define

$$\widehat{\Omega}_i(\mathsf{s}^{\triangleleft}_{\boldsymbol{\lambda}}) = \sum_{M \subseteq \mathrm{Rem}_i(\boldsymbol{\lambda})} (-q)^{|M|} \mathsf{s}^{\triangleleft}_{\widehat{\boldsymbol{\lambda}}_M}$$

Exactly we before, it now follows that $e_i\mathsf{s}^{\triangleleft}_{\boldsymbol{\lambda}} \in q\mathscr{F}^{\Lambda\triangleleft}_{\mathbb{A}_0}$ proving (a) with (b) following similarly. We leave the details to the reader. $\square$

As before, set $\mathscr{B}^{\triangleleft}_0(\Lambda) = \{\mathsf{s}^{\triangleleft}_{\boldsymbol{\nu}} + q^{-1}\mathscr{L}^{\triangleleft}_{\mathbb{A}_\infty}(\Lambda) \mid \boldsymbol{\nu} \in \mathsf{B}^{\triangleright}(\Lambda)\}$ and $\mathscr{B}^{\triangleright}_0(\Lambda) = \{\mathsf{s}^{\triangleright}_{\boldsymbol{\mu}} + q^{-1}\mathscr{L}^{\triangleright}_{\mathbb{A}_\infty}(\Lambda) \mid \boldsymbol{\mu} \in \mathsf{B}^{\triangleleft}(\Lambda)\}$. The argument of Corollary 6F.10 now yields:

Corollary 6F.18. Let $\Lambda \in P^+$. Then $(\mathscr{L}^{\triangleleft}_{\mathbb{A}_0}(\Lambda), \mathscr{B}^{\triangleleft}_0(\Lambda))$ and $(\mathscr{L}^{\triangleright}_{\mathbb{A}_0}(\Lambda), \mathscr{B}^{\triangleright}_0(\Lambda))$ are 0-crystal bases of $L(\Lambda)$.

By Theorem 6F.3, the crystal bases $\mathscr{B}^{\triangleleft}_0(\Lambda)$ and $\mathscr{B}^{\triangleright}_0(\Lambda)$ lift to canonical bases $\{\mathsf{G}^{\triangleleft}_{0,\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{B}^{\triangleleft}_0(\Lambda)\}$ of $\mathscr{L}^{\triangleleft}_{\mathcal{A}}(\Lambda)$, and $\{\mathsf{G}^{\triangleright}_{0,\boldsymbol{\nu}} \mid \boldsymbol{\nu} \in \mathscr{B}^{\triangleright}_0(\Lambda)\}$ of $\mathscr{L}^{\triangleright}_{\mathcal{A}}(\Lambda)$, that are uniquely determined by the properties:

$$\begin{aligned}
\overline{\mathsf{G}^{\triangleleft}_{0,\boldsymbol{\mu}}} = \mathsf{G}^{\triangleleft}_{0,\boldsymbol{\mu}} \qquad &\text{and} \qquad \mathsf{G}^{\triangleleft}_{0,\boldsymbol{\mu}} \equiv \mathsf{s}^{\triangleleft}_{\boldsymbol{\mu}} \pmod{q\mathscr{L}^{\triangleleft}_{\mathbb{A}_0}(\Lambda)} \\
\overline{\mathsf{G}^{\triangleright}_{0,\boldsymbol{\nu}}} = \mathsf{G}^{\triangleright}_{0,\boldsymbol{\nu}} \qquad &\text{and} \qquad \mathsf{G}^{\triangleright}_{0,\boldsymbol{\nu}} \equiv \mathsf{s}^{\triangleright}_{\boldsymbol{\nu}} \pmod{q\mathscr{L}^{\triangleright}_{\mathbb{A}_0}(\Lambda)}.
\end{aligned} \tag{6F.19}$$

for $\boldsymbol{\mu} \in \mathscr{B}^{\triangleleft}_0(\Lambda)$ and $\boldsymbol{\nu} \in \mathscr{B}^{\triangleright}_0(\Lambda)$. Now set $\mathscr{B}^{\triangleleft}_0(\Lambda) = \{\mathsf{s}^{\triangleleft}_{\boldsymbol{\nu}} + q\mathscr{L}^{\triangleleft}_{\mathbb{A}_0}(\Lambda) \mid \boldsymbol{\nu} \in \mathcal{K}^{\triangleleft}_n\}$ and $\mathscr{B}^{\triangleright}_0(\Lambda) = \{\mathsf{s}^{\triangleright}_{\boldsymbol{\mu}} + q\mathscr{L}^{\triangleright}_{\mathbb{A}_0}(\Lambda) \mid \boldsymbol{\mu} \in \mathcal{K}^{\triangleright}_n\}$.

Theorem 6F.20. Suppose that $\boldsymbol{\sigma}, \boldsymbol{\mu} \in \mathcal{K}^{\triangleleft}_n$ and $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathcal{K}^{\triangleright}_n$. Then $\mathsf{d}^{\triangleleft}(\mathsf{G}^{\triangleleft}_{0,\boldsymbol{\mu}}) = \mathbb{D}^{\triangleleft}_{\boldsymbol{\mu}}$, $\mathsf{d}^{\triangleright}(\mathsf{G}^{\triangleright}_{0,\boldsymbol{\nu}}) = \mathbb{D}^{\triangleright}_{\boldsymbol{\nu}}$,

$$\left(\mathsf{G}^{\triangleleft}_{\infty,\boldsymbol{\lambda}}, \mathsf{G}^{\triangleleft}_{0,\boldsymbol{\mu}}\right)^{\triangleleft} = \delta_{\boldsymbol{\lambda}\mathsf{m}(\boldsymbol{\mu})} \qquad \text{and} \qquad \left(\mathsf{G}^{\triangleright}_{\infty,\boldsymbol{\sigma}}, \mathsf{G}^{\triangleright}_{0,\boldsymbol{\nu}}\right)^{\triangleright} = \delta_{\mathsf{m}(\boldsymbol{\sigma})\boldsymbol{\nu}}.$$

Proof. By Theorem 6F.14, $\mathsf{B}^{\triangleleft}(\Lambda) = \mathcal{K}^{\triangleleft}_n$. Therefore, by Lemma 6D.17 and the uniqueness of canonical bases from [33, Theorem 5], if $\boldsymbol{\nu} \in \mathcal{K}^{\triangleleft}_n$ then we can write $\mathsf{d}^{\triangleleft}(\mathsf{G}^{\triangleleft}_{0,\boldsymbol{\nu}}) = \mathbb{D}^{\triangleleft}_{\boldsymbol{\mu}}$, for some $\boldsymbol{\mu} \in \mathcal{K}^{\triangleleft}_n$. By Theorem 6E.10, if $\boldsymbol{\mu} \in \mathcal{K}^{\triangleleft}_n$ then

$$\left(\mathbb{D}^{\triangleleft}_{\boldsymbol{\mu}}\right)^{\circledast} = \mathbb{D}^{\triangleleft}_{\boldsymbol{\mu}} \qquad \text{and} \qquad \mathbb{D}^{\triangleleft}_{\boldsymbol{\mu}} \equiv \left[S^{\triangleleft}_{\boldsymbol{\mu}}\right] \left(\bmod q\left[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}^{\Lambda}_{\bullet}(\mathbb{K}[x])\right]\right).$$

Hence, $\mathsf{d}^{\lhd}(\mathsf{G}^{\lhd}_{0,\boldsymbol{\mu}}) = \mathbb{D}^{\lhd}_{\boldsymbol{\mu}}$. Similarly, $\mathsf{d}^{\rhd}(\mathsf{G}^{\rhd}_{0,\boldsymbol{\nu}}) = \mathbb{D}^{\rhd}_{\boldsymbol{\nu}}$. Using Theorem 6F.14 and Lemma 6D.17, if $\boldsymbol{\tau} \in \mathcal{K}^{\lhd}_n$ then

$$
\begin{aligned}
\left(\mathsf{G}^{\lhd}_{\infty,\mathsf{m}(\boldsymbol{\tau})}, \mathsf{G}^{\lhd}_{0,\boldsymbol{\mu}}\right)^{\lhd} &= \left(\mathsf{d}^{\lhd}_T\left(q^{-\operatorname{def}\boldsymbol{\tau}}\mathbb{Y}^{\lhd}_{\boldsymbol{\tau}}\right), \mathsf{G}^{\lhd}_{0,\boldsymbol{\mu}}\right)^{\lhd} \\
&= q^{\operatorname{def}\boldsymbol{\tau}}\left\langle\left(q^{-\operatorname{def}\boldsymbol{\tau}}\mathbb{Y}^{\lhd}_{\boldsymbol{\tau}}\right)^{\#}, \mathbb{D}^{\lhd}_{\boldsymbol{\mu}}\right\rangle = \left\langle\mathbb{Y}^{\lhd}_{\boldsymbol{\tau}}, \mathbb{D}^{\lhd}_{\boldsymbol{\mu}}\right\rangle = \delta_{\boldsymbol{\tau}\boldsymbol{\mu}},
\end{aligned}
$$

where the last equality follows by Theorem 6E.16 and (6C.2). Setting $\boldsymbol{\lambda} = \mathsf{m}(\boldsymbol{\tau})$ gives the first inner product in the displayed equation. The inner product $(\mathsf{G}^{\rhd}_{\infty,\boldsymbol{\sigma}}, \mathsf{G}^{\rhd}_{0,\boldsymbol{\nu}})^{\rhd}$ can be computed in the same way. $\qquad\square$

6G. **Modular branching rules.** This section uses the results of the last section, and Theorem 2D.1, to prove precise forms of the modular branching theorem, which is Theorem **D** from the introduction. That is, we prove that the modular branching rules for $\mathsf{R}^{\Lambda}_n(\mathbb{K}[x])$ categorify the crystal graph of $L(\Lambda)$. In principle, this result has already been proved by Lauda and Vazirani [44], however, their theorem does not imply our result because it is not clear how to relate their labelling of the irreducible $\mathsf{R}^{\Lambda}_n(\mathbb{K}[x])$-modules to Corollary 6F.15. On the other hand, our results do imply those of [44] for the cyclotomic KLR algebras of types $A^{(1)}_{e-1}$ and $C^{(1)}_{e-1}$. Moreover, our approach to the modular branching rules is considerably shorter than the other routes in the literature because we have already established the link between the representation theory of $\mathsf{R}^{\Lambda}_n(\mathbb{K}[x])$ and the crystal graph of $L(\Lambda)$.

Suppose that M is an $\mathsf{R}^{\Lambda}_n(\mathbb{K}[x])$-module. Recall from subsection 5E that head M and soc M are the head of socle of M, respectively. For $i \in I$ and $k \geq 0$ inductively define $\mathsf{R}^{\Lambda}_n(\mathbb{K}[x])$-modules $\widetilde{e}^k_i M$ and $\widetilde{f}^k_i M$ by setting $\widetilde{e}^0_i M = M = \widetilde{f}^0_i M$ and if $k \geq 0$ define

$$
\widetilde{e}^{k+1}_i M = \operatorname{soc}\left(E_i\left(\widetilde{e}^k_i M\right)\right) \qquad \text{and} \qquad \widetilde{f}^{k+1}_i M = \operatorname{head}\left(F_i\left(\widetilde{f}^k_i M\right)\right).
$$

Using these operators attach the following non-negative integers to M:

$$
\varepsilon_i(M) = \max\left\{k \geq 0 \,\Big|\, \widetilde{e}^k_i M \neq 0\right\} \qquad \text{and} \qquad \varphi_i(M) = \max\left\{k \geq 0 \,\Big|\, \widetilde{f}^k_i M \neq 0\right\}.
$$

The key result that we need is the following, which lifts some of the easy preliminary results from Grojnowski's approach to the modular branching rules into our setting.

Proposition 6G.1. *Let $\boldsymbol{\mu} \in \mathcal{K}^{\lhd}_n$, $\boldsymbol{\nu} \in \mathcal{K}^{\rhd}_n$ and $i, j \in I$ and assume that $\varepsilon_i(D^{\lhd}_{\boldsymbol{\mu}}) > 0$ and $\varepsilon_j(D^{\rhd}_{\boldsymbol{\nu}}) > 0$.*

(a) *As $\mathsf{R}^{\Lambda}_{n-1}(\mathbb{K}[x])$-modules, $E_i(D^{\lhd}_{\boldsymbol{\mu}})$ is self-dual and $\widetilde{e}_i D^{\lhd}_{\boldsymbol{\mu}}$ is irreducible with $\varepsilon_i(\widetilde{e}_i D^{\lhd}_{\boldsymbol{\mu}}) = \varepsilon_i(D^{\lhd}_{\boldsymbol{\mu}}) - 1$. Moreover, if $[E_i D^{\lhd}_{\boldsymbol{\mu}} : L] > 0$ and $L \not\cong q^b \widetilde{e}_i D^{\lhd}_{\boldsymbol{\mu}}$ as $\mathsf{R}^{\Lambda}_{n-1}$-modules, then $\varepsilon_i(L) < \varepsilon_i(\widetilde{e}_i D^{\lhd}_{\boldsymbol{\mu}})$.*

(b) *As $\mathsf{R}^{\Lambda}_{n-1}(\mathbb{K}[x])$-modules, $E_j(D^{\rhd}_{\boldsymbol{\nu}})$ is self-dual and $\widetilde{e}_j D^{\rhd}_{\boldsymbol{\nu}}$ is irreducible with $\varepsilon_j(\widetilde{e}_j D^{\rhd}_{\boldsymbol{\nu}}) = \varepsilon_j(D^{\rhd}_{\boldsymbol{\nu}}) - 1$. Moreover, if $[E_j D^{\rhd}_{\boldsymbol{\nu}} : L] > 0$ and $L \not\cong q^b \widetilde{e}_j D^{\rhd}_{\boldsymbol{\nu}}$ as $\mathsf{R}^{\Lambda}_{n-1}$-modules, then $\varepsilon_j(L) < \varepsilon_j(\widetilde{e}_j D^{\rhd}_{\boldsymbol{\nu}})$.*

(c) *Let M be an irreducible $\mathsf{R}^{\Lambda}_n(\mathbb{K}[x])$-module. Then y_n acts nilpotently on $E_i M$ with nilpotency index $\varepsilon_i(M)$.*

Proof. The modules $E_i(D^{\lhd}_{\boldsymbol{\mu}})$ and $E_j(D^{\rhd}_{\boldsymbol{\nu}})$ are self-dual by Proposition 6B.3. The remaining claims in (a) follow from [16, 36]. In more detail, by construction any irreducible $\mathsf{R}^{\Lambda}_m(\mathbb{K}[x])$-module is an irreducible $\mathsf{R}_m(\mathbb{K}[x])$-module. Hence, $\widetilde{e}_i D^{\triangle}_{\boldsymbol{\mu}} = \operatorname{soc}(E_i D^{\triangle}_{\boldsymbol{\mu}})$ is an irreducible $\mathsf{R}^{\Lambda}_{n-1}(\mathbb{K}[x])$-module by [36, Corollary 3.12], which also shows that $\varepsilon_i(\widetilde{e}_i D^{\triangle}_{\boldsymbol{\mu}}) = \varepsilon_i(D^{\triangle}_{\boldsymbol{\mu}}) - 1$.

The remaining statements follow from [36, Lemma 3.9]. (The paper [36] assumes that the quiver Γ is simply-laced but the arguments apply without change in type $C_{e-1}^{(1)}$.)

Parts (b) now follows by symmetry.

Now consider (c). Since y_n has positive degree, it is a nilpotent element of $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$, so the real claim here is that y_n has nilpotency index $\varepsilon_i(M)$ when acting on $E_i M$. This can be proved by repeating the argument of [39, Theorem 3.5.1] using [36, Lemma 2.1 and Lemma 3.7]. $\qquad\square$

Corollary 6G.2. Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_\bullet^\lhd$ and $\boldsymbol{\sigma}, \boldsymbol{\nu} \in \mathcal{K}_\bullet^\rhd$ and fix $i, j \in I$ and $a, b \in \mathbb{Z}$.

(a) If $\operatorname{soc}(E_i D_{\boldsymbol{\mu}}^\lhd) \cong q^a D_{\boldsymbol{\lambda}}^\lhd$, then $\operatorname{head}(F_i D_{\boldsymbol{\lambda}}^\lhd) \cong q^{\mathsf{d}_i - d_i(\boldsymbol{\lambda}) - a} D_{\boldsymbol{\mu}}^\lhd$

(b) If $\operatorname{soc}(E_j D_{\boldsymbol{\nu}}^\rhd) \cong q^b D_{\boldsymbol{\sigma}}^\rhd$, then $\operatorname{head}(F_j D_{\boldsymbol{\sigma}}^\rhd) \cong q^{\mathsf{d}_j - d_i(\boldsymbol{\sigma}) - b} D_{\boldsymbol{\nu}}^\rhd$.

Proof. Let $\triangle \in \{\lhd, \rhd\}$ and suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_\bullet^\triangle$ and $i \in I$. By tensor-hom adjointness,

$$\operatorname{Hom}_{\mathsf{R}_n^\Lambda(\mathbb{K}[x])}\left(q^a F_i^\Lambda D_{\boldsymbol{\lambda}}^\triangle, D_{\boldsymbol{\mu}}^\triangle\right) \cong \operatorname{Hom}_{\mathsf{R}_{n-1}^\Lambda(\mathbb{K}[x])}\left(q^a D_{\boldsymbol{\lambda}}^\triangle, E_i^\Lambda D_{\boldsymbol{\mu}}^\triangle\right).$$

By assumption, the right-hand hom-space is nonzero if and only if $\operatorname{soc}(E_i D_{\boldsymbol{\mu}}^\triangle) \cong q^a D_{\boldsymbol{\lambda}}^\triangle$. On the other hand, $F_i^\Lambda D_{\boldsymbol{\lambda}}^\triangle = q^{\mathsf{d}_i - d_i(\boldsymbol{\lambda})} F_i D_{\boldsymbol{\lambda}}^\triangle$ and $F_i D_{\boldsymbol{\lambda}}^\triangle$ is self-dual by Proposition 6B.3. Therefore, the left-hand hom-space is nonzero if and only if $q^{\mathsf{d}_i - d_i(\boldsymbol{\lambda}) - a} D_{\boldsymbol{\mu}}^\triangle$ is a quotient of $F_i D_{\boldsymbol{\lambda}}^\triangle$. Moreover, since $\operatorname{soc}(E_i D_{\boldsymbol{\mu}}^\triangle)$ is irreducible by Proposition 6G.1, it follows that $\operatorname{head}(F_i D_{\boldsymbol{\lambda}}^\triangle)$ is irreducible, so this completes the proof. $\qquad\square$

By Proposition 6G.1, if L is a composition factor of $E_i D_{\boldsymbol{\mu}}^\triangle$ then $\varepsilon_i(L) < \varepsilon(\widetilde{e}_i D_{\boldsymbol{\mu}}^\lhd)$, so we also obtain:

Corollary 6G.3. Suppose that $i, j \in I$ and let $\boldsymbol{\mu} \in \mathcal{K}_n^\lhd$ and $\boldsymbol{\nu} \in \mathcal{K}_n^\rhd$. Then

$$\varepsilon_i(D_{\boldsymbol{\mu}}^\lhd) = \max\left\{k \geq 0 \,\Big|\, E_i^k D_{\boldsymbol{\mu}}^\lhd \neq 0\right\} \quad \text{and} \quad \varepsilon_i(D_{\boldsymbol{\nu}}^\rhd) = \max\left\{k \geq 0 \,\Big|\, E_i^k D_{\boldsymbol{\nu}}^\rhd \neq 0\right\}.$$

Recall the definition of the quantum integers $[k]_i$ and quantum factorials $[k]_i!$ from subsection 6D.

Kashiwara's theory of global crystal bases, combined with Corollary 6F.18 and Theorem 6F.17, gives:

Lemma 6G.4 (Kashiwara [34, Lemma 12.1])**.** Suppose that $i, j \in I$ and let $\boldsymbol{\mu} \in \mathcal{K}_n^\lhd$ and $\boldsymbol{\nu} \in \mathcal{K}_n^\rhd$. Then

$$E_i \mathbb{D}_{\boldsymbol{\mu}}^\lhd = [\varepsilon_i^\lhd(\boldsymbol{\mu})]_i \, \mathbb{D}_{e_i \boldsymbol{\mu}}^\lhd + \sum_{\substack{\boldsymbol{\lambda} \in \mathcal{K}_{n-1}^\lhd \\ \varepsilon_i^\lhd(\boldsymbol{\lambda}) < \varepsilon_i^\lhd(\boldsymbol{\mu}) - \mathsf{d}_i}} a_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd, i} \mathbb{D}_{\boldsymbol{\lambda}}^\lhd, \qquad E_j \mathbb{D}_{\boldsymbol{\nu}}^\rhd = \left[\varepsilon_j^\rhd(\boldsymbol{\nu})\right]_j \mathbb{D}_{e_j \boldsymbol{\nu}}^\rhd + \sum_{\substack{\boldsymbol{\sigma} \in \mathcal{K}_{n-1}^\rhd \\ \varepsilon_j^\rhd(\boldsymbol{\sigma}) < \varepsilon_j^\rhd(\boldsymbol{\nu}) - \mathsf{d}_i}} a_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\rhd, i} \mathbb{D}_{\boldsymbol{\sigma}}^\rhd,$$

$$F_i \mathbb{D}_{\boldsymbol{\mu}}^\lhd = [\varphi_i^\lhd(\boldsymbol{\mu})]_i \, \mathbb{D}_{f_i \boldsymbol{\mu}}^\lhd + \sum_{\substack{\boldsymbol{\lambda} \in \mathcal{K}_{n+1}^\lhd \\ \varphi_i^\lhd(\boldsymbol{\lambda}) < \varphi_i^\lhd(\boldsymbol{\mu}) - \mathsf{d}_j}} b_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd, j} \mathbb{D}_{\boldsymbol{\lambda}}^\lhd, \qquad F_j \mathbb{D}_{\boldsymbol{\nu}}^\rhd = \left[\varphi_j^\rhd(\boldsymbol{\nu})\right]_j \mathbb{D}_{f_j \boldsymbol{\nu}}^\rhd + \sum_{\substack{\boldsymbol{\sigma} \in \mathcal{K}_{n+1}^\rhd \\ \varphi_j^\rhd(\boldsymbol{\sigma}) < \varphi_j^\rhd(\boldsymbol{\nu}) - \mathsf{d}_j}} b_{\boldsymbol{\sigma}\boldsymbol{\nu}}^{\rhd, j} \mathbb{D}_{\boldsymbol{\sigma}}^\rhd.$$

for bar invariant Laurent polynomials $a_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd, i}, a_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\rhd, i}, b_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\lhd, j}, b_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\rhd, j} \in \mathcal{A}$.

Similar to Corollary 6G.3, we can use Lemma 6G.4 to argue by induction to determine the crystal data statistics $\varepsilon_i^\triangle(\boldsymbol{\mu})$ and $\varphi_i^\triangle(\boldsymbol{\mu})$ from (6F.11), for $\boldsymbol{\mu} \in \mathcal{K}_n^\triangle$:

$$\varepsilon_i^\triangle(\boldsymbol{\mu}) = \max\left\{k \geq 0 \,\Big|\, E_i^k \mathbb{D}_{\boldsymbol{\mu}}^\triangle \neq 0\right\} \quad \text{and} \quad \varphi_i^\triangle(\boldsymbol{\mu}) = \max\left\{k \geq 0 \,\Big|\, F_i^k \mathbb{D}_{\boldsymbol{\mu}}^\triangle \neq 0\right\}, \quad (6\text{G}.5)$$

Using the last two results we can prove the "modular restriction rules" for the simple $\mathsf{R}_n^\Lambda(\mathbb{K}[x])$-modules. By Proposition 6G.1, we already know that $\widetilde{e}_i D_{\boldsymbol{\mu}}$ is irreducible so the

next result precisely identifies which irreducible it is. We remind the reader that this result applies to any cyclotomic KLR algebra of type $A_{e-1}^{(1)}$, A_∞, $C_{e-1}^{(1)}$ or C_∞ by Corollary 4F.4.

For $\triangle \in \{\triangleleft, \triangleright\}$ define $\boldsymbol{\omega}_{\overline{n}}^{\triangle}$ to be the minimal element of $\mathcal{P}_n^\ell$ with respect to the partial order $\triangle$. That is, $\boldsymbol{\omega}_n^{\triangleleft} = (n|0|\ldots|0)$ when $\triangle = \triangleleft$, and $\boldsymbol{\omega}_n^{\triangleright} = (0|\ldots|0|1^n)$ when $\boldsymbol{\omega}^{\triangle} = \triangleright$.

Theorem 6G.6. Suppose that $i, j \in I$, $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\triangleright}$. Then $\varepsilon_i(D_{\boldsymbol{\mu}}^{\triangleleft}) = \varepsilon_i^{\triangleleft}(\boldsymbol{\mu})$ and $\varepsilon_j(D_{\boldsymbol{\nu}}^{\triangleright}) = \varepsilon_j^{\triangleright}(\boldsymbol{\nu})$. If $\varepsilon_i(\boldsymbol{\mu}) \neq 0$ and $\varepsilon_j(\boldsymbol{\nu}) \neq 0$, respectively, then as $\mathsf{R}_{n-1}^{\Lambda}(\mathbb{K}[x])$-modules,

$$\widetilde{e}_i D_{\boldsymbol{\mu}}^{\triangleleft} \cong q^{\mathsf{d}_i(\varepsilon_i^{\triangleleft}(\boldsymbol{\mu})-1)} D_{e_i\boldsymbol{\mu}}^{\triangleleft} \qquad \text{and} \qquad \widetilde{e}_j D_{\boldsymbol{\nu}}^{\triangleright} \cong q^{\mathsf{d}_j(\varepsilon_j^{\triangleright}(\boldsymbol{\nu})-1)} D_{e_j\boldsymbol{\nu}}^{\triangleright}.$$

Proof. It is enough to consider case $\widetilde{e}_i D_{\boldsymbol{\mu}}^{\triangleleft}$, because the result for $\widetilde{e}_i D_{\boldsymbol{\nu}}^{\triangleright}$ is then implied by symmetry. We argue, first, by induction on n and then on the $\triangleleft$-dominance order to show that $\varepsilon_i(D_{\boldsymbol{\mu}}^{\triangle}) = \varepsilon_i^{\triangleleft}(\boldsymbol{\mu})$ and that, *up to shift*, $\widetilde{e}_i D_{\boldsymbol{\mu}}^{\triangleleft} \cong D_{e_i\boldsymbol{\mu}}^{\triangleleft}$. First, suppose that $\boldsymbol{\mu} = \boldsymbol{\omega}_n^{\triangleright} = (n|0|\ldots|0)$, which is the maximal element of $\mathcal{K}_n^{\triangleleft}$ under dominance. Then $D_{\boldsymbol{\mu}}^{\triangleleft}$ is the one dimensional trivial module of $\mathsf{R}_n^{\Lambda}(\mathbb{K}[\underline{x}])$ and $[D_{\boldsymbol{\mu}}^{\triangleleft}] = \mathbb{D}_{\boldsymbol{\mu}}^{\triangleleft}$ by Theorem 6E.16. Hence, $\varepsilon_i(D_{\boldsymbol{\mu}}^{\triangleleft}) = \varepsilon_i(\boldsymbol{\mu})$ and $\widetilde{e}_i D_{\boldsymbol{\mu}}^{\triangleleft} = D_{e_i\boldsymbol{\mu}}^{\triangleleft}$ if $\varepsilon_i(D_{\boldsymbol{\mu}}^{\triangleleft}) \neq 0$, which is if and only if $i = \mathsf{r}_n(\boldsymbol{\mu})$, $e_i\boldsymbol{\omega}_n^{\triangleright} = e_i\boldsymbol{\mu} = \boldsymbol{\omega}_{n-1}^{\triangleright}$ and $\varepsilon_i(\boldsymbol{\mu}) = 1$. Therefore, the theorem holds when $\boldsymbol{\mu} = \boldsymbol{\omega}_n^{\triangleright}$.

Now suppose that $\boldsymbol{\mu} \neq \boldsymbol{\omega}_n^{\triangleright}$ is not maximal with respect to dominance in $\mathcal{K}_n^{\triangleleft}$. By induction we can assume that, up to shift, $\widetilde{e}_i D_{\boldsymbol{\sigma}}^{\triangleleft} = D_{e_i\boldsymbol{\sigma}}^{\triangleleft}$ whenever $\boldsymbol{\sigma} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\sigma} \triangleright \boldsymbol{\mu}$. Set $\varepsilon = \varepsilon_i(D_{\boldsymbol{\mu}}^{\triangleleft})$. By Corollary 6G.3 and Proposition 6G.1, there exists $\boldsymbol{\nu} \in \mathcal{K}_{n-\varepsilon}^{\triangleleft}$ and a polynomial $f(q) \in \mathbb{N}[q, q^{-1}]$ such that $E_i^{(\varepsilon)}[D_{\boldsymbol{\mu}}^{\triangleleft}] = f(q)[D_{\boldsymbol{\nu}}^{\triangleleft}]$. We will show that $\boldsymbol{\nu} = e_i^{\varepsilon}\boldsymbol{\mu}$. By Theorem 6E.16, we can write

$$\left[D_{\boldsymbol{\mu}}^{\triangleleft}\right] = \mathbb{D}_{\boldsymbol{\mu}}^{\triangleleft} + \sum_{\boldsymbol{\sigma} \triangleright \boldsymbol{\mu}} \mathfrak{o}_{\boldsymbol{\sigma}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(q)\mathbb{D}_{\boldsymbol{\sigma}}^{\triangleleft}.$$

Let $\varepsilon' = \max\{\varepsilon_i^{\triangleleft}(\boldsymbol{\sigma}) \mid \mathfrak{o}_{\boldsymbol{\sigma}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(q) \neq 0\}$. If $\varepsilon' > \varepsilon$ then, by Lemma 6G.4,

$$E_i^{(\varepsilon')}\left[D_{\boldsymbol{\mu}}^{\triangleleft}\right] = \sum_{\substack{\boldsymbol{\sigma} \trianglerighteq \boldsymbol{\mu} \\ \varepsilon_i^{\triangleleft}(\boldsymbol{\sigma}) = \varepsilon'}} \mathfrak{o}_{\boldsymbol{\sigma}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(q)\mathbb{D}_{e_i^{\varepsilon'}\boldsymbol{\sigma}}^{\triangleleft}.$$

In particular, $E_i^{(\varepsilon')}[D_{\boldsymbol{\mu}}^{\triangleleft}] \neq 0$ is a contradiction. Similarly, if $\varepsilon' < \varepsilon$ then $E_i^{(\varepsilon')}[D_{\boldsymbol{\mu}}^{\triangleleft}] = 0$, giving a second contradiction. Hence, $\varepsilon' = \varepsilon$ and we have

$$f(q)\left[D_{\boldsymbol{\nu}}^{\triangleleft}\right] = E_i^{(\varepsilon)}[D_{\boldsymbol{\mu}}^{\triangleleft}] = \sum_{\substack{\boldsymbol{\sigma} \trianglerighteq \boldsymbol{\mu} \\ \varepsilon_i^{\triangleleft}(\boldsymbol{\sigma}) = \varepsilon}} \mathfrak{o}_{\boldsymbol{\sigma}\boldsymbol{\mu}}^{\mathbb{K}\triangleleft}(q)\mathbb{D}_{e_i^{\varepsilon}\boldsymbol{\sigma}}^{\triangleleft}.$$

If $\varepsilon_i(\boldsymbol{\mu}) < \varepsilon = \varepsilon_i(D_{\boldsymbol{\mu}}^{\triangleleft})$ then $\boldsymbol{\nu} = e_i^{\varepsilon}\boldsymbol{\sigma}$, for some $\boldsymbol{\sigma} \trianglerighteq \boldsymbol{\mu}$. Applying Corollary 6G.2 and induction, it follows that $D_{\boldsymbol{\mu}}^{\triangleleft} \cong \widetilde{f}_i^{\varepsilon}D_{\boldsymbol{\nu}}^{\triangleleft} \cong D_{\boldsymbol{\sigma}}^{\triangleleft}$, up to shift. This is a contradiction since $\boldsymbol{\sigma} \triangleright \boldsymbol{\mu}$. Therefore, $\varepsilon_i(\boldsymbol{\mu}) = \varepsilon_i(D_{\boldsymbol{\mu}}^{\triangleleft})$ and $\widetilde{e}_i D_{\boldsymbol{\mu}}^{\triangleleft} = D_{e_i\boldsymbol{\mu}}^{\triangleleft}$, up to shift, completing the proof of the inductive step.

We have now shown that $\varepsilon_i(D_{\boldsymbol{\mu}}^{\triangleleft}) = \varepsilon_i^{\triangleleft}(\boldsymbol{\mu})$ and if $\varepsilon_i(\boldsymbol{\mu}) > 0$ then $\widetilde{e}_i D_{\boldsymbol{\mu}}^{\triangleleft} \cong q^d D_{e_i\boldsymbol{\mu}}^{\triangleleft}$, for some $d \in \mathbb{Z}$, and it remains to show that $d = \mathsf{d}_i(\varepsilon_i^{\triangleleft}(\boldsymbol{\mu})-1)$. To complete the proof, observe that because $\varepsilon_i(D_{\boldsymbol{\mu}}^{\triangleleft}) = \varepsilon_i^{\triangleleft}(\boldsymbol{\mu})$, Kashiwara's Lemma 6G.4 implies that $[E_i D_{\boldsymbol{\mu}}^{\triangleleft} : D_{e_i\boldsymbol{\mu}}^{\triangleleft}]_q = [\varepsilon_i^{\triangleleft}(\boldsymbol{\mu})]_i$. By (KLR3), y_n commutes with $\mathsf{R}_{n-1}^{\Lambda}(\mathbb{K}[x])$, so multiplication by y_n defines an $\mathsf{R}_{n-1}^{\Lambda}(\mathbb{K}[x])$-module endomorphism of $E_i D_{\boldsymbol{\mu}}^{\triangleleft}$. By Proposition 6G.1 (c), the nilpotency index of y_n acting on $E_i D_{\boldsymbol{\mu}}^{\triangleleft}$ is $\varepsilon_i^{\triangleleft}(\boldsymbol{\mu})$. Therefore,

$$\left[y_n^k D_{\boldsymbol{\mu}}^{\triangleleft}/y_n^{k+1} D_{\boldsymbol{\mu}}^{\triangleleft} : D_{e_i\boldsymbol{\mu}}^{\triangleleft}\right]_q \neq 0, \qquad \text{for } 0 \leq k < \varepsilon_i(\boldsymbol{\mu}). \tag{6G.7}$$

Moreover, every composition factor of $E_i D_{\boldsymbol{\mu}}^{\triangleleft}$ isomorphic to $D_{e_i\boldsymbol{\mu}}^{\triangleleft}$, up to shift, arises uniquely in this way by the remarks above. The module $E_i D_{\boldsymbol{\mu}}^{\triangleleft}$ is self-dual by Proposition 6G.1 (a). Consequently, $\mathrm{head}(E_i D_{\boldsymbol{\mu}}^{\triangleleft}) \cong q^d D_{e_i\boldsymbol{\mu}}^{\triangleleft}$, for some $d \in \mathbb{Z}$. Moreover, $\widetilde{e}_i D_{\boldsymbol{\mu}}^{\triangleleft} = \mathrm{soc}(E_i D_{\boldsymbol{\mu}}^{\triangleleft}) \cong q^{d + 2\mathsf{d}_i(\varepsilon_i^{\triangleleft}(\boldsymbol{\mu})-1)} D_{e_i\boldsymbol{\mu}}^{\triangleleft}$ by (6G.7). Hence, using self-duality again, $d = -\mathsf{d}_i(\varepsilon_i^{\triangleleft}(\boldsymbol{\mu}) - 1)$, so $\widetilde{e}_i D_{\boldsymbol{\mu}}^{\triangleleft} = q^{\mathsf{d}_i(\varepsilon_i^{\triangleleft}(\boldsymbol{\mu})-1)} D_{e_i\boldsymbol{\mu}}^{\triangleleft}$ as claimed. $\qquad\square$

Corollary 6G.8. Let $i, j \in I$, $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\triangleright}$. Then $\varphi_i(D_{\boldsymbol{\mu}}^{\triangleleft}) = \varphi_i^{\triangleleft}(\boldsymbol{\mu})$, $\varphi_j(D_{\boldsymbol{\nu}}^{\triangleright}) = \varphi_j^{\triangleright}(\boldsymbol{\nu})$ and

$$\widetilde{f}_i D_{\boldsymbol{\mu}}^{\triangleleft} \cong q^{\mathsf{d}_i(1 - \varphi_i^{\triangle}(\boldsymbol{\mu}))} D_{f_i\boldsymbol{\mu}}^{\triangleleft} \qquad \text{and} \qquad \widetilde{f}_j D_{\boldsymbol{\nu}}^{\triangleright} \cong q^{\mathsf{d}_j(1 - \varphi_j^{\triangle}(\boldsymbol{\nu}))} D_{f_j\boldsymbol{\nu}}^{\triangleright}$$

as $\mathsf{R}_{n+1}^{\Lambda}(\mathbb{K}[x])$-modules.

Proof. Let $\triangle \in \{\triangleleft, \triangleright\}$. By (6F.12), $d_i(\boldsymbol{\mu}) = \varphi_i^{\triangle}(\boldsymbol{\mu}) - \varepsilon_i^{\triangle}(\boldsymbol{\mu})$, so

$$\widetilde{f}_i D_{\boldsymbol{\mu}}^{\triangle} \cong q^{\mathsf{d}_i(1 - \varphi_i^{\triangle}(\boldsymbol{\mu}))} D_{f_i\boldsymbol{\mu}}^{\triangle}$$

by Theorem 6G.6 and Corollary 6G.2. In turn, this implies that $\varphi_i(D_{\boldsymbol{\mu}}^{\triangle}) = \varphi_i^{\triangle}(\boldsymbol{\mu})$. $\qquad\square$

Since $\varepsilon_i(D_{\boldsymbol{\mu}}^{\triangle}) = \varepsilon_i^{\triangle}(\boldsymbol{\mu})$ by Theorem 6G.6, and $\varphi_i(D_{\boldsymbol{\mu}}^{\triangle}) = \varphi_i^{\triangle}(\boldsymbol{\mu})$ by Corollary 6G.8, Lemma 6G.4 now implies:

Corollary 6G.9. Let $i, j \in I$, $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\triangleright}$. Then

$$E_i[D_{\boldsymbol{\mu}}^{\triangleleft}] = [\varepsilon_i^{\triangleleft}(\boldsymbol{\mu})]_i [D_{e_i\boldsymbol{\mu}}^{\triangleleft}] + \sum_{\substack{\boldsymbol{\lambda} \in \mathcal{K}_{n-1}^{\triangleleft} \\ \varepsilon_i^{\triangleleft}(\boldsymbol{\lambda}) < \varepsilon_i^{\triangleleft}(\boldsymbol{\mu}) - \mathsf{d}_i}} c_{\boldsymbol{\lambda\mu}}^{\triangleleft, i} [D_{\boldsymbol{\lambda}}^{\triangleleft}],$$

$$E_j[D_{\boldsymbol{\nu}}^{\triangleright}] = [\varepsilon_j^{\triangleright}(\boldsymbol{\nu})]_j [D_{e_j\boldsymbol{\nu}}^{\triangleright}] + \sum_{\substack{\boldsymbol{\sigma} \in \mathcal{K}_{n-1}^{\triangleright} \\ \varepsilon_j^{\triangleright}(\boldsymbol{\sigma}) < \varepsilon_j^{\triangleright}(\boldsymbol{\nu}) - \mathsf{d}_i}} c_{\boldsymbol{\sigma\nu}}^{\triangleright, i} [D_{\boldsymbol{\sigma}}^{\triangleright}],$$

$$F_i[D_{\boldsymbol{\mu}}^{\triangleleft}] = [\varphi_i^{\triangleleft}(\boldsymbol{\mu})]_i [D_{f_i\boldsymbol{\mu}}^{\triangleleft}] + \sum_{\substack{\boldsymbol{\lambda} \in \mathcal{K}_{n+1}^{\triangleleft} \\ \varphi_i^{\triangleleft}(\boldsymbol{\lambda}) < \varphi_i^{\triangleleft}(\boldsymbol{\mu}) - \mathsf{d}_i}} d_{\boldsymbol{\lambda\mu}}^{\triangleleft, i} [D_{\boldsymbol{\lambda}}^{\triangleleft}],$$

$$F_j[D_{\boldsymbol{\nu}}^{\triangleright}] = [\varphi_j^{\triangleright}(\boldsymbol{\nu})]_j [D_{f_j\boldsymbol{\nu}}^{\triangleright}] + \sum_{\substack{\boldsymbol{\sigma} \in \mathcal{K}_{n+1}^{\triangleright} \\ \varphi_j^{\triangleright}(\boldsymbol{\sigma}) < \varphi_j^{\triangleright}(\boldsymbol{\nu}) - \mathsf{d}_j}} d_{\boldsymbol{\sigma\nu}}^{\triangleright, j} [D_{\boldsymbol{\sigma}}^{\triangleright}].$$

for bar invariant Laurent polynomials $c_{\boldsymbol{\lambda\mu}}^{\triangleleft, i}, c_{\boldsymbol{\lambda\mu}}^{\triangleright, i}, d_{\boldsymbol{\lambda\mu}}^{\triangleleft, j}, d_{\boldsymbol{\lambda\mu}}^{\triangleright, j} \in \mathbb{N}[q, q^{-1}]$.

Many people have observed that the last result implies that the dimension of $D_{\boldsymbol{\mu}}^{\triangle}$ is at least the number of paths in the $\triangle$-crystal graph from $\underline{\mathbf{0}}_{\ell}$ to $\boldsymbol{\mu}$, but we can do much better. If $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangle}$ and $\underline{\mathbf{0}}_{\ell} \overset{\mathbf{i}\triangle}{\rightsquigarrow} \boldsymbol{\mu}$ is a good node sequence, define the bar invariant polynomial $[\varepsilon_{\mathbf{i}}] \in \mathbb{N}[q, q^{-1}]$ recursively by setting

$$[\varepsilon_{\mathbf{i}}^{\triangle}(q)] = \begin{cases} [\varepsilon_{i_n}^{\triangle}(\boldsymbol{\mu})]_{i_n} [\varepsilon_{\mathbf{i}'}(q)], & \text{if } n > 0 \text{ and } \mathbf{i}' = (i_1, \ldots, i_{n-1}), \\ 1 & \text{if } n = 0. \end{cases}$$

Given two characters $\chi, \chi' \in \mathbb{N}[q, q^{-1}][I^n]$ write $\chi \geq \chi'$ if $\chi - \chi' \in \mathbb{N}[q, q^{-1}][I^n]$.

Corollary 6G.10. Let $\boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft}$ and $\boldsymbol{\nu} \in \mathcal{K}_n^{\triangleright}$. Then

$$\mathrm{ch}\, D_{\boldsymbol{\mu}}^{\triangleleft} \geq \sum_{\underline{\mathbf{0}}_{\ell} \overset{\mathbf{i}\triangleleft}{\rightsquigarrow} \boldsymbol{\mu}} [\varepsilon_{\mathbf{i}}^{\triangleleft}(q)]\, \mathbf{i} \qquad \text{and} \qquad \mathrm{ch}\, D_{\boldsymbol{\nu}}^{\triangleright} \geq \sum_{\underline{\mathbf{0}}_{\ell} \overset{\mathbf{j}\triangleright}{\rightsquigarrow} \boldsymbol{\nu}} \left[\varepsilon_{\mathbf{j}}^{\triangleright}(q)\right] \mathbf{j}.$$

Proof. This follows easily from Corollary 6G.10 by induction on n. $\qquad\square$

This result is rarely sharp. When $\mathscr{R}_n^\Lambda(F)$ is semisimple and $S_{\boldsymbol\lambda}^\triangle = D_{\boldsymbol\lambda}^\triangle$ is concentrated in degree zero, then the $\triangle$-good residue sequences are in bijection with the standard $\boldsymbol\lambda$-tableaux and $[\varepsilon_{\mathbf i}^\triangle(q)] = 1$ (cf. [52, Proposition 2.4.6]). It follows that the right-hand side is the graded character of the Specht module, which is concentrated in degree zero in the semisimple case, so in this case $D_{\boldsymbol\mu}^\triangle = S_{\boldsymbol\mu}^\triangle$ and both bounds in corollary are sharp.

Corollary 6G.11. *Let* $i,j \in I$, $\boldsymbol\mu \in \mathcal{K}_n^\triangleleft$ *and* $\boldsymbol\nu \in \mathcal{K}_n^\triangleright$. *Then*

$$\mathrm{END}_{\mathsf{R}_{n-1}^\Lambda(F)}\left(E_i^\Lambda D_{\boldsymbol\mu}^\triangleleft\right) \cong F[y_n]/\left(y_n^{\varepsilon_i^\triangleleft(\boldsymbol\mu)}\right) \quad \text{and} \quad \mathrm{END}_{\mathsf{R}_{n-1}^\Lambda(F)}\left(E_i^\Lambda D_{\boldsymbol\nu}^\triangleright\right) \cong F[y_n]/\left(y_n^{\varepsilon_i^\triangleright(\boldsymbol\nu)}\right).$$

as $\mathbb{Z}$-*graded algebras.*

Proof. Let $\triangle \in \{\triangleleft, \triangleright\}$. As observed in the proof of Theorem 6G.6, multiplication by y_n defines an $\mathsf{R}_{n-1}^\Lambda(F)$-module homomorphism of $E_i D_{\boldsymbol\mu}^\triangle = E_i^\Lambda D_{\boldsymbol\mu}^\triangle$ and y_n acts on $E_i D_{\boldsymbol\mu}^\triangle$ as a nilpotent operator of index $\varepsilon_i^\triangle(\boldsymbol\mu)$. Hence, the image of y_n in the endomorphism ring $\mathrm{END}_{\mathsf{R}_{n-1}^\Lambda(F)}(E_i D_{\boldsymbol\mu}^\triangleleft)$ generates a subalgebra isomorphic to $F[y_n]/(y_n^{\varepsilon_i^\triangleleft(\boldsymbol\mu)})$. By (6G.7), the image of the endomorphism given by multiplication by y_n^k has head isomorphic to $q^{\mathsf d_i(2k+1-\varepsilon_i^\triangle(\boldsymbol\mu))} D_{e_i\boldsymbol\mu}^\triangle$, for $0 \le k < \varepsilon_i^\triangle(\boldsymbol\mu)$. On the other hand, if φ is a (homogeneous) $\mathsf{R}_{n-1}^\Lambda(\mathbb{K}[x])$-module endomorphism of $E_i D_{\boldsymbol\mu}^\triangle$ then φ then head(im φ) $\cong q^k D_{e_i\boldsymbol\mu}^\triangle$, for some $k \in \mathbb{Z}$. As $[E_i D_{\boldsymbol\mu}^\triangle : \varepsilon_i^\triangle(\boldsymbol\mu)]_q = [\varepsilon_i^\triangle(\boldsymbol\mu)]_i$, it follows that $\varphi(m) = y_n^k m$, for some k. $\qquad\square$

We are missing a description of the endomorphism rings

$$\mathrm{END}_{\mathsf{R}_{n+1}^\Lambda(F)}\left(F_i^\Lambda D_{\boldsymbol\mu}^\triangleleft\right) \quad \text{and} \quad \mathrm{END}_{\mathsf{R}_{n+1}^\Lambda(F)}\left(F_j^\Lambda D_{\boldsymbol\mu}^\triangleleft\right),$$

for $\boldsymbol\mu \in \mathcal{K}_n^\triangleleft$, $\boldsymbol\nu \in \mathcal{K}_n^\triangleleft$ and $i,j \in I$. Naively, we might expect that

$$\mathrm{END}_{\mathsf{R}_{n+1}^\Lambda(F)}\left(F_i^\Lambda D_{\boldsymbol\mu}^\triangleleft\right) \cong F[c_{n+1}]/\left(c_{n+1}^{\varphi_i^\triangleleft(\boldsymbol\mu)}\right)$$

and

$$\mathrm{END}_{\mathsf{R}_{n+1}^\Lambda(F)}\left(F_i^\Lambda D_{\boldsymbol\nu}^\triangleright\right) \cong F[c_{n+1}]/\left(c_{n+1}^{\varphi_j^\triangleright(\boldsymbol\nu)}\right),$$

where $c_{n+1} = y_1 + y_2 + \cdots + y_{n+1}$. In type $A_{e-1}^{(1)}$, this result was proved by Brundan and Kleshchev [11, Theorem 4.9]. Unfortunately, in type $C_{e-1}^{(1)}$, the element c_{n+1} is rarely homogeneous, so this statement needs to be modified. In any case, we do not see how to obtain a description of these endomorphism rings using the results of this paper.

<div align="center">INDEX OF NOTATION</div>

This index of notation gives a brief description of the main notation used in the paper, together with the section and page where the notation is first introduced.

§	Symbol	Description	Page
	$\Bbbk[\underline{x}]$	The positively graded polynomial ring $\Bbbk[\underline{x}]$, with $x \in \underline{x}$ in degree 1	199
	$\mathbb{K}[\underline{x}^{\pm}]$	The $\mathbb{Z}$-graded Laurent polynomial ring $\mathbb{K}[\underline{x}, \underline{x}^{-1}]$	199
	$\mathcal{A}$	The ring $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, where q is an indeterminate	199
	$\mathbb{A}$	The ring $\mathbb{Q}(q)$ of rational functions in q	199
	$q^d M$	The graded module obtained by shifting the grading on M by d	199
	$\mathrm{Hom}_A(M, N)$	The homogeneous A-module maps $M \to N$ of degree 0	199
	$\mathrm{HOM}_A(M, N)$	*All* homogeneous A-module maps $M \to N$	199
	$\mathrm{End}_A(M)$	The homogeneous A-module endomorphisms of M of degree 0	199
	$\mathrm{END}_A(M)$	*All* homogeneous A-module endomorphisms of M	199
2B	$\mathbb{N}$	The set of non-negative integers $\mathbb{Z}_{\geq 0}$	200
	Γ	A symmetrisable quiver, usually of type $A_{e-1}^{(1)}$ or $C_{e-1}^{(1)}$	200
	I	The vertex set $\{0, 1, \ldots, e-1\}$ of Γ	200
	$C = (\mathsf{c}_{ij})$	Cartan matrix of Γ	200
	d_i	$D = \mathrm{diag}(\mathsf{d}_0, \ldots, \mathsf{d}_{e-1})$ is the symmetriser of C	200
	α_i	Simple root, for $i \in I$	200
	Λ_i	Fundamental weight, for $i \in I$	200
	P^+	Dominant weight lattice	200
	Q^+	Positive root lattice	200
	$\mathfrak{S}_n$	Symmetric group on $\{1, 2, \ldots, n\}$	200
	σ_k	Simple reflection $\sigma_k = (k, k+1) \in \mathfrak{S}_n$, for $1 \leq k < n$	200
	$L(w)$	Coxeter length of $w \in \mathfrak{S}_n$	200
	$A_{e-1}^{(1)}$	Affine quiver of type A with vertex set I	201
	$C_{e-1}^{(1)}$	Affine quiver of type C with vertex set I	201
	$\mathbf{Q}_I$	Family $\mathbf{Q}_I = (Q_{i,j}(u,v))_{i,j \in I}$ of Rouquier's Q-polynomials	201
2C	$\mathbf{W}_I$	Family $\mathbf{W}_I = (W_i(u))_{i \in I}$ of weight polynomials, for $i \in I$	202
	Λ	The dominant weight in P^+ determined by $\mathbf{W}_I$	202
	I^α	The orbit $\{\mathbf{i} \in I^n \mid \alpha = \alpha_{i_1} + \cdots + \alpha_{i_n}\}$ for $\alpha \in Q^+$	202
	$\mathscr{R}_n^\Lambda, \mathscr{R}_\alpha^\Lambda$	A (standard) cyclotomic KLR algebra	202
	$\mathscr{R}_n, \mathscr{R}_\alpha$	A (standard) KLR algebra	202
	$1_{\mathbf{i}}$	An idempotent in, and generator of, R_n^Λ or $\mathscr{R}_n^\Lambda$, for $i \in I$	202
	$y_1, \ldots, y_n$	Generators of R_n^Λ or $\mathscr{R}_n^\Lambda$	202
	$\psi_1, \ldots, \psi_{n-1}$	Generators of R_n^Λ or $\mathscr{R}_n^\Lambda$	202
	$\deg$	Degree function on $\mathscr{R}_n^\Lambda$, R_n^Λ, graded rings, and tableaux	202
	$*$	The unique anti-isomorphism of R_n^Λ, or $\mathscr{R}_n^\Lambda$, that fixes each generator	202
	$M^{\circledast}$	Graded dual $M^{\circledast} = \mathrm{HOM}_A(M, K)$ of M	202
	$\mathbf{Q}_I^{\underline{x}}$	Family $(Q_{i,j}^{\underline{x}}(u,v))_{i,j \in I}$ of deformed Q-polynomials defining R_n^Λ	203

§	Symbol	Description	Page
	$\mathbf{W}_I^x$	Family $(W_i^x(u))_{i \in I}$ of deformed weight polynomials defining R_n^Λ	203
	R_n^Λ	Deformed cyclotomic KLR algebra determined by $(\Gamma, \mathbf{Q}_I^x, \mathbf{W}_I^x)$	204
	$\mathsf{R}_\alpha^\Lambda$	Block of cyclotomic KLR algebra R_n^Λ	204
2D	ψ_w	Element of R_n^Λ or $\mathscr{R}_n^\Lambda$ defined by a fixed reduced expression for $w \in \mathfrak{S}_n$	204
	φ_w	Element of R_n^Λ or $\mathscr{R}_n^\Lambda$ indexed by $w \in \mathfrak{S}_n$	205
3A	(c, r)	A content system for R_n^Λ	205
3B	$\mathcal{P}_n^\ell$	The poset of ℓ-partitions of n	210
	$\triangleleft, \triangleright$	Reverse dominance and dominance orders on $\mathcal{P}_n^\ell$	210
	$\triangle, \triangledown$	Throughout, $\triangle \in \{\triangleleft, \triangleright\}$ and $\{\triangle, \triangledown\} = \{\triangleleft, \triangleright\}$	210
	(k, r, c)	The node in component k, row r and column c	210
	$\leq, \geq$	Lexicographic orders on the set of nodes $\{(k, r, c)\}$	210
	$\mathrm{Std}(\boldsymbol{\lambda})$	Standard tableau of shape $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$	210
	$\mathrm{Std}^2(\mathcal{P})$	Pairs of standard tableaux $\bigcup_{\boldsymbol{\lambda} \in \mathcal{P}} \mathrm{Std}(\boldsymbol{\lambda}) \times \mathrm{Std}(\boldsymbol{\lambda})$, for $\mathcal{P} \subseteq \mathcal{P}_n^\ell$	210
	$\mathrm{Std}(\mathbf{i})$	Set of standard tableaux with residue sequence $\mathbf{i}$	211
	$\mathsf{c}(k, r, c)$	Content $\mathsf{c}(k, c - r)$ of the node (k, r, c)	211
	$\mathsf{r}(k, r, c)$	Residue $\mathsf{r}(k, c - r)$ of the node (k, r, c)	211
	$\mathsf{c}(\mathsf{t})$	Content sequence $\mathsf{c}(\mathsf{t}) = (\mathsf{c}_1(\mathsf{t}), \ldots, \mathsf{c}_n(\mathsf{t}))$ of the tableau t	211
	$\mathsf{r}(\mathsf{t})$	Residue sequence $\mathsf{r}(\mathsf{t}) = (\mathsf{r}_1(\mathsf{t}), \ldots, \mathsf{r}_n(\mathsf{t}))$ of the tableau t	211
	$Q_m(\mathsf{t})$	$Q_{\mathsf{r}_m(\mathsf{t}), \mathsf{r}_{m+1}(\mathsf{t})}^x (\mathsf{c}_m(\mathsf{t}), \mathsf{c}_{m+1}(\mathsf{t})) - \delta_{\mathsf{r}_m(\mathsf{t}), \mathsf{r}_{m+1}(\mathsf{t})} / (\mathsf{c}_{m+1}(\mathsf{t}) - \mathsf{c}_m(\mathsf{t}))^2$	212
3C	F_t	Semisimple idempotent in $\mathsf{R}_n^\Lambda(\mathbb{K}[\underline{x}^\pm])$, for $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$	212
3F	$\mathcal{S}_n^\ell$	Universal level ℓ semisimple algebra for content system	221
	Ψ_{st}	Basis elements of $\mathcal{S}_n^\ell(\mathbb{K})$	222
4A	$\mathsf{s}_{\downarrow m}$	Restriction of the tableau s to $\{1, \ldots, m\}$	226
	$\mathsf{s} \trianglelefteq \mathsf{u}$	dominance on standard tableaux	226
	$(\mathsf{s}, \mathsf{t}) \trianglelefteq (\mathsf{u}, \mathsf{v})$	Dominance on pairs of tableaux: $\mathsf{s} \trianglelefteq \mathsf{u}$ and $\mathsf{t} \triangleleft \mathsf{v}$	226
	$\boldsymbol{\lambda}'$	Conjugate ℓ-partition $\boldsymbol{\lambda}' = (\lambda^{(\ell)\prime} | \ldots | \lambda^{(1)\prime})$	226
	t'	Conjugate tableau: $\mathsf{t}'(k, r, c) = \mathsf{t}(\ell - k + 1, c, r)$	226
	$\mathsf{t}_{\boldsymbol{\lambda}}^\triangleleft, \mathsf{t}_{\boldsymbol{\lambda}}^\triangleright$	Initial tableau with respect to $\triangleleft$ and $\triangleright$	226
	$d_\mathsf{t}^\triangleleft, d_\mathsf{t}^\triangleright$	Permutations: $d_\mathsf{t}^\triangleleft \mathsf{t}_{\boldsymbol{\lambda}}^\triangleleft = \mathsf{t} = d_\mathsf{t}^\triangleright \mathsf{t}_{\boldsymbol{\lambda}}^\triangleright$, for $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$	226
	$\mathbf{i}_{\boldsymbol{\lambda}}^\triangleleft, \mathbf{i}_{\boldsymbol{\lambda}}^\triangleright$	Residue sequences: $\mathbf{i}_{\boldsymbol{\lambda}}^\triangleleft = \mathsf{r}(\mathsf{t}_{\boldsymbol{\lambda}}^\triangleleft)$ and $\mathbf{i}_{\boldsymbol{\lambda}}^\triangleright = \mathsf{r}(\mathsf{t}_{\boldsymbol{\lambda}}^\triangleright)$	227
	$y_{\boldsymbol{\lambda}}^\triangleleft, y_{\boldsymbol{\lambda}}^\triangleright$	Polynomials $y_{\boldsymbol{\lambda}}^\triangleleft, y_{\boldsymbol{\lambda}}^\triangleright \in \mathbb{k}[y_1, \ldots, y_n]$	227
	$\psi_{\mathsf{st}}^\triangleleft, \psi_{\mathsf{st}}^\triangleright$	The basis elements $\psi_{d_\mathsf{s}^\triangleleft} y_{\boldsymbol{\lambda}}^\triangleleft 1_{\mathbf{i}_{\boldsymbol{\lambda}}^\triangleleft} \psi_{d_\mathsf{t}^\triangleleft}^*$ and $\psi_{d_\mathsf{s}^\triangleright} y_{\boldsymbol{\lambda}}^\triangleright 1_{\mathbf{i}_{\boldsymbol{\lambda}}^\triangleright} \psi_{d_\mathsf{t}^\triangleright}^*$	227
	$f_{\mathsf{st}}^\triangleleft, f_{\mathsf{st}}^\triangleright$	The basis elements $f_{\mathsf{st}}^\triangleleft = F_\mathsf{s} \psi_{\mathsf{st}}^\triangleleft F_\mathsf{t}$ and $f_{\mathsf{st}}^\triangleright = F_\mathsf{s} \psi_{\mathsf{st}}^\triangleright F_\mathsf{t}$, for $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$	227
	$\rho_k(\mathsf{t})$	The difference $\mathsf{c}_{k+1}(\mathsf{s}) - \mathsf{c}_k(\mathsf{s}) \in \mathbb{k}[\underline{x}]$	228

§	Symbol	Description	Page
	$\gamma_{\mathsf{t}}^{\lhd}, \gamma_{\mathsf{t}}^{\rhd}$	Important monomials in $\mathbb{K}[\underline{x}^{\pm}]$, for $\mathsf{t} \in \mathrm{Std}(\mathcal{P}_n^{\ell})$	229
4C	$\deg^{\lhd}, \deg^{\rhd}$	Degree functions for the $\psi^{\lhd}$ and $\psi^{\rhd}$ bases	234
	$S_{\boldsymbol{\lambda}}^{\lhd}, S_{\boldsymbol{\lambda}}^{\rhd}$	Graded Specht modules for the $\psi^{\lhd}$ and $\psi^{\rhd}$ bases	236
4D	$\gamma_{\boldsymbol{\lambda}}^{\diamondsuit}$	The defect polynomial of $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$	237
	$\alpha_{\boldsymbol{\lambda}}$	The positive root $\sum_{A \in \boldsymbol{\lambda}} \alpha_{\mathsf{r}(A)} \in Q^+$	238
	$\mathcal{P}_{\alpha}^{\ell}$	The set of ℓ-partitions $\{\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell} \mid \alpha_{\boldsymbol{\lambda}} = \alpha\}$	238
	$\mathrm{def}(\boldsymbol{\lambda})$	The Λ-defect of $\boldsymbol{\lambda}$, which is $\mathrm{def}(\alpha_{\boldsymbol{\lambda}}) = (\Lambda, \alpha_{\boldsymbol{\lambda}}) - \frac{1}{2}(\alpha_{\boldsymbol{\lambda}}, \alpha_{\boldsymbol{\lambda}})$	238
	$d_A^{\lhd}(\boldsymbol{\lambda}), d_A^{\rhd}(\boldsymbol{\lambda})$	Number of addable minus removal i-nodes below/above A	238
	$d_i(\boldsymbol{\lambda})$	Number of addable minus removable i-nodes of $\boldsymbol{\lambda}$	238
4E	$\langle\ ,\ \rangle_{\alpha}$	Non-degenerate symmetric bilinear form on $\mathsf{R}_{\alpha}^{\Lambda}(\mathbb{K}[\underline{x}])$	240
	$z_{\boldsymbol{\lambda}}^{\lhd}, z_{\boldsymbol{\lambda}}^{\rhd}$	Distinguished generators for Specht submodules	241
5A	$\langle\ ,\ \rangle_{\boldsymbol{\lambda}}^{\lhd}, \langle\ ,\ \rangle_{\boldsymbol{\lambda}}^{\rhd}$	Bilinear forms on $S_{\boldsymbol{\lambda}}^{\lhd}$ and $S_{\boldsymbol{\lambda}}^{\rhd}$	246
	$D_{\boldsymbol{\mu}}^{\lhd}, D_{\boldsymbol{\nu}}^{\rhd}$	Simple R_n^{Λ}-modules defined by the $\psi_{\mathsf{st}}^{\lhd}$ and $\psi_{\mathsf{st}}^{\rhd}$ bases	246
	$\mathcal{K}_n^{\lhd}, \mathcal{K}_n^{\rhd}$	Indexing sets for simple R_n^{Λ}-modules	246
5B	$\mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\mathbb{K}\lhd}(q), \mathsf{d}_{\boldsymbol{\lambda}\boldsymbol{\nu}}^{\mathbb{K}\rhd}(q)$	Graded decomposition numbers for R_n^{Λ}	249
	$Y_{\boldsymbol{\mu}}^{\lhd}, Y_{\boldsymbol{\nu}}^{\rhd}$	Projective covers of $D_{\boldsymbol{\mu}}^{\lhd}$ and $D_{\boldsymbol{\nu}}^{\rhd}$, respectively	249
5C	$\mathrm{ch}\, M$	Formal character in $\mathcal{A}[I^n]$, for the R_n^{Λ}-module M	250
	$\overline{}$	The bar involution on $\mathcal{A} + \mathbb{Z}[q, q^{-1}]$ given by $\overline{f(q)} = f(q^{-1})$	250
5D	$\mathsf{m}(\boldsymbol{\mu})$	Bijection $\mathsf{m} \colon \mathcal{K}_n^{\lhd} \longrightarrow \mathcal{K}_n^{\rhd}$ such that $D_{\boldsymbol{\mu}}^{\lhd} \cong D_{\mathsf{m}(\boldsymbol{\mu})}^{\rhd}$	251
5E	ε	Sign automorphism of Γ and associated maps on R_n^{Λ}, $U_q(\mathfrak{g}_{\Gamma}), \ldots$	254
	$\mathrm{soc}\, M$	The socle of M	256
	$\mathrm{head}\, M$	The head of M	256
6A	$\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$	Category of graded R_n^{Λ}-modules, which are finite dimensional over $\mathbb{K}$	257
	$\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$	Full subcategory of $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$ of projective modules	257
	E_i^{Λ}	The i-restriction functor $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_{\alpha+\alpha_i}^{\Lambda}\, to\, \mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_{\alpha}^{\Lambda}$	257
	F_i^{Λ}	The i-induction functor $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_{\alpha}^{\Lambda} \to \mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_{\alpha+\alpha_i}^{\Lambda}$	257
6B	$M^{\#}$	The projective dual: $M^{\#} = \mathrm{Hom}_{\mathsf{R}_n^{\Lambda}(\mathbb{K}[x])}(M, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x]))$	262
6C	$[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])]$	Grothendieck group of $\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$	263
	$[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])]$	Grothendieck group of $\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])$	263
	$[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$	$\bigoplus_{n \geq 0}[\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])]$	263
	$[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])]$	$\bigoplus_{n \geq 0}[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_n^{\Lambda}(\mathbb{K}[x])]$	263
	$\langle\ ,\ \rangle$	Cartan pairing $[\mathrm{Proj}_{\mathbb{K}}\, \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])] \times [\mathrm{Rep}_{\mathbb{K}}\, \mathsf{R}_{\bullet}^{\Lambda}(\mathbb{K}[x])] \to \mathcal{A}$	264
6D	q_i	For $i \in I$, $q_i = q^{\mathsf{d}_i}$	264
	$[k]_i$	For $k \in \mathbb{Z}$, $[k]_i$ is the quantum integer $(q_i^k - q_i^{-k})/(q_i - q_i^{-1}) \in \mathcal{A}$	264

Acknowledgments

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