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A class of finite-by-cocommutative Hopf algebras

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ABSTRACT. We present a rich source of Hopf algebras starting from a cofinite central extension of a Noetherian Hopf algebra and a subgroup of the algebraic group of characters of the central Hopf subalgebra. The construction is transparent from a Tannakian perspective. We determine when the new Hopf algebras are co-Frobenius, or cosemisimple, or Noetherian, or regular, or have finite Gelfand-Kirillov dimension.

1. Introduction

A Hopf algebra H is *commutative-by-finite* if it has a normal Hopf subalgebra A such that A is commutative and H is a finitely generated A-module. In other words there is an exact sequence of Hopf algebras $A \hookrightarrow H \twoheadrightarrow \mathfrak{u}$ where A is commutative and $\mathfrak{u}$ is finite-dimensional. There are various remarkable families of commutative-by-finite Hopf algebras arising from the theory of quantum groups. A systematic study of *affine* commutative-by-finite Hopf algebras was started in [14]; here "affine" means that H is a finitely generated algebra.

A Hopf algebra K is *finite-by-cocommutative* if it fits into an exact sequence of Hopf algebras $\mathfrak{a} \hookrightarrow K \twoheadrightarrow U$ where U is cocommutative and $\mathfrak{a}$ is finite-dimensional. Lusztig's quantum groups at roots of 1 are finite-by-cocommutative. An example of a finite-by-cocommutative Hopf algebra appeared in [6] to disprove a conjecture on co-Frobenius Hopf algebras. A family of examples containing that one and characterized by suitable properties was presented in [36]. The finite dual H° of an affine commutative-by-finite Hopf algebra H was studied in [15]; it is finite-by-cocommutative.

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The goal of this paper is to present and study a family of finite-by-cocommutative Hopf algebras. We assume that the base field $\Bbbk$ is algebraically closed and has characteristic 0. To start with, consider a Noetherian Hopf algebra H with a central Hopf subalgebra A; we set $H_\varepsilon = H/HA^+$, where $A^+ = \ker \varepsilon_{|A}$. Thus we have an extension of Hopf algebras

$$A \hookrightarrow H \twoheadrightarrow H_\varepsilon. \tag{$\mathcal{E}$}$$

In Section 3, for any subgroup Γ of the pro-affine algebraic group $G = \operatorname{Alg}(A, \Bbbk)$, we define a suitable subgroup Γ_{fd} of Γ and a Hopf subalgebra $\mathcal{H}(\Gamma)$ of the finite dual H°, which is an extension of Hopf algebras

$$H_\varepsilon^\circ \hookrightarrow \mathcal{H}(\Gamma) \twoheadrightarrow \Bbbk\Gamma_{\mathrm{fd}}. \tag{$\mathcal{F}^\Gamma$}$$

From a Tannakian perspective, the category of finite-dimensional comodules over $\mathcal{H}(\Gamma)$ is equivalent to the full subcategory $\mathscr{C}_\Gamma$ of the category $\operatorname{rep} H$ of finite-dimensional H-modules such that the action of A is semisimple and by characters in Γ; thus the objects of $\mathscr{C}_\Gamma$ bear a Γ-grading.

In Section 4 we assume further that the extension $(\mathcal{E})$ is cleft and that $\dim H_\varepsilon < \infty$. Then H is a finitely generated A-module, A is Noetherian, G is an algebraic group, $\Gamma_{\mathrm{fd}} = \Gamma$, $(\mathcal{F}^\Gamma)$ becomes $H_\varepsilon^* \hookrightarrow \mathcal{H}(\Gamma) \twoheadrightarrow \Bbbk\Gamma$ and so $\mathcal{H}(\Gamma)$ is finite-by-cocommutative. We establish several properties of $\mathcal{H}(\Gamma)$:

Theorem 1.1.

(i) If Γ is finitely-generated, then $\mathcal{H}(\Gamma)$ is affine.
(ii) $\mathcal{H}(\Gamma)$ is co-Frobenius.
(iii) $\mathcal{H}(\Gamma)$ is cosemisimple if and only if H_ε is semisimple.
(iv) If Γ is finitely-generated, then $\operatorname{GKdim} \mathcal{H}(\Gamma) < \infty$ *if and only if Γ is nilpotent-by-finite.*
(v) $\mathcal{H}(\Gamma)$ is Noetherian if and only if Γ is polycyclic-by-finite.
(vi) If $\mathcal{H}(\Gamma)$ is Noetherian, then it is regular iff H_ε is semisimple.

Theorem 1.1 is proved in Section 4, see Remark 4.11 and Theorems 4.16, 4.17, 4.20, 4.23 and 4.26. The keys to these results are that $\mathcal{H}(\Gamma)$ is *strongly* Γ-graded and that Γ, being a subgroup of an algebraic group, is linear, i.e., embedable into $GL(n, \Bbbk)$ for some n.

The contents of the paper are organized in the following way. Section 2 contains expositions of known facts needed along the article. In Section 3 we present the Hopf algebras $\mathcal{H}(\Gamma)$ and the extensions $(\mathcal{F}^\Gamma)$ in a general context and establish some basic properties. In Section 4 we study the Hopf algebras $\mathcal{H}(\Gamma)$ under the restrictions above. Section 5 is devoted to examples.

Notations. The natural numbers are denoted by $\mathbb{N}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given $m < n \in \mathbb{N}_0$, we set $\mathbb{I}_{m,n} = \{i \in \mathbb{N}_0 : m \leq i \leq m\}$ and $\mathbb{I}_n = \mathbb{I}_{1,n}$. “Algebra” means associative unital algebra. The space of algebra homomorphism from a $\Bbbk$-algebra A to a $\Bbbk$-algebra B is denoted by $\operatorname{Alg}(A, B)$. The category of finite-dimensional left R-modules, where R is an algebra, is denoted by $\operatorname{rep} R$. All Hopf algebras are supposed to have bijective antipode. We write $M \leq N$ to express that M is a subobject of N in a given category. The notation for Hopf algebras is standard: Δ is the comultiplication, ε is the counit, $\mathcal{S}$ is the antipode. For the comultiplication and the coactions we use the Heynemann-Sweedler notation.

2. Preliminaries

We refer the reader to [42, 45] for the basic facts about Hopf algebras used throughout the paper. Given a coalgebra K and an algebra H, the group of invertible elements in $\mathrm{Hom}(K, H)$ with respect to the convolution is denoted by $\mathrm{Reg}(K, H)$.

2.1. **Cleft comodule algebras.** These were studied in [13, 25]; we recall the relevant facts. Let H be a Hopf algebra, let R be a right comodule algebra with coaction $\rho : R \to R \otimes H$ and let $R^{\mathrm{co}\, H} = \{x \in R : \rho(x) = x \otimes 1\}$. One defines similarly ${}^{\mathrm{co}\, H}T$ for a left comodule algebra T.

For instance, if $\pi : C \to B$ is a Hopf algebra map, then C is a right, respectively left, comodule algebra via $\rho = (\mathrm{id} \otimes \pi)\Delta$, resp. $\lambda = (\pi \otimes \mathrm{id})\Delta$. The algebras of right and left coinvariants of π are

$$C^{\mathrm{co}\, \pi} = C^{\mathrm{co}\, B} = \{x \in C : (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\},$$
$${}^{\mathrm{co}\, \pi}C = {}^{\mathrm{co}\, B}C = \{x \in C : (\pi \otimes \mathrm{id})\Delta(x) = 1 \otimes x\}.$$

We consider three properties of the extension of algebras $R^{\mathrm{co}\, H} \subset R$:

(i) $R^{\mathrm{co}\, H} \subset R$ is H-Galois, if the canonical map $\mathrm{can} : R \otimes_{R^{\mathrm{co}\, \pi}} R \to R \otimes H$, given by $x \otimes y \mapsto (x \otimes 1)\rho(y)$, is bijective.
(ii) $R^{\mathrm{co}\, H} \subset R$ has the normal basis property if $R \simeq R^{\mathrm{co}\, H} \otimes H$ as left $R^{\mathrm{co}\, H}$-modules and right H-comodules.
(iii) $R^{\mathrm{co}\, H} \subset R$ is cleft if there exists $\chi \in \mathrm{Reg}(H, R)$ such that χ is a morphism of H-comodules, i.e., $\rho\chi = (\chi \otimes \mathrm{id})\Delta$.

Theorem 2.1 ([25]). *The extension $R^{\mathrm{co}\, H} \subset R$ is cleft if and only if it is H-Galois and has the normal basis property.*

Example 2.2 ([55]). Let G be a group with unit e and let R be an algebra. A $\Bbbk G$-comodule algebra structure on R is the same as a G-grading of algebras $R = \oplus_{g \in G} R_g$; here $R^{\mathrm{co}\, H} = R_e$. Now "$R$ is strongly G-graded" means that

$$R_g R_h = R_{gh}, \qquad \text{for all } \ g, h \in G.$$

Then $R_e \subset R$ is $\Bbbk G$-Galois if and only if R is strongly G-graded.

2.2. **Extensions of Hopf algebras.** This notion was considered in many papers. see e.g. [33, 8, 49, 31, 50, 52, 38, 16, 17]. Following [1, 8] together with [49] we say that the sequence of morphisms of Hopf algebras

$$A \overset{\iota}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$$

is exact if the following conditions hold:

(i) ι is injective.
(ii) π is surjective.
(iii) $\ker \pi = C\iota(A)^+$.
(iv) $\iota(A) = C^{\mathrm{co}\, \pi}$.

In this case we also say that C is an extension of B by A.

The left and right adjoint actions of C, denoted by $\mathrm{ad}_\ell, \mathrm{ad}_r : C \to \mathrm{End}\, C$, are given by

$$\mathrm{ad}_\ell(x)(y) = x_{(1)} y \mathcal{S}(x_{(2)}), \quad \mathrm{ad}_r(x)(y) = \mathcal{S}(x_{(1)}) y x_{(2)}, \quad x, y \in C.$$

Notice that

$$\mathrm{ad}_r(x)(y) = \mathcal{S}^{-1}\left(\mathrm{ad}_\ell(\mathcal{S}(x))\mathcal{S}(y)\right), \qquad x, y \in C.$$

A Hopf subalgebra A of a Hopf algebra C is *normal* if it stable under one of, hence both, the adjoint actions.

Lemma 2.3. *Let A be Hopf subalgebra of a Hopf algebra C and $B \coloneqq C/A^+C$.*
- *(i) If A is normal, then $A^+C = CA^+$.*
- *(ii) If $A^+C = CA^+$, then it is a Hopf ideal, B is a Hopf algebra and the quotient map $\pi : C \to B$ is a morphism of Hopf algebras.*
- *(iii) If $A^+C = CA^+$ and C is a faithfully flat A-module (under left or right multiplication), then $A \overset{\iota}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ is exact, π is faithfully coflat and A is normal.*

The converse in (i) and whether Hopf algebras are faithfully flat over Hopf subalgebras are open questions.

Proof. (i), (ii) are easy; see [42, 3.4.3] for (iii); cf. [8, 1.2.4], [49, 1.4] [53]. □

Remark 2.4. A Hopf algebra C is faithfully flat over a Hopf subalgebra A provided that either of the following conditions hold:
- (i) $\dim C < \infty$, in fact C is a free A-module [44];
- (ii) [49, 3.3] A is central and C is Noetherian;
- (iii) [49, 2.1] A is normal and $\dim A < \infty$; in fact C is a free A-module.

Let C be a Hopf algebra. The left and right coadjoint actions of C are the (left and right) comodule structures $\varrho_\ell, \varrho_r : C \to C \otimes C$ given by

$$\varrho_\ell(x) = x_{(1)}\mathcal{S}(x_{(3)}) \otimes x_{(2)}, \qquad \varrho_r(x) = x_{(2)} \otimes \mathcal{S}(x_{(1)})x_{(3)}, \qquad x \in C.$$

Let τ be the usual flip. Notice that

$$\varrho_r(x) = \left(\mathcal{S}^{-1} \otimes \mathcal{S}^{-1}\right) \tau \left(\varrho_\ell(\mathcal{S}(x))\right), \qquad x \in C.$$

A surjective Hopf algebra map $\pi : C \to B$ is *conormal*, or simply B is a conormal quotient of C, if $\ker \pi$ is a subcomodule for one of, hence both, ϱ_ℓ and ϱ_r (notice a change of terminology with respect to [8]).

Lemma 2.5. *Let $\pi : C \to B$ be a surjective morphism of Hopf algebras.*
- *(i) If π is conormal, then $C^{\operatorname{co}\pi} = {}^{\operatorname{co}\pi}C$.*
- *(ii) $A \coloneqq C^{\operatorname{co}\pi}$ equals ${}^{\operatorname{co}\pi}C$ iff A is a Hopf subalgebra of C.*
- *(iii) If π is conormal and C is a faithfully coflat (left or right) B-comodule, then $A \coloneqq C^{\operatorname{co}\pi} \overset{\iota}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ is exact and ι is faithfully flat.*

Proof. (i): [8, 1.1.7]; (ii) [8, 1.1.4]; (iii): [8, 1.2.14], [49, 1.4], [53]. □

Lemma 2.6. *If $\pi : C \to B$ is a surjective morphism of Hopf algebras and B is cosemisimple, then C is a faithfully coflat B-comodule.*

Proof. C is coflat because B is cosemisimple; but coflatness for a surjective Hopf algebra map implies faithful coflatness by [23, Remark, p. 247]. □

2.3. **Cleft extensions of Hopf algebras.** These were considered e. g. in [1, 8, 16, 17, 47, 48]. Given Hopf algebras H and K, we consider the subgroups of $\operatorname{Reg}(K, H)$ given by

$$\begin{aligned}
\operatorname{Reg}_1(K, H) &= \{\phi \in \operatorname{Reg}(K, H) : \phi(1) = 1\}, \\
\operatorname{Reg}_\varepsilon(K, H) &= \{\phi \in \operatorname{Reg}(K, H) : \varepsilon\phi = \varepsilon\}, \\
\operatorname{Reg}_{1,\varepsilon}(K, H) &= \operatorname{Reg}_1(K, H) \cap \operatorname{Reg}_\varepsilon(K, H).
\end{aligned}$$

Definition–Lemma 2.7. *An exact sequence* $A \overset{\iota}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ *is* cleft *if satisfies one of the following equivalent conditions:*

(i) there exists $\chi \in \operatorname{Reg}_1(B, C)$ *such that* $(\operatorname{id} \otimes \pi)\Delta\chi = (\chi \otimes \operatorname{id})\Delta$*;*
(ii) there exists $\xi \in \operatorname{Reg}_\varepsilon(C, A)$ *such that* $\xi(ac) = a\xi(c), \quad \forall\, a \in A, c \in C$*;*
(iii) there exist a morphism of A*-modules* $\xi : C \to A$ *and a morphism of* B*-comodules* $\chi : B \to C$ *such that* $\xi\chi = \varepsilon_B 1_A$ *and* $(\iota\xi) * (\chi\pi) = \operatorname{id}_C$.

If this happens, then $\xi(1) = 1$, $\varepsilon\chi = \varepsilon$, hence $\pi\chi = \operatorname{id}_B$ and $\xi\iota = \operatorname{id}_A$. See [1, 3.1.14] for a proof.

2.4. **Hopf center and Hopf cocenter.** Let H be a Hopf algebra. We recall a few facts from [1].

- [1, 2.2.3] There exists a maximal (with respect to the inclusion) central Hopf subalgebra $\mathcal{HZ}(H)$ of H, called the *Hopf center* of H.
- [1, 2.3.8] A quotient Hopf algebra map $q : H \to K$ is cocentral if it satisfies

$$(q \otimes \operatorname{id})\Delta = (q \otimes \operatorname{id})\Delta^{\mathrm{op}}.$$

  There exists a minimal cocentral Hopf algebra map $\mathfrak{q} : H \to \mathcal{HC}(H)$; by abuse of notation, $\mathcal{HC}(H)$ is called the *Hopf cocenter* of H. Here *minimal* means that any cocentral map $q : H \to K$ factorizes through $\mathfrak{q}$.
- If $\dim H < \infty$, then $\mathcal{HZ}(H)^* \simeq \mathcal{HC}(H^*)$ and $\mathcal{HZ}(H^*) \simeq \mathcal{HC}(H)^*$.

Lemma 2.8 ([1, 3.3.9])**.** *Let* $A \overset{\iota}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ *be an exact sequence.*

(i) If A *is central in* C *and* $\mathcal{HZ}(B) \simeq \Bbbk$*, then* A *is the Hopf center of* C*.*
(ii) If $\pi : C \to B$ *is cocentral and* $\mathcal{HC}(A) \simeq \Bbbk$*, then* $B \simeq \mathcal{HC}(C)$*.*

Remark 2.9. Let H be a finite-dimensional simple Hopf algebra (i.e., without proper non-trivial normal Hopf subalgebras).

(i) If H is not commutative, then $\mathcal{HZ}(H) \simeq \Bbbk$.
(ii) If H is not cocommutative, then $\mathcal{HC}(H) \simeq \Bbbk$.
(iii) (N. A. and H.-J.Schneider, Appendix to [1]). The small quantum groups associated to simple Lie algebras and their parabolic subalgebras are simple Hopf algebras, hence their Hopf centers and cocenters are trivial.

2.5. **Cocycles and twists.** Let H be a Hopf algebra. A Hopf 2-cocycle [24] or simply a cocycle is a convolution invertible inear map $\sigma : H \otimes H \to \Bbbk$ that satisfies

$$\sigma\left(x_{(1)}, y_{(1)}\right) \sigma\left(x_{(2)} y_{(2)}, z\right) = \sigma\left(y_{(1)}, z_{(1)}\right) \sigma\left(x, y_{(2)} z_{(2)}\right), \sigma(x, 1) = \sigma(1, x) = \varepsilon(x),$$

for all $x, y, z \in H$. Then we have a new Hopf algebra H_σ, the coalgebra H with multiplication conjugated by σ, i. e.

$$x \cdot_\sigma y = \sigma\left(x_{(1)}, y_{(1)}\right) x_{(2)} y_{(2)}\, \sigma^{-1}\left(x_{(3)}, y_{(3)}\right), \qquad x, y \in H. \tag{2.1}$$

Dually, a twist for H is an element $F = F^{(1)} \otimes F^{(2)} \in H \otimes H$ that satisfies

$$(1 \otimes F)(\operatorname{id} \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes \operatorname{id})(F), \qquad (\operatorname{id} \otimes \varepsilon)(F) = (\varepsilon \otimes \operatorname{id})(F) = 1$$

and has an inverse $F^{-1} = F^{(-1)} \otimes F^{(-2)}$. Then H^F, the algebra H with the comultiplication $\Delta^F \coloneqq F\Delta F^{-1}$ is a Hopf algebra. These definitions are compatible with duals, i.e. the transpose of a twist is a cocycle etc.

Example 2.10. Let now Γ be a group with unit e and A a Hopf algebra. Assume that $A = \oplus_{\kappa \in \Gamma} A_\kappa$ is a Γ-graded algebra, each homogeneous component A_κ is a subcoalgebra and $\mathcal{S}(A_\kappa) = A_{\kappa^{-1}}$. Then A_e is a Hopf subalgebra of A. Let $F = F^{(1)} \otimes F^{(2)} \in A_e \otimes A_e$ be a twist for A_e. Then F is a twist for A, which remains a Γ-graded algebra: $A^F = \oplus_{\kappa \in \Gamma} A_\kappa^F$, where A_κ^F is again a sucoalgebra, namely A_κ with comultiplication conjugated by F.

3. Hopf algebras with a central Hopf subalgebra

3.1. **Hopf systems.** Let A be a central Hopf subalgebra of a Noetherian Hopf algebra H and let $G = \operatorname{Alg}(A, \Bbbk)$ be the pro-affine algebraic group defined by A; its unit is the counit ε. Given $\kappa \in G$, let

$$\mathfrak{M}_\kappa = \ker \kappa \in \operatorname{Specmax} A, \qquad \mathfrak{I}_\kappa = H\mathfrak{M}_\kappa = \mathfrak{M}_\kappa H, \qquad H_\kappa = H/\mathfrak{I}_\kappa.$$

Since A is central, H_κ is an algebra (with multiplication m_κ and unit u_κ) and the natural projection $p_\kappa : H \to H_\kappa$ is an algebra map. If $\kappa, \gamma \in G$, then

$$\Delta(\mathfrak{M}_{\kappa\gamma}) \subset \mathfrak{M}_\kappa \otimes A + A \otimes \mathfrak{M}_\gamma \implies \Delta(\mathfrak{I}_{\kappa\gamma}) \subset \mathfrak{I}_\kappa \otimes H + H \otimes \mathfrak{I}_\gamma,$$
$$\mathcal{S}(\mathfrak{M}_\kappa) = \mathfrak{M}_{\kappa^{-1}} \implies \mathcal{S}(\mathfrak{I}_\kappa) = \mathfrak{I}_{\kappa^{-1}}.$$

Hence for any $\kappa, \gamma \in G$ there are well-defined algebra morphisms

$$\Delta_{\kappa,\gamma} : H_{\kappa\gamma} \to H_\kappa \otimes H_\gamma, \tag{3.1}$$
$$\mathcal{S}_\kappa : H_\kappa \to H^{\mathrm{op}}_{\kappa^{-1}}, \tag{3.2}$$

that satisfy

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "p_{\kappa\gamma}"'] & H \otimes H \arrow[d, "p_\kappa \otimes p_\gamma"] \\ H_{\kappa\gamma} \arrow[r, "\Delta_{\kappa,\gamma}"] & H_\kappa \otimes H_\gamma, \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\mathcal{S}"] \arrow[d, "p_\kappa"'] & H^{\mathrm{op}} \arrow[d, "p_{\kappa^{-1}}"] \\ H_\kappa \arrow[r, "\mathcal{S}_\kappa"] & H^{\mathrm{op}}_{\kappa^{-1}}. \end{tikzcd} \tag{3.3}$$

By the coassociativity and antipode axioms, for any $\kappa, \gamma, \nu \in G$ we have

$$(\Delta_{\kappa,\gamma} \otimes \mathrm{id}_{H_\nu})\Delta_{\kappa\gamma,\nu} = (\mathrm{id}_{H_\kappa} \otimes \Delta_{\gamma,\nu})\Delta_{\kappa,\gamma\nu} : H_{\kappa\gamma\nu} \to H_\kappa \otimes H_\gamma \otimes H_\nu, \tag{3.4}$$
$$(\mathrm{id}_{H_\kappa} \otimes \varepsilon)\Delta_{\kappa,\varepsilon} = \mathrm{id}_{H_\kappa} = (\varepsilon \otimes \mathrm{id}_{H_\kappa})\Delta_{\varepsilon,\kappa}, \tag{3.5}$$
$$m_\kappa(\mathrm{id} \otimes \mathcal{S}_{\kappa^{-1}})\Delta_{\kappa,\kappa^{-1}} = u_\kappa\varepsilon = m_\kappa(\mathcal{S}_{\kappa^{-1}} \otimes \mathrm{id})\Delta_{\kappa^{-1},\kappa} \tag{3.6}$$

In particular H_ε is a quotient Hopf algebra of H. In other words, $(H_\kappa)_{\kappa \in G}$ is a Hopf system in the sense of [1]. By Lemma 2.3 (iii) and Remark 2.4 (ii), we have an exact sequence of Hopf algebras

$$A \overset{\iota}{\hookrightarrow} H \twoheadrightarrow H_\varepsilon.$$

Given $\kappa \in G$, H_κ a H_ε-bicomodule algebra via

$$\varrho_\kappa := \Delta_{\kappa,\varepsilon} : H_\kappa \to H_\kappa \otimes H_\varepsilon, \qquad \lambda_\kappa := \Delta_{\varepsilon,\kappa} : H_\kappa \to H_\varepsilon \otimes H_\kappa;$$
$$\text{clearly } \varrho_\kappa p_\kappa = (p_\kappa \otimes p_\varepsilon)\Delta_H, \qquad \lambda_\kappa p_\kappa = (p_\varepsilon \otimes p_\kappa)\Delta_H.$$

Lemma 3.1 ([4, Lemma 3.1])**.** *If H is H_ε-cleft, then so is $H_\kappa, \forall\, \kappa \in G$.*

Remark 3.2. If H is H_ε-cleft, then $H_\kappa \neq 0$, i.e., $\mathfrak{I}_\kappa \neq H$. For this, consider $\xi \in \operatorname{Reg}_\varepsilon(H, A)$ satisfying $\xi(ah) = a\xi(h)$, for all $a \in A, h \in H$; see Definition–Lemma 2.7. Then $\xi(\mathfrak{I}_\kappa) = \xi(\mathfrak{M}_\kappa H) \subset \mathfrak{M}_\kappa$. Thus, if $\mathfrak{I}_\kappa = H$, then $1 \in \mathfrak{I}_\kappa$ and therefore $1 = \xi(1) \in \xi(\mathfrak{I}_\kappa) = \mathfrak{M}_\kappa$, a contradiction.

3.2. **Tensor categories.** Let $\mathscr{C}$ be the full subcategory of $\operatorname{rep} H$ whose objects are those where A acts in a semisimple way. Thus, if $V \in \mathscr{C}$, then

$$V = \bigoplus_{\kappa \in G} V_\kappa, \qquad \text{where} \qquad V_\kappa = \{v \in V : z \cdot v = \kappa(z)v \quad \forall\, z \in A\}.$$

Given $\kappa \in G$, we identify $\operatorname{rep} H_\kappa$ with a subcategory $\mathscr{C}_\kappa$ of $\operatorname{rep} H$ via restriction along p_κ. Then $\mathscr{C}_\kappa$ is indeed a full subcategory of $\mathscr{C}$; if $V \in \mathscr{C}$, then $V_\kappa \in \mathscr{C}_\kappa$. In other words

$$\mathscr{C} = \bigoplus_{\kappa \in G} \mathscr{C}_\kappa. \tag{3.7}$$

Because of (3.3), $\mathscr{C}$ is closed under tensor products and duality. Clearly it is a full subcategory closed under taking subquotients, hence it is a tensor subcategory of $\operatorname{rep} H$ [26, 4.11.1]. Notice however that $\mathscr{C}$ is not closed under extensions, see the discussion on $\operatorname{rep} H$ in Subsection 3.4. By (3.3), we also see that (3.7) is a grading of tensor categories, that is

$$\mathscr{C}_\kappa \otimes \mathscr{C}_\gamma \to \mathscr{C}_{\kappa\gamma}, \qquad\qquad (\mathscr{C}_\kappa)^* = \mathscr{C}_{\kappa^{-1}}.$$

By Remark 3.2 this grading is faithful if H is H_ε-cleft. This statement is also true when $\dim H_\varepsilon$ is finite, as we show next.

Proposition 3.3. *If $\dim H_\varepsilon$ is finite, then the grading $\mathscr{C} = \bigoplus_{\kappa \in G} \mathscr{C}_\kappa$ is faithful. Hence, $H_\kappa \neq 0$ for all $\kappa \in \Gamma$.*

Proof. Let $\mathcal{A}$ be the full subcategory of $\operatorname{rep} A$ whose objects W are semisimple, i.e.,

$$W = \bigoplus_{\kappa \in G} W_\kappa, \qquad \text{where} \qquad W_\kappa = \{w \in W : z \cdot w = \kappa(z)w, \quad \forall\, z \in A\}.$$

We have a grading $\mathcal{A} = \bigoplus_{\kappa \in G} \mathcal{A}_\kappa$, where $\mathcal{A}_\kappa$ is the full subcategory of $\operatorname{rep} A$ whose objects are those W with $W = W_\kappa$. This grading is faithful since the one-dimensional representation supported by κ belongs to $\mathcal{A}_\kappa$.

The restriction functor $\operatorname{rep} H \to \operatorname{rep} A$ induces a functor $F : \mathscr{C} \to \mathcal{A}$. We claim that F is dominant, that is, for every object $W \in \mathcal{A}$ there exists an object $V \in \mathscr{C}$ such that W is a subobject of $F(V)$. To see this, recall that H is faithfully flat over A, cf. Remark 2.4 (ii). Then the inclusion $A \hookrightarrow H$ induces a monomorphism $W = A \otimes_A W \hookrightarrow H \otimes_A W$ in $\operatorname{rep} A$, the latter being the restriction of the finite dimensional H-module $H \otimes_A W$, by assumption. Clearly, the image of W is contained in the A-socle V of $H \otimes_A W$, which is an object of $\mathscr{C}$, because A is central in H. The faithfulness of the grading (3.7) follows from the fact that $F(\mathscr{C}_\kappa) \subseteq \mathcal{A}_\kappa$, for all $\kappa \in G$. ☐

Definition 3.4. Let $\Gamma \leq G$. Then $\mathscr{C}_\Gamma$ denotes the full subcategory of $\mathscr{C}$ generated by $\mathscr{C}_\kappa$, $\kappa \in \Gamma$, that is the full subcategory of $\operatorname{rep} H$ with objects where A acts in a semisimple way by characters in Γ. In other words

$$\mathscr{C}_\Gamma = \bigoplus_{\kappa \in \Gamma} \mathscr{C}_\kappa. \tag{3.8}$$

3.3. **Matrix coefficients.** Let R be an algebra. Given $V \in \operatorname{rep} R$ with associated representation $\vartheta_V : R \to \operatorname{End} V$, the image of the transpose map ${}^t\vartheta_V : (\operatorname{End} V)^* \to R^*$ is denoted by C_V; its elements are the matrix coefficients of V. Clearly C_V is a subcoalgebra of the finite dual R° and

$$R^\circ = \bigcup_{V \in \operatorname{rep} R} C_V.$$

Given a morphism of algebras $\phi : R \to T$, the transpose ${}^t\phi : T^* \to R^*$ induces a morphism of coalgebras $\phi^\circ : T^\circ \to R^\circ$. Clearly, if $V \in \operatorname{rep} T$ and $V^\phi \in \operatorname{rep} R$ is obtained by restriction along ϕ, then $\phi^\circ(C_V) = C_{V^\phi}$.

If in addition R and T are Hopf algebras and ϕ is a morphism of Hopf algebras, then so is $\phi^\circ : T^\circ \to R^\circ$.

We consider the subcoalgebras of H°

$$C(\kappa) \coloneqq \bigcup_{W \in \operatorname{rep} H_\kappa} C_{W^{p_\kappa}}, \quad \kappa \in G. \tag{3.9}$$

In other words $p_\kappa^\circ : H_\kappa^\circ \to C(\kappa)$ is an isomorphism of coalgebras. In particular $C(\varepsilon) \simeq H_\varepsilon^\circ$ as Hopf algebras.

Let Γ be any subgroup of G. We introduce

$$\mathcal{H}(\Gamma) \coloneqq \sum_{\kappa \in \Gamma} C(\kappa). \tag{3.10}$$

Then $\mathcal{H}(\Gamma)$ is a Hopf subalgebra of H° by (3.3). Notice that the subcategory $\mathscr{C}_\kappa$ in Subsection 3.2 is equivalent to the category of finite-dimensional right $C(\kappa)$-comodules. Also $\mathscr{C}_\Gamma$ is tensor-equivalent to the tensor category of finite-dimensional $\mathcal{H}(\Gamma)$-comodules.

Let $\Gamma_{\text{fd}} \coloneqq \{\kappa \in \Gamma : \operatorname{rep} H_\kappa \neq 0\} = \{\kappa \in \Gamma : C(\kappa) \neq 0\}$. Clearly Γ_{fd} is a subgroup of Γ but it could be strictly smaller.

Example 3.5. Let $\mathfrak{h}_n$ be the n^{th} Heisenberg Lie algebra, with basis x_i, y_i, z, $i \in \mathbb{I}_n$ where z is central, $[x_i, x_j] = [y_i, y_j] = 0$, $[x_i, y_j] = \delta_{ij} z$, $i, j \in \mathbb{I}_n$. By Lemma 2.3 (ii), we have an exact sequence of Hopf algebras

$$\Bbbk[z] \hookrightarrow U(\mathfrak{h}_n) \twoheadrightarrow R$$

where R is the polynomial ring in $2n$ variables. Now $\mathbb{G} = \operatorname{Alg}(\Bbbk[z], \Bbbk)$ is the algebraic group $(\Bbbk, +)$; for any $\kappa \in \mathbb{G} \setminus 0$, $H_\kappa = U(\mathfrak{h}_n)/\langle z - \kappa \rangle$ is isomorphic to the Weyl algebra in $2n$ variables which has no non-zero finite-dimensional representation, being simple and infinite-dimensional. Thus for any $\Gamma \leq \mathbb{G}$, Γ_{fd} is trivial.

The following is the main result of this Section. Recall that $\iota : A \to H$ is the inclusion. Let $\varpi : \mathcal{H}(\Gamma) \to A^\circ$ be the restriction of $\iota^\circ : H^\circ \to A^\circ$ and let $\imath : H_\varepsilon^\circ \simeq C(\varepsilon) \to \mathcal{H}(\Gamma)$ be the inclusion.

Theorem 3.6. *The algebra $\mathcal{H}(\Gamma) = \bigoplus_{\kappa \in \Gamma_{\text{fd}}} C(\kappa)$ is faithfully Γ_{fd}-graded and the following is an exact sequence of Hopf algebras:*

$$H_\varepsilon^\circ \overset{\imath}{\hookrightarrow} \mathcal{H}(\Gamma) \overset{\varpi}{\twoheadrightarrow} \Bbbk\Gamma_{\text{fd}}. \tag{3.11}$$

Proof. Let (κ_j) be a finite family of different elements in G. Then

$$\bigcap_{j \neq i} \mathfrak{M}_{\kappa_j} + \mathfrak{M}_{\kappa_i} \overset{(\star)}{=} A \overset{(\#)}{\Longrightarrow} \bigcap_{j \neq i} \mathfrak{I}_{\kappa_j} + \mathfrak{I}_{\kappa_i} = H \overset{(*)}{\Longrightarrow} \left(\sum_{j \neq i} C(\kappa_j) \right) \cap C(\kappa_i) = 0.$$

Here $(\star)$ is a standard fact in commutative algebra and $(\#)$ is evident. Let us prove $(*)$: If $f \in C(\kappa)$, then $f_{|\mathfrak{I}_\kappa} = 0$. Thus if

$$f \in \left(\sum_{j \neq i} C(\kappa_j) \right) \cap C(\kappa_i), \quad \text{then} f_{|\bigcap_{j \neq i} \mathfrak{I}_{\kappa_j} + \mathfrak{I}_{\kappa_i}} = 0, \quad \text{hence} \quad f = 0.$$

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Therefore the sum $\sum_{\kappa\in\Gamma} C(\kappa)$ is direct. Since the multiplication of H° is the transpose of Δ, (3.3) implies that $C(\kappa)\cdot C(\gamma)\subseteq C(\kappa\gamma)$, for all $\kappa,\gamma\in\Gamma$.

We next claim that $\operatorname{Im}\varpi\simeq \Bbbk\Gamma_{\mathrm{fd}}$, so that the map ϖ in (3.11) makes sense. Indeed, if $V\in\operatorname{rep}H_\kappa$, then A acts on V via κ and thus $C_{\operatorname{Res}_A V}\subseteq\Bbbk\kappa$ and the equality holds iff $V\neq 0$. This implies that $\operatorname{Im}\varpi\subseteq\Bbbk\Gamma_{\mathrm{fd}}$ and the equality follows by definition of Γ_{fd}.

Our next goal is to show that

$$H^\circ_\varepsilon=\mathcal{H}(\Gamma)^{\operatorname{co}\varpi}. \tag{3.12}$$

We start with the following observation. By standard arguments on matrix coefficients, if $W\in\operatorname{rep}H_\kappa$ and $V=W^{p_\kappa}$, then

$$\varpi(f)=\boldsymbol{\varepsilon}(f)\kappa,\qquad\qquad f\in C_V \tag{3.13}$$

where the counit of H° is denoted by $\boldsymbol{\varepsilon}$. Now (3.12) follows because each $C(\kappa)$ is a subcoalgebra: given $f\in\mathcal{H}(\Gamma)$, write $f=\sum_{\kappa\in\Gamma} f_\kappa$ with $f_\kappa\in C(\kappa)$. Then by (3.13),

$$(\mathrm{id}\otimes\varpi)\Delta(f)=\sum_{\kappa\in\Gamma} f_\kappa\otimes\kappa,$$

hence $f\in\mathcal{H}(\Gamma)^{\operatorname{co}\varpi}$ if and only if $f\in C(\varepsilon)$.

For the exactness of (3.11), it remains to see that $\ker\varpi=\mathcal{H}(\Gamma)(H^\circ_\varepsilon)^+$. Now $\mathcal{H}(\Gamma)$ is faithfully coflat over $\Bbbk\Gamma_{\mathrm{fd}}$ by Lemma 2.6. By Lemma 2.5 (iii) we are reduced to show that ϖ is conormal, but this follows from the centrality of A: take $f\in\mathcal{H}(\Gamma)$, then $(\mathrm{id}\otimes\varpi)\varrho_\ell(f)=f_{(1)}\mathcal{S}(f_{(3)})\otimes\varpi(f_{(2)})=0$ if and only if $\langle f_{(1)}\mathcal{S}(f_{(3)}),h\rangle\langle\varpi(f_{(2)}),a\rangle=0$ for all $h\in H$, $a\in A$. Now

$$\begin{aligned}\Big\langle f_{(1)}\mathcal{S}(f_{(3)}),h\Big\rangle\Big\langle\varpi(f_{(2)}),a\Big\rangle&=\Big\langle f_{(1)},h_{(1)}\Big\rangle\Big\langle f_{(3)},\mathcal{S}(h_{(2)})\Big\rangle\Big\langle\varpi(f_{(2)}),a\Big\rangle\\&=\Big\langle f,h_{(1)}a\mathcal{S}(h_{(2)})\Big\rangle=\varepsilon(h)\langle f,a\rangle.\end{aligned}$$

Thus, if $f\in\ker\varpi$, then $f_{(1)}\mathcal{S}(f_{(3)})\otimes\varpi(f_{(2)})=0$, hence $\ker\varpi$ is a subcomodule with respect to ϱ_ℓ, i.e. ϖ is conormal. ☐

Remark 3.7. If $\Gamma'\leq\Gamma$, then $\mathcal{H}(\Gamma')$ is a Hopf subalgebra of $\mathcal{H}(\Gamma)$, $\Gamma'_{\mathrm{fd}}\leq\Gamma_{\mathrm{fd}}$ and there is a morphism of exact sequences

$$\begin{tikzcd} H^\circ_\varepsilon \arrow[r, hook] \arrow[d, equal] & \mathcal{H}(\Gamma') \arrow[r, two heads] \arrow[d, hook] & \Bbbk\Gamma'_{\mathrm{fd}} \arrow[d, hook] \\ H^\circ_\varepsilon \arrow[r, hook] & \mathcal{H}(\Gamma) \arrow[r, two heads] & \Bbbk\Gamma_{\mathrm{fd}}. \end{tikzcd}$$

Remark 3.8. The projection $\mathcal{H}(\Gamma)\to\Bbbk\Gamma_{\mathrm{fd}}$ is cocentral.

Proof. This follows from the centrality of A in H. ☐

Finally we establish some properties of $\mathcal{H}(\Gamma)$, see § 4.2. First we recall:

Theorem 3.9 ([5, 2.10, 2.13])**.** *Let $A\overset{\iota}{\hookrightarrow}C\overset{\pi}{\twoheadrightarrow}B$ be an exact sequence of Hopf algebras with C faithfully coflat as a B-comodule. Then C is co-Frobenius (respectively cosemisimple) if and only if A and B are co-Frobenius (respectively cosemisimple).*

Theorem 3.9 applied to the exact sequence (3.11), see Lemma 2.6, gives:

Corollary 3.10. *$\mathcal{H}(\Gamma)$ is co-Frobenius (respectively cosemisimple) if and only if H°_ε is co-Frobenius (respectively cosemisimple).*

3.4. **A grading of** $\operatorname{rep} H$**.** Given $V \in \operatorname{rep} H$, we set

$$V_{(\kappa)} = \{v \in V : (z - \kappa(z))^n \cdot v = 0 \quad \forall\, z \in A, \quad \text{for some } n \in \mathbb{N}\}.$$

Hence $V = \bigoplus_{\kappa \in G} V_{(\kappa)}$ by a classical argument. Given $\kappa \in G$, we introduce the full subcategory $\mathscr{C}_{(\kappa)}$ of $\operatorname{rep} H$ whose objects are those $V \in \operatorname{rep} H$ with $V = V_{(\kappa)}$. Therefore

$$\operatorname{rep} H = \bigoplus_{\kappa \in G} \mathscr{C}_{(\kappa)}. \tag{3.14}$$

Since $\mathscr{C}_\kappa \subset \mathscr{C}_{(\kappa)}$, the grading (3.7) of $\mathscr{C}$ is inherited from (3.14).

Proposition 3.11. *The decomposition* (3.14) *is a grading of tensor categories. If* $\dim H_\varepsilon < \infty$*, then this grading is faithful.*

Proof. Let $V, W \in \operatorname{rep} H$. We first observe that $V \in \mathscr{C}_{(\kappa)}$ iff there exists a filtration

$$0 = V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that $V_i/V_{i-1} \in \mathscr{C}_\kappa$ for all $i \in \mathbb{I}_{0,n}$. Thus, if $V \in \mathscr{C}_{(\kappa)}$ and $W \in \mathscr{C}_{(\gamma)}$ with an analogous filtration $0 = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_m = W$, then the filtration of $V \otimes W$ given by $(V \otimes W)_{-1} = 0$ and

$$(V \otimes W)_j = \sum_{r,s \,\geq\, 0, r+s=j} V_r \otimes W_s, \quad j \geq 0,$$

is exhaustive and satisfies $(V \otimes W)_j/(V \otimes W)_{j-1} \in \mathscr{C}_{\kappa\gamma}$. Indeed, given r and s, take $v \in V_r$ and $w \in W_s$. By hypothesis, for any $z \in A$ we have

$$z \cdot v \in \kappa(z)v + V_{r-1}, \qquad\qquad z \cdot w \in \gamma(z)w + W_{s-1},$$

hence

$$\begin{aligned} z \cdot (v \otimes w) = z_{(1)} \cdot v \otimes z_{(2)} \cdot w &\in \Big(\kappa(z_{(1)})v + V_{r-1}\Big) \otimes \Big(\gamma(z_{(2)})w + W_{s-1}\Big) \\ &\subseteq \kappa\gamma(z)(v \otimes w) + V_r \otimes W_{s-1} + V_{r-1} \otimes W_s + V_{r-1} \otimes W_{s-1} \\ &\subseteq \kappa\gamma(z)(v \otimes w) + (V \otimes W)_{r+s-1}. \end{aligned}$$

Thus $\mathscr{C}_{(\kappa)} \otimes \mathscr{C}_{(\gamma)} \hookrightarrow \mathscr{C}_{(\kappa\gamma)}$. Now, if $v \in V_i$ and $f \in V^*$, then for any $z \in A$ we have $\langle z \cdot f, v\rangle = \langle f, \mathcal{S}(z) \cdot v\rangle = \langle f, \kappa^{-1}(z) \cdot v + v'\rangle$ for some $v' \in V_{i-1}$. Hence

$$\Big\langle \Big(z - \kappa^{-1}(z)\Big) \cdot f, v \Big\rangle = \langle f, v'\rangle.$$

Iterating we see that $(z - \kappa^{-1}(z))^{n+1} \cdot f = 0$, for n as above. Therefore $(\mathscr{C}_{(\kappa)})^* = \mathscr{C}_{(\kappa^{-1})}$.

Observe that the previous arguments imply that the category $\operatorname{rep} A$ also bears a grading $\operatorname{rep} A = \bigoplus_{\kappa \in G} \mathcal{A}_{(\kappa)}$, where $\mathcal{A}_{(\kappa)}$ is the full subcategory of $\operatorname{rep} A$ whose objects are those V with $V = V_{(\kappa)}$. Now this grading is faithful since the one-dimensional representation supported by κ belongs to $\mathcal{A}_{(\kappa)}$.

Assume now that $\dim H_\varepsilon < \infty$. As in Proposition 3.3, the restriction functor $F : \operatorname{rep} H \to \operatorname{rep} A$ is dominant, implying the faithfulness of the grading (3.14). ☐

Definition 3.12. Let $\varGamma \leq G$. Then $\mathscr{C}_{(\varGamma)}$ denotes the full subcategory of $\operatorname{rep} H$ generated by $\mathscr{C}_{(\kappa)}$, $\kappa \in \varGamma$, in other words $\mathscr{C}_{(\varGamma)} = \bigoplus_{\kappa \in \varGamma} \mathscr{C}_{(\kappa)}$. We introduce

$$\mathcal{H}((\varGamma)) \coloneqq \sum_{W \in \mathscr{C}_{(\varGamma)}} C_W;$$

this is a Hopf subalgebra of H° and $\mathscr{C}_{(\varGamma)}$ is tensor-equivalent to the tensor category of finite-dimensional $\mathcal{H}((\varGamma))$-comodules.

The Hopf algebra $\mathcal{H}((\varGamma))$ will be studied elsewhere.

4. Finite-by-cocommutative Hopf algebras

4.1. **Discussion of the assumptions.** As in the previous Section we fix a Noetherian Hopf algebra H with a central Hopf subalgebra A; thus we have an exact sequence of Hopf algebras $(\mathcal{E})$: $A \overset{\iota}{\hookrightarrow} H \overset{p_\varepsilon}{\twoheadrightarrow} H_\varepsilon$. Let $G = \mathrm{Alg}(A, \Bbbk)$.

4.1.1. *Assumptions on H_ε.* Consider the following assumptions:

Assumption 4.1. *H is a finitely generated A-module, say $(h_i)_{i \in \mathbb{I}_N}$ generate H over A.*

Assumption 4.2. *H_ε is finite-dimensional.*

Remark 4.3. Assumption 4.1 implies 4.2. In general, we have for any $\kappa \in G$

$$\dim H_\kappa \leq N. \tag{4.1}$$

Conversely, Assumption 4.2 implies Assumption 4.1 if moreover the extension $(\mathcal{E})$ is *cleft*.

Question 4.4. Does Assumption 4.2 imply Assumption 4.1 always?

Example 4.5. Assumption 4.2 does not imply cleftness of $(\mathcal{E})$; see the example by Oberst and Schneider in [42, Section 3].

If (4.1) holds, then the subcoalgebra $C(\kappa)$ of the finite dual H° defined in (3.9) is identified with the finite-dimensional coalgebra H_κ^*, via the injective map of coalgebras $p_\kappa^t : H_\kappa^* \to H^\circ$. In particular, $C(\varepsilon) \simeq H_\varepsilon^* = H_\varepsilon^\circ$.

Lemma 4.6. *If Assumption 4.2 holds, then* $\dim H_\kappa = \dim H_\varepsilon$*, for all* $\kappa \in G$.

Compare with [26, Theorem 3.5.2].

Proof. First recall that for any Hopf algebra K and right K-comodule V, one has an isomorphism of right K-comodules

$$V \otimes K \to V_{\text{trivial}} \otimes K, \qquad v \otimes k \mapsto v_{(0)} \otimes v_{(1)}k, \qquad v \in V,\ k \in K.$$

Hence, if $\dim V < \infty$, then $V \otimes K \simeq K^{\dim V}$. Similarly, $K \otimes V \simeq K^{\dim V}$. Consider the coalgebra decomposition

$$\mathcal{H}(\varGamma) = \bigoplus_{\kappa \in \varGamma} C(\kappa),$$

where $\dim C(\kappa) < \infty$ by (4.1). Recall that $\mathscr{C}_\kappa$ is the category of finite-dimensional $C(\kappa)$-comodules. By the previous discussion, for any $\gamma \in \varGamma$,

$$\bigoplus_{\kappa \in \varGamma} (C(\kappa) \otimes C(\gamma)) \simeq \mathcal{H}(\varGamma) \otimes C(\gamma) \simeq \mathcal{H}(\varGamma)^{\dim C(\gamma)} \simeq \bigoplus_{\kappa \in \varGamma} C(\kappa)^{\dim C(\gamma)}$$

as $\mathcal{H}(\varGamma)$-comodules. Notice that $C(\kappa) \otimes C(\gamma) \in \mathscr{C}_{\kappa\gamma}$; while $C(\kappa)^{\dim C(\gamma)} \in \mathscr{C}_\kappa$. Therefore

$$C(\kappa) \otimes C(\gamma) \simeq C(\kappa\gamma)^{\dim C(\gamma)}, \qquad \text{for all } \kappa, \gamma \in \varGamma. \tag{4.2}$$

By a similar argument, using the isomorphism $C(\kappa) \otimes \mathcal{H}(\varGamma) \simeq \mathcal{H}(\varGamma)^{\dim C(\kappa)}$, we also obtain that

$$C(\kappa) \otimes C(\gamma) \simeq C(\kappa\gamma)^{\dim C(\kappa)}, \qquad \text{for all } \kappa, \gamma \in \varGamma. \tag{4.3}$$

Now $\dim C(\kappa) \neq 0$, by Proposition 3.3. Comparing dimensions on (4.2) and (4.3), we get $\dim C(\kappa) = \dim C(\gamma)$, for all $\kappa, \gamma \in \varGamma$. This proves the lemma. ☐

In the rest of the paper, we shall assume that Assumption 4.2 holds and that the extension $(\mathcal{E})$ is cleft, hence Assumption 4.1 also holds.

4.1.2. *Assumptions on H.* Let H be a Hopf algebra with a central Hopf subalgebra A such that H is a finitely generated A-module. Then

$$H \text{ is Noetherian} \iff A \text{ is Noetherian} \iff A \text{ is affine.}$$

In particular, $G = \operatorname{Alg}(A, \Bbbk)$ is an algebraic group. These equivalences follow from Lemma 4.9 below whose proof requires two results.

Theorem 4.7 ([27])**.** *If R is a right Notherian ring which is finitely generated as a right module over a commutative subring S, then S is Noetherian.*

Theorem 4.8 ([41])**.** *Let A be a commutative Hopf algebra. Then A is Noetherian if and only if it is affine.*

Lemma 4.9. *Let H be a Hopf algebra with a commutative Hopf subalgebra A such that H is a finitely generated A-module. The following are equivalent:*

(a) H is Noetherian.
(b) A is Noetherian.
(c) A is affine.

Proof. (a) $\Rightarrow$ b is Theorem 4.7. (b) $\Rightarrow$ (a): By assumption H is a Noetherian A-module and so it is a Noetherian algebra. (b) $\Leftrightarrow$ (c) is Theorem 4.8. $\square$

4.1.3. *Structure of $\mathcal{H}(\Gamma)$.* Let $\Gamma \leq G$; then $\Gamma_{\mathrm{fd}} = \Gamma$ because of (4.1) and Remark 3.2. Our main object of interest is the Hopf algebra $\mathcal{H}(\Gamma) = \bigoplus_{\kappa \in \Gamma} C(\kappa)$. By Theorem 3.6 it fits into the exact sequence

$$H_\varepsilon^* \overset{\imath}{\hookrightarrow} \mathcal{H}(\Gamma) \overset{\varpi}{\twoheadrightarrow} \Bbbk\Gamma. \qquad (\mathcal{F}^\Gamma)$$

Lemma 4.10. *The extension $(\mathcal{F}^\Gamma)$ is cleft; hence $\mathcal{H}(\Gamma)$ is strongly Γ-graded.*

Proof. Let $\chi : H_\varepsilon \to H$ be a cleaving map and let $\boldsymbol{\xi} : \mathcal{H}(\Gamma) \to H_\varepsilon^*$ be the restriction of the transpose $\chi^t : H^* \to H_\varepsilon^*$. Then $\boldsymbol{\xi}$ satisfies the conditions of Definition–Lemma 2.7 (ii), hence $(\mathcal{F}^\Gamma)$ is cleft. The second claim follows from Theorem 2.1 and Example 2.2. $\square$

Remark 4.11. If the family $(\kappa_i)_{i \in I}$ generates Γ, then $\bigoplus_{i \in I} C(\kappa_i)$ generates the algebra $\mathcal{H}(\Gamma)$. Hence, if Γ is finitely generated, then $\mathcal{H}(\Gamma)$ is affine.

4.1.4. *Pairing.* The natural pairing $\langle\,,\,\rangle : H^* \times H \to \Bbbk$ descends to a pairing $\langle\,,\,\rangle : \mathcal{H}(\Gamma) \times H \to \Bbbk$. Clearly it is non-degenerate on one side. Let $I = \mathcal{H}(\Gamma)^\perp \subseteq H$; this is a Hopf ideal since $\langle\,,\,\rangle$ is a Hopf pairing. We have

$$I = \cap_{\kappa \in \Gamma} C(\kappa)^\perp = \cap_{\kappa \in \Gamma} \mathfrak{I}_\kappa,$$

by (3.10) and since all H_κ's are finite-dimensional. Let $\kappa \in \Gamma$, $W \in \operatorname{rep} H_\kappa$ and let $\mathsf{c} \in C_{W^{p_\kappa}}$ be a matrix coefficient. Then $\langle \mathsf{c}, a\rangle = \kappa(a)\mathsf{c}$, for any $a \in A$. Hence

$$I \cap A = \cap_{\kappa \in \Gamma} C(\kappa)^\perp \cap A = \{a \in A : \kappa(a) = 0 \text{ for all } \kappa \in \Gamma\}.$$

That is, $I \cap A$ is the ideal of functions on G that vanish in the Zariski closure $\overline{\Gamma}$ of Γ. Hence, if $\overline{\Gamma} \neq G$, then $I \cap A \neq 0$ and so $I \neq 0$. We have proved:

Lemma 4.12. *If the pairing $\langle\,,\,\rangle : \mathcal{H}(\Gamma) \times H \to \Bbbk$ is non-degenerate, then Γ is Zariski-dense in G.*

Question 4.13. *If Γ is Zariski-dense in G, is the pairing $\langle\,,\,\rangle : \mathcal{H}(\Gamma) \times H \to \Bbbk$ non-degenerate? That is, does $I \cap A = 0$ imply that $I = 0$?*

4.1.5. *Comparison.* Consider another cleft exact sequence of Hopf algebras $A' \overset{\iota}{\hookrightarrow} H' \overset{p_\varepsilon}{\twoheadrightarrow} H'_\varepsilon$ with H' Noetherian and $\dim H'_\varepsilon < \infty$; set $G' = \mathrm{Alg}(A', \Bbbk)$. Given subgroups $\Gamma \leq G$ and $\Gamma' \leq G'$ it is natural to ask whether the Hopf algebras $\mathcal{H}(\Gamma)$ and $\mathcal{H}(\Gamma')$ are isomorphic (here is an abuse of notation, as the initial data are different). We offer a partial answer, applicable in many examples. Recall that $\varpi : \mathcal{H}(\Gamma) \to \Bbbk\Gamma$ is cocentral, cf. Remark 3.8.

Proposition 4.14. *Let $f : \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma')$ be an isomorphism of Hopf algebras. If $\mathcal{HC}(\mathcal{H}(\Gamma)) = \Bbbk\Gamma$ and $\mathcal{HC}(\mathcal{H}(\Gamma')) = \Bbbk\Gamma'$, then f induces isomorphisms $\varphi : \Gamma \to \Gamma'$ and $f_0 : H_\varepsilon^* \simeq H'^*_\varepsilon$ such that the following diagram commutes:*

$$\begin{tikzcd} H_\varepsilon^* \arrow[r, hook] \arrow[d, "f_0"] & \mathcal{H}(\Gamma) \arrow[r, two heads, "\varpi"] \arrow[d, "f"] & \Bbbk\Gamma \arrow[d, "\varphi"] \\ H'^*_\varepsilon \arrow[r, hook] & \mathcal{H}(\Gamma') \arrow[r, two heads, "\varpi'"] & \Bbbk\Gamma'. \end{tikzcd}$$

Proof. Since the morphism $\varpi' f : \mathcal{H}(\Gamma) \to \Bbbk\Gamma'$ is cocentral, there exists a unique morphism of Hopf algebras $\varphi : \Bbbk\Gamma \to \Bbbk\Gamma'$ such that $\varphi\varpi = \varpi' f$. Then φ must be an isomorphism which, combined with the fact that $H_\varepsilon^* = \mathcal{H}(\Gamma)^{co\varpi}$ and $H'^*_\varepsilon = \mathcal{H}(\Gamma')^{co\varpi'}$, implies that f induces by restriction an isomorphism $f_0 : H_\varepsilon^* \simeq H'^*_\varepsilon$ making the diagram commute. $\square$

Remark 4.15. By Lemma 2.8 (ii), the assumptions of Proposition 4.14 are satisfied if H_ε and H'_ε are simple and noncommutative or, more generally, if $\mathcal{HZ}(H_\varepsilon) \simeq \Bbbk \simeq \mathcal{HZ}(H'_\varepsilon)$.

4.2. **Coradical filtration and co-Frobenius property.** Recall that a Hopf algebra K is co-Frobenius if it admits a non-zero (right) integral, i.e., a linear functional $\int : K \to \Bbbk$, $\int \neq 0$, invariant under the dual of the left regular representation. See e.g. [5, 6] for details and a list of equivalent characterizations.

Let $\Gamma \leq G$. Corollary 3.10 implies that $\mathcal{H}(\Gamma)$ is co-Frobenius. The next Theorem gives a refinement of this fact.

Theorem 4.16. *The coradical filtration of $\mathcal{H}(\Gamma)$ is*

$$\operatorname{corad}_n \mathcal{H}(\Gamma) = \bigoplus_{\kappa \in \Gamma} \operatorname{corad}_n C(\kappa), \tag{4.4}$$

hence $\mathcal{H}(\Gamma)$ is co-Frobenius.

Proof. The proof of (4.4) is by induction on $n \geq 0$. For $n = 0$ this is a consequence of [45, 3.4.3], while the inductive step follows from [45, 2.4.3]. Then (4.1) implies that $\operatorname{corad}_N C(\kappa) = C(\kappa)$ and therefore $\operatorname{corad}_N \mathcal{H}(\Gamma) = \mathcal{H}(\Gamma)$. Finally, by finiteness of the coradical filtration, $\mathcal{H}(\Gamma)$ is co-Frobenius cf. [7, Theorem 2.1]. $\square$

By the Larson–Radford theorem [35], Corollary 3.10 also implies the following result.

Theorem 4.17. *$\mathcal{H}(\Gamma)$ is cosemisimple if and only if H_ε is semisimple.*

4.3. **Gelfand–Kirillov dimension.** We refer to [34] for the definition and basic properties of this notion.

The main result of this subsection characterizes the algebras $\mathcal{H}(\Gamma)$ with finite Gelfand–Kirillov dimension. We start by a general remark. Let Γ be a group and let $R = \bigoplus_{\kappa \in \Gamma} R_\kappa$ be a strongly Γ-graded algebra.

Lemma 4.18. *If* $\dim R_\kappa < \infty$ *for all* $\kappa \in \Gamma$*, then* $\operatorname{GKdim} R \geq \operatorname{GKdim} \Bbbk\Gamma$*. Furthermore the equality holds if there exists* $N \in \mathbb{N}$ *such that*

$$\dim R_\kappa \leq N, \qquad \textit{for all } \kappa \in \Gamma. \tag{4.5}$$

In particular, if $\operatorname{GKdim} R < \infty$, then Γ has polynomial growth.

Proof. First notice that $R_\kappa \neq 0$ for all $\kappa \in \Gamma$ since $0 \neq R_e = R_\kappa R_{\kappa^{-1}}$. Given a finite subset $X \subset \Gamma$, set $R_X = \bigoplus_{\kappa \in X} R_\kappa$. Then $|X| \leq \dim R_X < \infty$ by the preceding. We have $R_X R_Y = R_{XY}$ for any $Y \subset \Gamma$, since R is strongly graded. Hence $(R_X)^n = R_{X^n}$ and

$$|X^n| \overset{\spadesuit}{\leq} \dim(R_X)^n \overset{\clubsuit}{\leq} N|X^n|, \qquad n \in \mathbb{N},$$

where in the second inequality we assume (4.5). Now $\spadesuit$ implies that

$$\limsup_{n\to\infty} \log_n |X^n| \leq \operatorname{GKdim} R, \qquad \text{hence } \operatorname{GKdim} \Bbbk\Gamma \leq \operatorname{GKdim} R.$$

Assume that (4.5) holds. Let V be a finite-dimensional subspace of R; clearly there exists a finite $X \subset \Gamma$ such that $V \subset R_X$. Then $\clubsuit$ implies that

$$\limsup_{n\to\infty} \log_n \dim V^n \leq \limsup_{n\to\infty} \log_n \dim(R_X)^n \leq \operatorname{GKdim} \Bbbk\Gamma,$$

and the equality $\operatorname{GKdim} R = \operatorname{GKdim} \Bbbk\Gamma$ follows. □

Recall that a finitely generated group is *nilpotent-by-finite* if it has a normal nilpotent subgroup of finite index. Here is a celebrated result by Gromov:

Theorem 4.19 ([28])**.** *If* Γ *is a finitely generated group, then* $\operatorname{GKdim} \Bbbk\Gamma < \infty$ *if and only if* Γ *is nilpotent-by-finite.*

Assume next that Γ is a not necessarily finitely generated group. Then $\Bbbk\Gamma$ has finite GKdim if and only if there exists $N \in \mathbb{N}$ such that

$$\operatorname{GKdim} \Bbbk\Upsilon < N \text{ for any finitely generated } \Upsilon \leq \Gamma.$$

In particular any finitely generated subgroup of Γ should be nilpotent-by-finite. We then conclude:

Theorem 4.20. *Let* $\Gamma \leq G$*. Then*

$$\operatorname{GKdim} \mathcal{H}(\Gamma) = \operatorname{GKdim} \Bbbk\Gamma. \tag{4.6}$$

Thus $\operatorname{GKdim} \mathcal{H}(\Gamma) < \infty$ *iff there exists* $N \in \mathbb{N}$ *such that any finitely generated* $\Upsilon \leq \Gamma$ *is nilpotent-by-finite and* $\operatorname{GKdim} \Bbbk\Upsilon < N$*.*

Proof. Assume first that Γ is finitely generated. Then (4.6) follows from Lemma 4.18 since (4.1) gives (4.5). Therefore $\operatorname{GKdim} \mathcal{H}(\Gamma) < \infty$ iff Γ has polynomial growth iff Γ is nilpotent-by-finite by Theorem 4.19.

In general any finitely generated subalgebra of $\mathcal{H}(\Gamma)$ is contained in $\mathcal{H}(\Upsilon)$ for some finitely generated $\Upsilon \leq \Gamma$, cf. Remark 3.7. Thus

$$\operatorname{GKdim} \mathcal{H}(\Gamma) = \sup_{\substack{\Upsilon \leq \Gamma \\ \text{fin. gen.}}} \operatorname{GKdim} \mathcal{H}(\Upsilon) = \sup_{\substack{\Upsilon \leq \Gamma \\ \text{fin. gen.}}} \operatorname{GKdim} \Bbbk\Upsilon = \operatorname{GKdim} \Bbbk\Gamma.$$

So, (4.6) holds in general and the theorem follows from Theorem 4.19 by the considerations above. □

We list a few examples for illustration:

(a) $\operatorname{GKdim} \Bbbk\mathbb{Q} = 1$ because any finitely generated subgroup of $\mathbb{Q}$ is cyclic (and torsion-free). Since $\mathbb{Q}$ is a subgroup of the additive group $\Bbbk$, it embeds in any algebraic group which is not a torus.
(b) Let $\mathbb{G}_\infty \leq \Bbbk^\times$ be the group of all roots of 1. Then $\operatorname{GKdim} \Bbbk\mathbb{G}_\infty = 0$ because any finitely generated subgroup is cyclic (and torsion). Now $\mathbb{G}_\infty$ embeds in any algebraic group that contains a torus.

4.4. **Noetherianity.** A general reference for this Subsection is [40]. Recall that a solvable group is polycyclic if every subgroup is finitely generated; equivalently, if it admits a subnormal series with cyclic factors. Also, a group is *polycyclic-by-finite* if it has a normal polycyclic subgroup of finite index. The Hirsch number of a polycyclic group is the number of infinite factors in any subnormal series; the Hirsch number of a polycyclic-by-finite group is that of a polycyclic normal subgroup with finite index.

It is a classical result that the group algebra of a polycyclic-by-finite group is Noetherian [29]; a well-known open question is whether the converse holds.

Recall that a group is Noetherian if it satisfies the maximal condition on subgroups; if the group algebra of a given group is Noetherian, then so is the group but the converse is not true, see [32]. However for linear groups there is a remarkable result of Tits:

Theorem 4.21 ([54])**.** *A linear Noetherian group is polycyclic-by-finite.*

In consequence, if $\varGamma$ is a linear group, then the following are equivalent:

$$\Bbbk\varGamma \text{ is Noetherian} \quad \Leftrightarrow \quad \varGamma \text{ is Noetherian} \quad \Leftrightarrow \quad \varGamma \text{ is polycyclic-by-finite.}$$

Theorem 4.22 ([43, Thm. 5.5])**.** *If $\varGamma$ is a polycyclic-by-finite group and R is a strongly $\varGamma$-graded ring, then R is right Noetherian when R_e is so.*

Theorem 4.23. *Let $\varGamma \leq G$. The following are equivalent:*
(a) The algebra $\mathcal{H}(\varGamma)$ is Noetherian.
(b) The group $\varGamma$ is polycyclic-by-finite (in particular it is solvable).

(a) $\Rightarrow$ (b). Recall that $\varGamma$, being a subgroup of an affine algebraic group, is linear. Since $\Bbbk\varGamma$ is Noetherian, being a quotient of $\mathcal{H}(\varGamma)$, Theorem 4.21 applies. (b) $\Rightarrow$ (a): By Lemma 4.10, Theorem 4.22 and (4.1). ☐

4.5. **Regularity.** A reference for this Subsection is [40, Chapter 7]. As in *loc. cit.* we use the following abbreviations: pd stands for projective dimension; r. gldim, l. gldim, gldim stand for right global dimension, respectively left global dimension, and global dimension. Given an algebra R, if $\operatorname{r.gldim} R = \operatorname{l.gldim} R$ then we set $\operatorname{gldim} R = \operatorname{l.gldim} R$; we say that R is regular if $\operatorname{gldim} R < \infty$.

To start with, recall from [37] the estimate

$$\operatorname{l.gldim} C \leq \operatorname{r.gldim} B + \operatorname{l.gldim} A \tag{4.7}$$

for a crossed product $C = A\#_\sigma B$ of an associative algebra A with a Hopf algebra B (with antipode not necessarily bijective). See [18] for the dual version of this result. Clearly if the antipode of a Hopf algebra B is bijective, then $\operatorname{l.gldim} B = \operatorname{r.gldim} B$. But this equality holds even if the antipode is not bijective, see [56, Proposition A.1].

We apply the estimate (4.7) to a cleft exact sequence of Hopf algebras (all with bijective antipode) $A \overset{\iota}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ where we assume that $\dim A < \infty$. First, we record

Lemma 4.24. *In the situation above, if A is semisimple and B is regular, then C is regular and* $\operatorname{gldim} C \leq \operatorname{gldim} B$.

Now any finite-dimensional Hopf algebra is Frobenius, thus either A is semisimple, or $\operatorname{gldim} A = \infty$. The latter case is dealt with the following result, that relies on a theorem by Schneider and a well-known argument.

Lemma 4.25. *A Hopf algebra K having a finite-dimensional normal non-semisimple Hopf subalgebra L is not regular.*

Proof. By [49, Theorem 2.1], K is free as L-module. Therefore

$$\infty = \operatorname{gldim} L \overset{\heartsuit}{=} \operatorname{pd} \Bbbk_L \overset{\diamondsuit}{\leq} \operatorname{pd} \Bbbk_K + \operatorname{pd} K_L = \operatorname{pd} \Bbbk_K,$$

that is, K is not regular. Here $\heartsuit$ is e.g. by [37, 2.4] and $\diamondsuit$ by [40, 7.2.1]. $\square$

Let $\Gamma \leq G$. Turning to the cleft extension $(\mathcal{F}^{\Gamma})$, we get the following characterization.

Theorem 4.26. *Assume that $\mathcal{H}(\Gamma)$ is Noetherian. Then $\mathcal{H}(\Gamma)$ is regular if and only if H_ε is semisimple; in this case,* $\operatorname{gldim} \mathcal{H}(\Gamma) \leq h$ *where h is the Hirsch number of Γ.*

Proof. By [35], H_ε is semisimple iff H_ε^* is so. If H_ε^* is not semisimple, then $\operatorname{gldim} \mathcal{H}(\Gamma) = \infty$ by Lemma 4.25. Otherwise Lemma 4.24 implies that $\operatorname{gldim} \mathcal{H}(\Gamma) \leq \operatorname{gldim} \Bbbk\Gamma$. When $\mathcal{H}(\Gamma)$ is Noetherian, Γ is polycyclic-by-finite by Theorem 4.23, thus $\operatorname{gldim} \Bbbk\Gamma \leq h$ by [40, 7.5.6]. $\square$

5. Examples

5.1. **First considerations.** In this Section we discuss examples of Hopf algebras $\mathcal{H}(\Gamma)$ as defined in Section 3, see (3.10), and more specifically as assumed in Section 4. Once a substantial list of examples is obtained, to have a comprehensive picture one needs to address the following questions:

Question 5.1. Given an algebraic group G with algebra of functions $\mathcal{O}(G)$, find all Hopf algebras H with a central Hopf subalgebra $A \simeq \mathcal{O}(G)$ such that the extension $A \overset{\iota}{\hookrightarrow} H \overset{p_\varepsilon}{\twoheadrightarrow} H_\varepsilon$ is cleft and $\dim H_\varepsilon < \infty$.

Question 5.1 can be rephrased as follows:

Question 5.2. Given an algebraic group G and a Hopf algebra $\mathfrak{u}$, $\dim \mathfrak{u} < \infty$, find all cleft extensions $\mathcal{O}(G) \overset{\iota}{\hookrightarrow} H \twoheadrightarrow \mathfrak{u}$ such that $\iota(\mathcal{O}(G))$ is central in H.

As is known [8, 31, 38], such an extension can be described by a collection $(\rightharpoonup, \sigma, \rho, \tau)$ made up of a weak action, a cocycle, a weak coaction and a cycle that satisfies a long set of axioms; centrality slightly simplifies the requirements (e.g. the weak action is trivial) but otherwise this situation seems to be difficult to handle.

Then we also need information on the following classical problem.

Question 5.3. Given an algebraic group G (that admits a Noetherian Hopf algebra H as in Question 5.1), describe its subgroups, in particular those that are finitely generated nilpotent-by-finite, or polycyclic-by-finite.

Towards this question, it is worth recalling the following celebrated result.

Theorem 5.4 ([12, 51])**.** *Any polycyclic-by-finite group is linear.*

5.2. **Quantum algebras of functions.** Let G be a semisimple simply connected algebraic group with Lie algebra $\mathfrak{g}$ and algebra of functions $\mathcal{O}(G)$. Let ℓ be an odd integer (prime to 3 if G has a component of type G_2), and ϵ a primitive ℓ^{th} root of unity. Recall the quantized algebra of functions $\mathcal{O}_\epsilon(G)$, see e.g. [21], and the small quantum group $\mathfrak{u}_\epsilon(\mathfrak{g})$. There is an exact sequence of Hopf algebras

$$\mathcal{O}(G) \hookrightarrow \mathcal{O}_\epsilon(G) \xrightarrow{\pi} \mathfrak{u}_\epsilon(\mathfrak{g})^*$$

which is cleft by [46, 3.4.3]. By Theorem 5.4 any polycyclic-by-finite group Γ is a subgroup of a suitable G and so gives rise to a Hopf algebra $\mathcal{H}(\Gamma)$.

More examples are given by the Hopf algebra quotients of the quantized algebras of functions classified in [9].

5.3. **Pointed Hopf algebras.** Recall that a Hopf algebra is *pointed* if every simple comodule is one-dimensional. See [10] for this notion and [2] for the related notion of a Nichols algebra. We start by a result needed later.

Theorem 5.5 ([39, 1.3])**.** *Let $\pi : U \to \mathfrak{u}$ be a surjective map of Hopf algebras. If U is pointed, then the $\mathfrak{u}$-comodule algebra U, with coaction* $(\mathrm{id} \otimes \pi)\Delta$*, is cleft.*

Let (V, c) be a braided vector space. Then the tensor algebra $T(V)$ is a braided graded Hopf algebra. A *pre-Nichols algebra* of V is a factor of $T(V)$ by a graded Hopf ideal supported in degrees $\geqslant 2$. The Nichols algebra of V is $\mathscr{B}(V) = T(V)/\mathcal{J}(V)$, where $\mathcal{J}(V)$ is the maximal Hopf ideal among those. Thus any pre-Nichols algebra $\mathcal{B}$ of V lies between $T(V)$ and $\mathscr{B}(V)$.

We refer to [3, 4, 11] for details on the following material. Let (V, c) be a braided vector space of diagonal type with braiding matrix $\mathfrak{q} = (q_{ij}) \in (\Bbbk^\times)^{\mathbb{I} \times \mathbb{I}}$, where $\mathbb{I} = \{1, \dots, \theta\}$, $\theta \in \mathbb{N}$. Assume that the Nichols algebra $\mathscr{B}_\mathfrak{q} := \mathscr{B}(V)$ has finite dimension, thus $\mathfrak{q}$ belongs to the classification in [30]. As shown in [11], (V, c), i.e., the matrix $\mathfrak{q}$, gives rise to the following data:

- The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_\mathfrak{q}$.
- The Hopf algebra $U_\mathfrak{q}$ (the Drinfeld double of the bosonization of $\widetilde{\mathcal{B}}_\mathfrak{q}$).
- The subalgebra $\mathcal{Z}_\mathfrak{q}$ of $U_\mathfrak{q}$ as modified in [4, § 4.5].

The class of Examples of this Subsection arises from the following result.

Theorem 5.6. *If $\mathfrak{q}$ satisfies the technical condition* [4, (4.26)]*, then $\mathcal{Z}_\mathfrak{q}$ is a central Hopf subalgebra of $U_\mathfrak{q}$, $M_\mathfrak{q} := \mathrm{Alg}(\mathcal{Z}_\mathfrak{q}, \Bbbk)$ is a solvable algebraic group, $\widetilde{\mathfrak{u}}_\mathfrak{q} = U_\mathfrak{q}/U_\mathfrak{q}\mathcal{Z}_\mathfrak{q}^+$ is finite-dimensional, and the exact sequence of Hopf algebras $\mathcal{Z}_\mathfrak{q} \hookrightarrow U_\mathfrak{q} \twoheadrightarrow \widetilde{\mathfrak{u}}_\mathfrak{q}$ is cleft.*

Proof. This follows from [11, Theorem 33 & Remark 11], see also the discussion in [4, § 4.5]. Theorem 5.5 implies the cleftness of the exact sequence. □

Let $\mathfrak{g}$ be a semisimple Lie algebra and q a root of 1 of odd order, coprime with 3 if $\mathfrak{g}$ has an ideal of type G_2. Then there is a suitable $\mathfrak{q}$ such that $U_\mathfrak{q}$ is isomorphic to the De Concini–Kac–Procesi quantized enveloping algebra $U_q(\mathfrak{g})$, see [19, 20, 22]. But this class of examples covers also quantum supergroups and more, see [3, 4].

5.4. **Bosonizations.** See e.g. [10] for this notion due to Radford and interpreted categorically by Majid. The category of Yetter–Drinfeld modules over a Hopf algebra K is denoted by ${}^K_K\mathcal{YD}$.

We fix a finite group L, $V \in {}^{\Bbbk L}_{\Bbbk L}\mathcal{YD}$ such that $\dim \mathscr{B}(V) < \infty$ and set $K = \mathscr{B}(V)\#\Bbbk L$. We consider three different settings for pairs (H, A) with H Noetherian, A central in H, the extension $(\mathcal{E})$ cleft and $H_\varepsilon \simeq K$.

5.4.1. *Blowing-up the group.* We assume that there exist a group Λ, a surjective map of groups $\wp : \Lambda \to L$ and a map $\varsigma : \operatorname{supp} V \to \Lambda$ such that $V \in {}^{\Bbbk\Lambda}_{\Bbbk\Lambda}\mathcal{YD}$ with the grading induced by ς and the action induced by $\wp$. Set $\Lambda_0 := \ker \wp$. We claim that the following sequence is exact and cleft:

$$A' := \Bbbk\Lambda_0 \hookrightarrow H' := \mathscr{B}(V)\#\Bbbk\Lambda \xrightarrow{\operatorname{id}\#\wp} K.$$

Indeed tensoring the exact sequence of vector spaces $(\Bbbk\Lambda)\Bbbk\Lambda_0^+ \hookrightarrow \Bbbk\Lambda \twoheadrightarrow \Bbbk L$ with $\mathscr{B}(V)$, we conclude that $\ker(\operatorname{id}\#\wp) = H'(A')^+$. It is also easy to see that

$$A' \overset{\diamond}{\subseteqq} (H')^{\operatorname{co}\operatorname{id}\#\wp}.$$

Now H' is K-cleft by Theorem 5.5. Hence we have equality in $\diamond$ by a dimension argument. Finally we need to check whether Λ_0 is central in H' in each example.

5.4.2. *Blowing-up the algebra.* Here we suppose that there exists a pre-Nichols algebra $\mathcal{B}$ of V with projection $\pi : \mathcal{B} \to \mathscr{B}(V)$ such that $\mathcal{B}^{\operatorname{co}\pi}$ is a (usual) Hopf subalgebra and the following sequence is exact and cleft:

$$A'' := \mathcal{B}^{\operatorname{co}\pi} \hookrightarrow H'' := \mathcal{B}\#\Bbbk L \xrightarrow{\pi\#\operatorname{id}} K.$$

We also assume that A'' is central in H''.

5.4.3. *Blowing-up both.* We assume that there exist a group Λ and a pre-Nichols algebra $\mathcal{B}$ as previously such that $\mathcal{B}^{\operatorname{co}\pi}\#\Bbbk\Lambda_0$ is a central Hopf subalgebra and the following sequence is exact and cleft:

$$A := \mathcal{B}^{\operatorname{co}\pi}\#\Bbbk\Lambda_0 \hookrightarrow H := \mathcal{B}\#\Bbbk\Lambda \xrightarrow{\pi\#\wp} K.$$

5.4.4. *Example: the enveloping group of a rack.* For simplicity we fix $n \in \mathbb{I}_{3,5}$ and assume that $L = \mathbb{S}_n$ and that $V \in {}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathcal{YD}$ has $\mathscr{B}(V) \simeq \mathtt{FK}_n$ (the Fomin–Kirillov algebra of rank n), see e.g. [2]. Then V has support $X = \mathcal{O}_2^n$ (the conjugacy class of transpositions in $\mathbb{S}_n$); this is a rack with operation $\triangleright$ given by conjugation. The enveloping group of X is

$$\mathbb{G}_{\mathbb{X}} := \langle e_x : x \in X \mid e_x\, e_y = e_{x\triangleright y}\, e_x, x, y \in X \rangle\,.$$

Then there exist

(i) a unique morphism of groups $\wp : \mathbb{G}_{\mathbb{X}} \to \mathbb{S}_n$ such that $\wp(e_x) = x$, $x \in X$
(ii) a realization of V in ${}^{\mathbb{G}_{\mathbb{X}}}_{\mathbb{G}_{\mathbb{X}}}\mathcal{YD}$,

inducing a Hopf algebra map $H' := \mathscr{B}(V)\#\Bbbk\mathbb{G}_{\mathbb{X}} \xrightarrow{\operatorname{id}\#\wp} \mathscr{B}(V)\#\Bbbk\mathbb{S}_n$. It can be shown that $z := e_x^2 = e_y^2$, for any $x, y \in X$ and that $Z := \ker \wp = \langle z \rangle \simeq \mathbb{Z}$. Now it is clear that $e_x^2 \cdot v = v$ for every $v \in V$, hence Z is central in H'. Thus we have a cleft exact sequence

$$A' := \Bbbk Z \hookrightarrow H' \xrightarrow{\operatorname{id}\#\wp} K = \mathscr{B}(V)\#\Bbbk\mathbb{S}_n.$$

Now it is well-known that

$$\mathrm{Alg}(\Bbbk Z, \Bbbk) \simeq \widehat{Z} \simeq \Bbbk^{\times}, \qquad\qquad K^* \simeq \mathscr{B}(W)\#\Bbbk^{\mathbb{S}_n},$$

where $W \simeq V^*$ linearly. Hence for any $\varGamma \leq \Bbbk^{\times}$, we have an extension

$$\mathscr{B}(W)\#\Bbbk^{\mathbb{S}_n} \hookrightarrow \mathcal{H}(\varGamma) \twoheadrightarrow \Bbbk\varGamma.$$

5.4.5. *Example: (finite) quantum linear spaces.* The input for this example is a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ whose entries are roots of 1 that satisfy

$$q_{ij}q_{ji} = 1, \qquad i \neq j \in \mathbb{I}_\theta, \qquad N_i \coloneqq \operatorname{ord} q_{ii} > 1, \qquad i \in \mathbb{I}_\theta.$$

Let M_i be the least common multiple of $\{\operatorname{ord} q_{ij} : j \in \mathbb{I}_\theta\}$, for every $i \in \mathbb{I}_\theta$. Thus N_i divides M_i. To this input we attach:

- The finite abelian group $L = \langle g_1\rangle \oplus \cdots \oplus \langle g_\theta\rangle$ where $\operatorname{ord} g_i = M_i$, $i \in \mathbb{I}_\theta$.
- The characters $\chi_1, \dots \chi_\theta \in \widehat{L}$, given by $\chi_j(g_i) = q_{ij}$, $i, j \in \mathbb{I}_\theta$.
- The free abelian group Λ of rank θ, with basis $\mathbf{g}_1, \dots \mathbf{g}_\theta$. The kernel of the map $\wp : \Lambda \to L$ given by $\wp(\mathbf{g}_i) = g_i$, $i \in \mathbb{I}_\theta$, is denoted by Λ_0; that is, $\Lambda_0 = \langle \mathbf{g}_1^{M_1}, \dots \mathbf{g}_\theta^{M_\theta}\rangle$.
- The characters $\boldsymbol{\chi}_1, \dots \boldsymbol{\chi}_\theta \in \widehat{\Lambda}$, given by $\boldsymbol{\chi}_j(\mathbf{g}_i) = q_{ij}$, $i, j \in \mathbb{I}_\theta$.
- A vector space V with a basis $x_1, \dots x_\theta$, realized in ${}^{\Bbbk L}_{\Bbbk L}\mathcal{YD}$ and ${}^{\Bbbk\Lambda}_{\Bbbk\Lambda}\mathcal{YD}$ by

$$\delta(x_i) = g_i \otimes x_i, \qquad g \cdot x_i = \chi_i(g)x_i, \qquad \boldsymbol{\delta}(x_i) = \mathbf{g}_i \otimes x_i, \qquad \mathbf{g} \cdot x_i = \boldsymbol{\chi}_i(x)x_i,$$

  for $i \in \mathbb{I}_\theta$, $g \in L$, $\mathbf{g} \in \Lambda$. Here $\delta : V \otimes \Bbbk L$ and $\boldsymbol{\delta} : V \otimes \Bbbk\Lambda$ are the coactions.
- The algebra $\mathcal{B} = \Bbbk\langle x_1, \dots, x_\theta | x_ix_j - q_{ij}\, x_jx_i, i \neq j \in \mathbb{I}_\theta\rangle$.

Then the following facts hold:

- $\mathscr{B}(V)$ and $\mathcal{B}$ are both Hopf algebras in ${}^{\Bbbk L}_{\Bbbk L}\mathcal{YD}$ and ${}^{\Bbbk\Lambda}_{\Bbbk\Lambda}\mathcal{YD}$. The natural projection π from $\mathcal{B}$ to the Nichols algebra $\mathscr{B}(V)$ induces an isomorphism $\mathscr{B}(V) \simeq \mathcal{B}/\langle x_1^{N_1}, \dots, x_\theta^{N_\theta}\rangle$.
- The subalgebra of $\mathcal{B}$ generated by $x_1^{N_1}, \dots, x_\theta^{N_\theta}$ coincides with $\mathcal{B}^{\mathrm{co}\,\pi}$ and we have an exact sequence of Hopf algebras

$$A \coloneqq \mathcal{B}^{\mathrm{co}\,\pi}\#\Bbbk\Lambda_0 \hookrightarrow H \coloneqq \mathcal{B}\#\Bbbk\Lambda \xrightarrow{\pi\#\wp} K. \tag{5.1}$$

Lemma 5.7. *If $N_i = M_i$, for all $i \in \mathbb{I}_\theta$, then A is central in H.*

Proof. The algebra H is generated by $x_1, \dots, x_\theta, \mathbf{g}_1^{\pm 1}, \dots, \mathbf{g}_\theta^{\pm 1}$ with relations

$$x_ix_j = q_{ij}\, x_jx_i, \qquad i \neq j \in \mathbb{I}_\theta, \qquad \mathbf{g}_ix_j = q_{ij}\, x_j\mathbf{g}_i, \qquad \mathbf{g}_i\mathbf{g}_j = \mathbf{g}_j\mathbf{g}_i, \qquad i, j \in \mathbb{I}_\theta.$$

The subalgebra A is generated by $x_1^{N_1}, \dots, x_\theta^{N_\theta}, \mathbf{g}_1^{\pm N_1}, \dots, \mathbf{g}_\theta^{\pm N_\theta}$. Now

$$x_i^{N_i}x_j = q_{ij}^{N_i}\, x_jx_i^{N_i}, \qquad \mathbf{g}_i^{N_i}x_j = q_{ij}^{N_i}\, x_j\mathbf{g}_i^{N_i}, \qquad \mathbf{g}_ix_j^{N_j} = \begin{cases} q_{ii}^{N_i}\, x_i^{N_i}\mathbf{g}_i, & \text{if } i = j; \\ q_{ji}^{-N_j}\, x_j^{N_j}\mathbf{g}_i & \text{if } i \neq j, \end{cases}$$

for all $i, j \in \mathbb{I}_\theta$. This implies the Lemma. $\square$

Now the comultiplication of A is determined by

$$\Delta\left(\mathbf{g}_i^{N_i}\right) = \mathbf{g}_i^{N_i} \otimes \mathbf{g}_i^{N_i}, \qquad \Delta\left(x_i^{N_i}\right) = x_i^{N_i} \otimes 1 + \mathbf{g}_i^{N_i} \otimes x_i^{N_i}, \qquad i \in \mathbb{I}_\theta.$$

Therefore $G \coloneqq \mathrm{Alg}(A, \Bbbk)$ is isomorphic to $\mathbb{B}^\theta$, where $\mathbb{B}$ is the Borel subgroup of $SL(2, \Bbbk)$. In conclusion, for any $\varGamma \leq \mathbb{B}^\theta$, we have an extension

$$K^* \hookrightarrow \mathcal{H}(\varGamma) \twoheadrightarrow \Bbbk\varGamma.$$

5.5. **Twisting $\mathcal{H}(\Gamma)$.** Let A be a central Hopf subalgebra of a Noetherian Hopf algebra H, so that one has the exact sequence $A \overset{\iota}{\hookrightarrow} H \overset{p_\varepsilon}{\twoheadrightarrow} H_\varepsilon$ of Hopf algebras. Given $\Gamma \leq G = \mathrm{Alg}(A, \Bbbk)$, let $\mathcal{H}(\Gamma)$ be as in (3.10). Let $\sigma : H_\varepsilon \otimes H_\varepsilon \to \Bbbk$ be a cocycle, cf. § 2.5. Then $\widetilde{\sigma} := \sigma(p_\varepsilon \otimes p_\varepsilon) : H \otimes H \to \Bbbk$ is a cocycle. Then

$$z \cdot_\sigma h = zh, \qquad h \cdot_\sigma z = hz, \qquad z \in A,\, h \in H.$$

Hence A is central in $H_{\widetilde{\sigma}}$, Assume that H is a finitely generated A-module. Then $H_{\widetilde{\sigma}}$, being a finitely generated A-module, is Noetherian and we have an exact sequence of Hopf algebras

$$A \overset{\iota}{\hookrightarrow} H_{\widetilde{\sigma}} \overset{p_\varepsilon}{\twoheadrightarrow} (H_\varepsilon)_\sigma \ .$$

Thus we may repeat the constructions and get a Hopf algebra that we denote $\mathcal{H}_\sigma(\Gamma)$.

The twist $F = {}^t\sigma$ for $H_\varepsilon^* \simeq C(\varepsilon)$ is naturally a twist for $\mathcal{H}(\Gamma)$ and for H°. See Example 2.10 for twists of graded algebras.

Proposition 5.8. *$\mathcal{H}_\sigma(\Gamma) \simeq \mathcal{H}(\Gamma)^F$.*

Proof. Since the subcoalgebras $C(\kappa)$ of $\mathcal{H}(\Gamma)$ are $C(\varepsilon)$-bimodules, then $C(\kappa)^F$ is a subcoalgebra of $\mathcal{H}(\Gamma)^F$, for all $\kappa \in \Gamma$. Notice that $(H_{\widetilde{\sigma}})._{\widetilde{\sigma}}\mathfrak{M}_\kappa = H\mathfrak{M}_\kappa$ for all $\kappa \in \Gamma$. Hence the subcoalgebra $C_{\widetilde{\sigma}}(\kappa) := (H_{\widetilde{\sigma}}/(H_{\widetilde{\sigma}})._{\widetilde{\sigma}}\mathfrak{M}_\kappa)^*$ of $(H_{\widetilde{\sigma}})^\circ$ is contained in $(H^\circ)^F$ and it coincides with the subcoalgebra $C(\kappa)^F$. Therefore

$$\mathcal{H}(\Gamma)^F = \oplus_{\kappa \in \Gamma} C(\kappa)^F = \oplus_{\kappa \in \Gamma} C_{\tilde{\sigma}}(\kappa) = \mathcal{H}_\sigma(\Gamma). \qquad \square$$

Proposition 5.8 has the following application. Let $\mathfrak{u}$ be a finite-dimensional pointed Hopf algebra. In all known examples, the graded Hopf algebra $\operatorname{gr}\mathfrak{u}$ with respect to the coradical filtration is of the form $\mathscr{B}(V)\#\Bbbk L$ as in §5.4 and $\mathfrak{u} \simeq (\operatorname{gr}\mathfrak{u})_\sigma$ for a suitable cocycle σ. Then for any pair (H, A) with H Noetherian, A central in H, the extension $(\mathcal{E})$ cleft and $H_\varepsilon \simeq \operatorname{gr}\mathfrak{u}$, the pair $(H_{\widetilde{\sigma}}, A)$ has analogous properties and $(H_\varepsilon)_\sigma \simeq \mathfrak{u}$.

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