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ABSTRACT. Let Q be a quiver of A_n type and $\mathbb{K}$ be an algebraically closed field. A nilpotent endomorphism of a quiver representation induces a linear transformation of the vector space at each vertex. Generically, among all nilpotent endomorphisms of a fixed representation X , there exists a well-defined Jordan form of each of these linear transformations $\text{GenJF}(X)$, called the generic Jordan form data of X . A subcategory of $\text{rep}(Q)$ is Jordan recoverable if we can recover X up to isomorphism from its generic Jordan form data.

There is a procedure that allows one to invert the map from representations to generic Jordan form data. The subcategories for which this procedure applies are called canonically Jordan recoverable. We focus on the subcategories of $\text{rep}(Q)$ that are canonically Jordan recoverable, and we give a combinatorial characterization of them.

1. INTRODUCTION

1.1. Jordan recoverability and canonical Jordan recoverability. Let Q be an A_n type quiver. Consider X a finite-dimensional representation of Q over an algebraically closed field $\mathbb{K}$. Denote by $\text{NEnd}(X)$ the set of nilpotent endomorphisms of X . Fix $N \in \text{NEnd}(X)$. For each vertex $q \in Q_0$, the morphism N induces a nilpotent endomorphism N_q of X_q . We can extract from N a sequence of integer partitions $\lambda^q \vdash \dim(X_q)$, which correspond to the Jordan block sizes of the Jordan form of each N_q . Write $\text{JF}(N) = \boldsymbol{\lambda} = (\lambda^q)_{q \in Q_0}$. Thanks to a result from [13], for any $X \in \text{rep}(Q)$, there is a dense open set in $\text{NEnd}(X)$ on which JF is constant. We denote $\text{GenJF}(X)$ this constant that we will refer to as the *generic Jordan form data* of X .

Throughout the article, by subcategory, we mean a full subcategory closed under direct sums and direct summands. Our interest is to characterize the subcategories $\mathcal{C}$ of $\text{rep}(Q)$

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such that we can recover up to isomorphism $X \in \mathcal{C}$ from $\text{GenJF}(X)$. Such a subcategory $\mathcal{C}$ is called *Jordan recoverable*.

In general, determining which subcategories of $\text{rep}(Q)$ are Jordan recoverable remains difficult. For some cases, one can reconstruct X from $\text{GenJF}(X)$ thanks to the existence of a generic choice of a representation Y in $\text{rep}(Q)$ such that Y admits a nilpotent endomorphism of Jordan form $\text{GenJF}(X)$, and then we can ask if Y is isomorphic to X .

Concretely, for all $\#Q_0$ -tuples of integer partitions λ , denote $\text{rep}(Q, \lambda)$ the variety of representations of Q which admit a nilpotent endomorphism of Jordan form λ . For a fixed subcategory $\mathcal{C}$ of $\text{rep}(Q)$, we could try to find if for any $X \in \mathcal{C}$ there is a (Zariski) dense open set Ω in $\text{rep}(Q, \text{GenJF}(X))$, such that any $Y \in \Omega$ is isomorphic to X . Such a subcategory $\mathcal{C}$ is said to be *canonically Jordan recoverable*.

Note that a subcategory that is Jordan recoverable is not necessarily canonically Jordan recoverable.

Example 1.1. Let Q be the following A_2 type quiver.

$$Q = 1 \xrightarrow{\alpha} 2$$

The only subcategory of $\text{rep}(Q)$ which is not Jordan recoverable is $\text{rep}(Q)$ itself. Indeed, any strict subcategory $\mathcal{C}$ of $\text{rep}(Q)$ is generated by at most two indecomposable representations, and the dimension vectors of these indecomposable representations are linearly independent. It means that we can recover a representation $\mathcal{C}$ from its dimension vector, and a fortiori, from its generic Jordan form.

However, $\text{rep}(Q)$ is not Jordan recoverable: take for instance $X = S_1 \oplus S_2$ and $Y = P_1$; they do not admit a nonzero nilpotent endomorphism (X_i, Y_i are 1-dimensional $\mathbb{K}$ -vector-spaces for $i \in \{1, 2\}$) and hence $\text{GenJF}(S_1 \oplus S_2) = ((1), (1)) = \text{GenJF}(P_1)$.

Now we give an example of a category that is Jordan recoverable but not canonically Jordan recoverable. Let $\mathcal{C} = \text{add}(S_1, S_2)$. Consider $X = S_1^a \oplus S_2^b$ with $a, b \in \mathbb{N}$. Any pair of nilpotent endomorphisms (N_1, N_2) , with $N_i : X_i \rightarrow X_i$ for $i \in \{1, 2\}$, endows X with a nilpotent endomorphism $N = (N_1, N_2)$. A generic nilpotent endomorphism admits a Jordan form given by the tuple $((a), (b))$ of integer partitions. So $\text{GenJF}(X) = ((a), (b))$ and we can check again that $\mathcal{C}$ is Jordan recoverable.

However, $\mathcal{C}$ is not canonically Jordan recoverable. Fix $X = S_1 \oplus S_2$. Then $\text{GenJF}(X) = ((1), (1))$. Let $Y \in \text{rep}(Q)$ such that Y admits a nilpotent endomorphism N of the Jordan form $((1), (1))$. Thus $N = 0$. In such case, $Y_1 \cong \mathbb{K} \cong Y_2$ and $Y_\alpha = k \text{Id}$. The endomorphism N does not give any restriction on the value $k \in \mathbb{K}$.

$$\begin{array}{ccc} Y \cong \mathbb{K} & \xrightarrow{k} & \mathbb{K} \\ \downarrow 0 & & \downarrow 0 \\ Y \cong \mathbb{K} & \xrightarrow{k} & \mathbb{K} \end{array}$$

Only two choices give different representations Y up to isomorphism: $k = 0$ and $k \neq 0$. The first case returns X , while the second returns P_1 . We get a dense open set Ω in the collection of representations admitting a nilpotent endomorphism of Jordan form $((1), (1))$ in which all the representations are isomorphic to P_1 . Hence, we did not recover X , and $\mathcal{C}$ is not canonically Jordan recoverable as we claimed.

This paper aims to give a combinatorial description of all canonically Jordan recoverable subcategories of $\text{rep}(Q)$ for Q being any A_n type quiver and for $n \in \mathbb{N}^*$.

1.2. Adjacency-avoiding interval subsets. Call *intervals* of $\{1, \dots, n\}$ the sets $\{i, i+1, \dots, j\}$ with $1 \leq i \leq j \leq n$. Fix an A_n quiver Q . The intervals in $\{1, \dots, n\}$ provide a natural description of $\text{rep}(Q)$: the indecomposable representations are in one-to-one correspondence with the intervals of $\{1, \dots, n\}$, and morphisms between two indecomposable representations are completely described in terms of specific subintervals of both corresponding intervals. Section 2 gives the precise statement. Denote by X_K the indecomposable representation of $\text{rep}(Q)$ corresponding to the interval K .

For any interval K , write $b(K)$ as the lower bound and $e(K)$ as the upper bound of K . Two intervals K and L are *adjacent* if either $b(K) = e(L) + 1$ or $b(L) = e(K) + 1$. We have the following result inspired by a previous work [5] and by Example 1.1.

Proposition 1.2. *Let $\mathcal{C}$ be a subcategory of $\text{rep}(Q)$. Write $\mathcal{J}$ for the interval set corresponding to the indecomposable representations that additively generate $\mathcal{C}$. If two intervals exist $K, L \in \mathcal{J}$ such that K and L are adjacent, then $\mathcal{C}$ is not canonically Jordan recoverable.*

Let us first prove this lemma, which will be helpful.

Lemma 1.3. *Fix an A_n type quiver Q . Let $K_1, \dots, K_p$ be $p \in \mathbb{N}^*$ disjoint intervals. Write $J = K_1 \cup \dots \cup K_p$. Then $\text{GenJF}(X_{K_1} \oplus \dots \oplus X_{K_p}) = (\lambda^q)_{q \in Q_0}$ with:*

- $\lambda^q = (1)$ for $q \in J$;
- $\lambda^q = (0)$ otherwise.

Proof. Since, for $q \in Q_0$, $\dim(X_q) \leq 1$, we must have $N = 0$. The result follows. $\square$

Proof of Proposition 1.2. Let $K, L \in \mathcal{J}$ be two adjacent intervals. Write $X = X_K \oplus X_L$ and $J = K \cup L$. Note that J is an interval. We get $\text{GenJF}(X) = (\lambda^q)_{q \in Q_0}$ as defined in the previous lemma.

First, note that the only nilpotent endomorphism N such that $\text{JF}(N) = \lambda$ is the zero morphism. Therefore, choosing $Y \in \text{rep}(Q, \lambda)$ is equivalent to taking a representation Y such that $Y_q \cong \mathbb{K}$ if $q \in J$, and $Y_q = 0$ otherwise, without any other restrictions. We get that

$$\Omega = \{Z \in \text{rep}(Q, \lambda) \mid Z_\alpha \neq 0, \forall \alpha \in Q_1, \{s(\alpha), t(\alpha)\} \subset J\}$$

is a dense open set of $\text{rep}(Q, \lambda)$. Following this last statement, and by observing that $Z \cong X_J \not\cong X$ for all $Z \in \Omega$, we conclude that $\mathcal{C}$ is not canonically Jordan recoverable. $\square$

This result highlights the necessary condition to avoid the existence of two adjacent intervals among the set of intervals corresponding to indecomposable representations that generate $\mathcal{C}$. We define an *adjacency-avoiding* interval set as an interval set with no pair of adjacent intervals.

We aim to prove that the adjacency-avoiding property also gives a sufficient combinatorial criterion to detect canonical Jordan recoverability.

Theorem 1.4. *Let Q be an A_n type quiver, and $\mathcal{C}$ be a subcategory of $\text{rep}_{\mathbb{K}}(Q)$. Write $\mathcal{J}$ for the interval set corresponding to the indecomposable representations that additively generate $\mathcal{C}$. Then $\mathcal{C}$ is canonically Jordan recoverable if and only if $\mathcal{J}$ is adjacency-avoiding.*

This theorem completely characterizes the canonically Jordan recoverable subcategories of $\text{rep}(Q)$ and specializes to give a previous result of Garver, Patrias, and Thomas for A_n type quivers.

Corollary 1.5 ([13]). *Let Q be an A_n type quiver. For any $m \in Q_0$, the category $\mathcal{C}_{Q,m}$ generated by the indecomposable representations X_K for K intervals containing m is canonically Jordan recoverable.*

Proof. Let K and L be two intervals corresponding to two indecomposable representations of $\mathcal{C}_{Q,m}$. By definition, $K \cap L \supseteq \{m\}$. Therefore, K and L are not adjacent, as two adjacent intervals must have an empty intersection. We conclude the desired result by applying Theorem 1.4. $\square$

To prove Theorem 1.4, we first describe the maximal adjacency-avoiding interval sets (for inclusion). After that, following a revised version of the work of [13], we give a recursive construction of the subcategories generated by indecomposable representations provided by these interval sets. Then, we prove the main result for the linearly oriented case by showing that operations applied during the construction of those subcategories preserve the canonical Jordan recoverability. We conclude the result in the general case by reducing it to the linearly oriented case.

1.3. A combinatorial motivation. In [13, Section 6], Garver, Patrias, and Thomas made some links with the Robinson–Schensted–Knuth (RSK) correspondence. Thanks to Gansner’s combinatorics [12, 11], they prove that if Q is the A_n type quiver where only the vertex m is a sink, applying GenJF on $\mathcal{C}_{Q,m}$ coincides with applying the RSK on an integer matrix recording the multiplicities of the indecomposables in $\mathcal{C}_{Q,m}$.

Theorem 1.4 can be applied to define an extended RSK correspondence. This extended RSK recovers both the scrambled RSK of [13, 4] and Gansner’s version of RSK, which applies to fillings of any partition shape, while being more general than either. The reader can find more details in [6]. A FPSAC extended abstract [7] is also available as a shorter version.

1.4. Outline of the paper. We start, in Section 2, with some fundamental reminders on type A quiver representations.

Then, in Section 3, in the context of background work on Jordan recoverability, we highlight a combinatorial computation of the generic Jordan form data via the Greene–Kleitman invariant.

Afterward, in Section 4, we introduce the notion of storability for tuples of integer partitions, which extracts the general behavior of the generic Jordan form data of any representation in a candidate category.

Following that, in Section 5, we show the behavior of the generic Jordan form data under elementary operations, such as adding copies of a well-chosen simple module and applying reflection functors.

In Section 6, we study the combinatorial behavior of the sets of adjacency-avoiding interval sets, by first describing all of them, and then exhibiting their reaction under the action mimicking the reflection functors in the algebraic world.

Finally, we prove the main result in Section 7. First, we use algebraic and combinatorial algorithms, based on the previous elementary operations, to construct categories arising from the maximal adjacency-avoiding interval sets in the linear case. Then,

we get to the conclusion by induction using the behavior of those subcategories, as shown combinatorially.

In Section 8, we discuss some future directions to explore following this work.

2. SOME GENERALITIES ABOUT A_n TYPE QUIVER REPRESENTATIONS

In this section, we give an overview of A type quiver representations. For more details, we refer the reader to the following references: [1, 3, 20].

2.1. A_n type quivers. A quiver Q is a 4-tuple (Q_0, Q_1, s, t) where Q_0 is the set of vertices, Q_1 is the set of arrows and $s, t : Q_1 \rightarrow Q_0$ are respectively source and target functions. The opposite quiver of Q , denoted Q^{op} , is the quiver obtained from Q by reversing the direction of all the arrows of Q . We say that Q is a finite quiver whenever Q_0 and Q_1 are finite sets. The underlying graph of a finite quiver Q is a pair $\mathcal{G}(Q) = (\mathcal{G}_0, \mathcal{G}_1)$ where $\mathcal{G}_0 = Q_0$ is the set of vertices of the graph and $\mathcal{G}_1 = \{\{s(\alpha), t(\alpha)\} \mid \alpha \in Q_1\}$ is the (multi)set of edges of the graph. Note that a finite quiver Q can be seen as the graph $\mathcal{G}(Q)$ endowed with an orientation for each edge.

Let Q be a finite quiver, and $n > 0$ be an integer. Assume that $Q_0 = \{1, \dots, n\}$. The quiver Q is said to be of A_n type whenever $\mathcal{G}(Q)$ is of the following shape.

$$1 \text{ --- } 2 \text{ --- } \dots \text{ --- } n$$

We denote $\overrightarrow{A_n}$ the A_n type quiver where all the arrows of Q are exactly $i \rightarrow i+1$ for $i \in \{1, \dots, n-1\}$. An A_n type quiver Q is called linearly oriented if either $Q = \overrightarrow{A_n}$ or $Q = \overrightarrow{A_n}^{\text{op}}$.

2.2. Representations. Let $\mathbb{K}$ be an algebraically closed field. This assumption is a restriction that we need to use the results of [13]. They need it because some of their arguments rely on algebraic geometry.

A representation of Q (over $\mathbb{K}$) is a pair $X = ((X_q)_{q \in Q_0}, (X_\alpha)_{\alpha \in Q_1})$ where:

- for each $q \in Q_0$, X_q is a $\mathbb{K}$ -vector space;
- for each $\alpha \in Q_1$, $X_\alpha : X_{s(\alpha)} \rightarrow X_{t(\alpha)}$ is a $\mathbb{K}$ -linear map.

We say that such a representation X is finite dimensional if $\dim X_q < \infty$ for all $q \in Q_0$. We denote $\mathbf{dim}(X) = (\dim X_q)_{q \in Q_0}$ the dimension vector of X . From now on, when we talk about representations of a quiver, we mean finite-dimensional representations.

Let X and Y be two representations of Q . A morphism ϕ from X to Y is a collection of linear maps $(\phi_q)_{q \in Q_0}$ such that for any $\alpha \in Q_1$, we have $\phi_{t(\alpha)} X_\alpha = Y_\alpha \phi_{s(\alpha)}$. Write $X \cong Y$ whenever X and Y are isomorphic. Denote $\text{Hom}(X, Y)$ the homomorphism space from X to Y and $\text{End}(X) = \text{Hom}(X, X)$ the endomorphism space of X .

Recall that the representations of Q endowed with the morphisms between them form a category functorially equivalent to the category of (finite-dimensional) left modules over the path algebra associated to Q . Denote $\text{rep}(Q)$ the category of finite-dimensional representations of Q . Remember that $\text{rep}(Q)$ depends on the field $\mathbb{K}$, but for simplicity we suppress it from the notation.

A representation $M \neq 0$ is indecomposable if either $X = 0$ or $Y = 0$, whenever $M \cong X \oplus Y$. We write $\text{Ind}(Q)$ for the indecomposable representations in $\text{rep}(Q)$ up to isomorphism.

Any $M \in \text{rep}(Q)$ admits a unique decomposition into a finite sum of indecomposable representations up to isomorphism. Given $X \in \text{Ind}(Q)$, denote $\text{mult}(X, M)$ the multiplicity

of X in M defined as the number of the indecomposable representations isomorphic to X appearing in the decomposition of M .

We now recall a complete description of the indecomposable representations of an A_n type quiver and the morphisms between them. We will state these results in terms of intervals of $\{1, \dots, n\}$.

Let n be a positive integer. The *intervals* of $\{1, \dots, n\}$ are the sets $\llbracket i, j \rrbracket := \{i, i+1, \dots, j\}$ given by all $1 \leq i \leq j \leq n$. If $i = j$, we write $\llbracket i, i \rrbracket = \llbracket i \rrbracket$. Denote $\mathcal{I}_n$ the set of intervals in $\{1, \dots, n\}$. For $K = \llbracket i, j \rrbracket \in \mathcal{I}_n$, write $b(K) = i$ and $e(K) = j$. Call *interval set* any subset of $\mathcal{I}_n$.

Definition 2.1. Let Q be an A_n type quiver. Consider $K, L \in \mathcal{I}_n$ such that $K \subseteq L$. We say that:

- K is *above* L (relative to Q) if the two following assertions are satisfied:
 - $b(K) = b(L)$ or we have the arrow $b(K) - 1 \leftarrow b(K)$ in Q ;
 - $e(K) = e(L)$ or we have the arrow $e(K) \rightarrow e(K) + 1$ in Q .
- K is *below* L (relative to Q) if the two following assertions are satisfied:
 - $b(K) = b(L)$ or we have the arrow $b(K) - 1 \rightarrow b(K)$ in Q ;
 - $e(K) = e(L)$ or we have the arrow $e(K) \leftarrow e(K) + 1$ in Q .

Example 2.2. Consider the following quiver.

$$\overrightarrow{A}_3 = 1 \longrightarrow 2 \longrightarrow 3$$

Then $\llbracket 2 \rrbracket$ is above $\llbracket 2, 3 \rrbracket$ and below $\llbracket 1, 2 \rrbracket$.

Note that any interval is above and below itself, relative to all A_n type quivers. Let Q be a quiver of A_n type. To any interval $K \in \mathcal{I}_n$, we consider X_K the representation of Q defined as it follows:

- $(X_K)_q = \mathbb{K}$ if $q \in K$, $(X_K)_q = 0$ otherwise;
- $(X_K)_\alpha = \text{Id}_{\mathbb{K}}$ if α is such that $\{s(\alpha), t(\alpha)\} \subseteq K$, $(X_K)_\alpha = 0$ otherwise;

Note that X_K is an indecomposable representation of Q for all $K \in \mathcal{I}_n$.

Theorem 2.3 ([9, 18]). *Let Q be an A_n type quiver.*

- The isomorphism classes of indecomposable representations of Q are in bijection with $\mathcal{I}_n$; more precisely, they are described by indecomposable representations X_K for $K \in \mathcal{I}_n$;*
- The homomorphism space between two indecomposable representations of Q is of dimension at most one; more precisely, $\text{Hom}(X_K, X_L)$ is nonzero if and only if there exists an interval J such that J is above K and below L relative to Q ; if such an interval exists, it is unique and $\text{Hom}(X_K, X_L)$ consists of scalar multiples of the morphism $\phi = (\phi_q)_{q \in Q_0}$ such that $\phi_q = \text{Id}_{\mathbb{K}}$ if $q \in J$ and $\phi_q = 0$ otherwise.*

In the light of the previous result :

- for all interval sets $\mathcal{J} \subseteq \mathcal{I}_n$ and all quivers Q of A_n type, write $\text{Cat}_Q(\mathcal{J})$ the subcategory of $\text{rep}(Q)$ additively generated by X_K for $K \in \mathcal{J}$;
- for all quivers Q of A_n type and for all nonzero subcategories $\mathcal{C}$ of $\text{rep}(Q)$, let $\text{Int}(\mathcal{C})$ be the interval set of $\mathcal{I}_n$ consisting of intervals K such that $X_K \in \mathcal{C}$.

Recall that we only consider full subcategories closed under direct sums and direct summands. Such subcategories are additively generated by X_K for $K \in \mathcal{J}$ for some $\mathcal{J} \subseteq \mathcal{I}_n$.

Hence, for any A_n type quiver Q , for all $\mathcal{J} \subseteq \mathcal{I}_n$ and for all subcategories $\mathcal{C} \subseteq \text{rep}(Q)$, we have $\mathcal{J} = \text{Int}(\text{Cat}_Q(\mathcal{J}))$ and $\mathcal{C} = \text{Cat}_Q(\text{Int}(\mathcal{C}))$.

2.3. Reflection functors. In this subsection, we will recall the definition of *reflection functors* for any quiver Q . For our purposes in this paper, defining those functors only on objects is sufficient.

Let Q be an arbitrary quiver and v be a vertex of Q . Denote $\sigma_v(Q)$ the quiver obtained from Q by reversing the directions of the arrows incident to v . If $\alpha \in Q_1$ such that $v \in \{s(\alpha), t(\alpha)\}$, denote $\tilde{\alpha}$ the reversed arrow of α in $\sigma_v(Q)$.

Now assume that v is a sink of Q . Consider $\Xi = \sigma_v(Q)$. The *reflection functor*

$$\mathcal{R}_v^+ : \text{rep}(Q) \longrightarrow \text{rep}(\Xi)$$

is defined as follows. Let $X = ((X_q)_{q \in Q_0}, (X_\beta)_{\beta \in Q_1}) \in \text{rep}(Q)$. We set $\mathcal{R}_v^+(X) = ((Y_q)_{q \in \Xi_0}, (Y_\beta)_{\beta \in \Xi_1}) \in \text{rep}(\Xi)$ where

- $Y_q = X_q$ for $q \neq v$ and

$$Y_v = \text{Ker} \left(\bigoplus_{\alpha \in Q_1, t(\alpha)=v} X_\alpha : \bigoplus_{\alpha \in Q_1, t(\alpha)=v} X_{s(\alpha)} \longrightarrow X_v \right);$$

- $Y_\beta = X_\beta$ if $\beta \in Q_1$ such that $t(\beta) \neq v$, otherwise $Y_{\tilde{\beta}} : Y_v \rightarrow X_{s(\beta)}$ is the composition of the kernel inclusion of Y_v to $\bigoplus_{\alpha \in Q_1, t(\alpha)=v} X_{s(\alpha)}$ with the projection onto the direct summand $X_{s(\beta)}$.

If v is a source of Q , the *reflection functor*

$$\mathcal{R}_v^- : \text{rep}(Q) \longrightarrow \text{rep}(\sigma_v(Q))$$

is defined dually.

Example 2.4. Let Q be a quiver and $X \in \text{rep}(Q)$ as below.

$$Q = 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4$$

$$X = \begin{array}{ccccc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \mathbb{K}^3 & \xrightarrow{\quad} & \mathbb{K}^4 & \xleftarrow{\quad} & \mathbb{K}^3 & \xleftarrow{\quad} & \mathbb{K}^2 \end{array}$$

Apply the reflection functor $\mathcal{R}_2^+$. The arrows α and β are the only ones with 2 as a target. First we get $\Xi = \sigma_2(Q)$ as follows.

$$\Xi = 1 \xleftarrow{\tilde{\alpha}} 2 \xrightarrow{\tilde{\beta}} 3 \xleftarrow{\gamma} 4$$

We calculate afterward

$$\text{Ker}(X_\alpha \oplus X_\beta) = \text{Ker} \left(\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \cong \mathbb{K}^3.$$

Thus we get $\mathcal{R}_2^+(X)$ by replacing the vector space at 2 by $\mathbb{K}^3$ and defining the morphisms for $\tilde{\alpha}$ and $\tilde{\beta}$ by the composition of the kernel inclusion $\mathbb{K}^3 \rightarrow \mathbb{K}^6$ and the respective projection to X_1 and X_3 . It gives the following result.

$$\mathcal{R}_2^+(X) = \begin{array}{ccccc} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & & \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \mathbb{K}^3 & \xleftarrow{\quad} & \mathbb{K}^3 & \xrightarrow{\quad} & \mathbb{K}^3 & \xleftarrow{\quad} & \mathbb{K}^2 \end{array}$$

The reflection functors are additive, meaning we can understand their actions on objects by knowing their actions on indecomposable objects.

By the following proposition, we recall the action of the reflection functors on $\text{Ind}(Q)$, for Q an A_n type quiver.

Proposition 2.5. *Let Q be an A_n type quiver, $v \in Q_0$ and $\llbracket v \rrbracket \neq K \in \mathcal{I}_n$. Write $\Xi = \sigma_v(Q)$. If v is a sink of Q , then $\mathcal{R}_v^+(X_K) \cong X_{K'} \in \text{rep}(\Xi)$ where*

$$K' = \begin{cases} K \cup \{v\} & \text{if either } e(K) = v - 1 \quad \text{or} \quad b(K) = v + 1; \\ K \setminus \{v\} & \text{if either } e(K) = v \quad \text{or} \quad b(K) = v; \\ K & \text{otherwise.} \end{cases}$$

If v is a source of Q , then $\mathcal{R}_v^-(X_K) = X_{K'}$ where K' is defined as above.

Note that, if v is a sink of Q , $\mathcal{R}_v^+(X_{\llbracket v \rrbracket}) = 0$, and if v is a source of Q , $\mathcal{R}_v^-(X_{\llbracket v \rrbracket}) = 0$. We also recall the following result, which will be helpful later.

Theorem 2.6. *Let Q be a quiver, and v be one of its sinks. Write $\Xi = \sigma_v(Q)$. The reflection functor $\mathcal{R}_v^+ : \text{rep}(Q) \rightarrow \text{rep}(\Xi)$ induces a category equivalence between the full subcategory of $\text{rep}(Q)$ additively generated by the indecomposable representations of Q except the simple projective representation at v and the full subcategory of $\text{rep}(\Xi)$ additively generated by indecomposable representations of Ξ except the simple injective representation at v . The quasi-inverse is induced by the reflection functor $\mathcal{R}_v^- : \text{rep}(\Xi) \rightarrow \text{rep}(Q)$.*

See [1, Theorem VII.5.3] for more details.

3. JORDAN RECOVERABILITY AND CANONICAL JORDAN RECOVERABILITY

3.1. The Greene–Kleitman invariant. In this subsection, for a given A_n type quiver Q , we introduce a combinatorial invariant for any representation of Q , whose representation-theoretic meaning will be shown in the next subsection. Before introducing this invariant, we need to recall some definitions and give some notations.

Recall that, given a quiver Q , the *Auslander–Reiten quiver* of $\text{rep}(Q)$ is a quiver whose vertex set is the set of isomorphism classes of indecomposable representations of Q and whose arrow set is the set of irreducible morphisms between the indecomposable representations.

Given a positive integer m , an *integer partition* of m is a finite weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_1 + \dots + \lambda_k = m$. The *length* $\ell(\lambda)$ of such an integer partition λ is k . Its *size* is $|\lambda| = \lambda_1 + \dots + \lambda_k = m$. We write $\lambda \vdash m$ when λ is a partition of m .

Let Q be an A_n type quiver. For all $q \in Q_0$, define the subcategory $\mathcal{C}_{Q,q}$ of $\text{rep}(Q)$ by $\mathcal{C}_{Q,q} = \text{Cat}_Q(\{K \in \mathcal{I}_n \mid q \in K\})$. Fix $X \in \text{rep}(Q)$. We decompose X as below, with $m_K = \text{mult}(X_K, X) \in \mathbb{N}$.

$$X \cong \bigoplus_{K \in \mathcal{I}_n} X_K^{m_K}$$

Consider the full subquiver of the Auslander–Reiten quiver whose vertices are isomorphism classes of indecomposable representations of Q in $\mathcal{C}_{Q,q}$. For $\ell \geq 1$, we consider Π_q^ℓ the set of all ℓ -tuples of maximal paths in this subquiver. Note that these paths start at the vertex corresponding to the projective representation P_q and end at the injective representation I_q . For all ℓ -tuples of paths in the Auslander–Reiten quiver $\gamma = (\gamma_1, \dots, \gamma_\ell)$, we write $\text{Supp}(\gamma)$ for the set of vertices passed through by some γ_i . For all $\gamma \in \Pi_q^\ell$, we consider a *weight* depending on X defined as follows:

$$\text{wt}_X(\gamma) = \sum_{X_K \in \text{Supp}(\gamma)} m_K.$$

Definition 3.1. The *Greene–Kleitman invariant* of X , denoted $\text{GK}(X)$, is the n -tuple of partitions $\lambda = (\lambda^q)_{q \in Q_0}$ with λ^q such that:

- $\lambda_1^q = \max_{\gamma \in \Pi_q^1} \text{wt}_X(\gamma)$;
- $\forall i \geq 2, \lambda_i^q = \max_{\gamma \in \Pi_q^i} \text{wt}_X(\gamma) - \max_{\gamma \in \Pi_q^{i-1}} \text{wt}_X(\gamma)$.

Remark 3.2. This definition is a restatement of [14], already made in [13].

Example 3.3. Consider $Q = \overrightarrow{A_5}$. Let $X \in \text{rep}(Q)$. We can picture X , up to isomorphism, as a *filling* of the Auslander–Reiten quiver: meaning a function ϕ_X which associates each X_K to $m_K = \text{mult}(X_K, X)$. We pictured the Auslander–Reiten quiver of $\overrightarrow{A_5}$, and an example of a representation X in Figure 3.1.

In Figure 3.2, we represent how we calculate the integer partition λ^3 . One can calculate λ^q following the same process for $q \in Q_0$. By doing all the calculations, we get $\text{GK}(X) = ((10), (11, 6), (\mathbf{11}, \mathbf{17} - \mathbf{11} = 6, \mathbf{18} - \mathbf{17} = 1), (11, 5), (6))$.

Remark 3.4. We can note that if $X \cong Y$ then $\text{GK}(X) = \text{GK}(Y)$ by definition: the filling of the Auslander–Reiten quivers for X and for Y are the same. This property explains why GK is an *invariant* of $\text{rep}(Q)$.

Following this remark, one can be interested in determining for which subcategories $\mathcal{C}$ of $\text{rep}(Q)$ the Greene–Kleitman invariant is complete.

In the next subsection, we will see that answering this question is equivalent to characterizing all the Jordan recoverable subcategories of $\text{rep}(Q)$. In this paper, we aim to partially answer this question by characterizing all the canonically Jordan recoverable subcategories of $\text{rep}(Q)$.

3.2. Jordan recoverability. Let Q be an A_n type quiver. Consider $X \in \text{rep}(Q)$. A *nilpotent endomorphism* $N : X \rightarrow X$ is an endomorphism such that $N^k = 0$ for some integer $k > 0$. One can think of a nilpotent endomorphism N as a collection of nilpotent transformations $(N_q)_{q \in Q_0}$ satisfying an additional compatibility relation. Denote $\text{NEnd}(X)$ the set of nilpotent endomorphisms of X .

Let $\mathbf{dim}(X) = \mathbf{d} = (d_q)_{q \in Q_0}$. For any $N \in \text{NEnd}(X)$, we can consider the Jordan form of N_q at each vertex q . It induces a sequence of partitions $\lambda^q \vdash d_q$. We refer to $(\lambda^q)_{q \in Q_0}$ as the *Jordan form data* of N . Denote it by $\text{JF}(N)$.

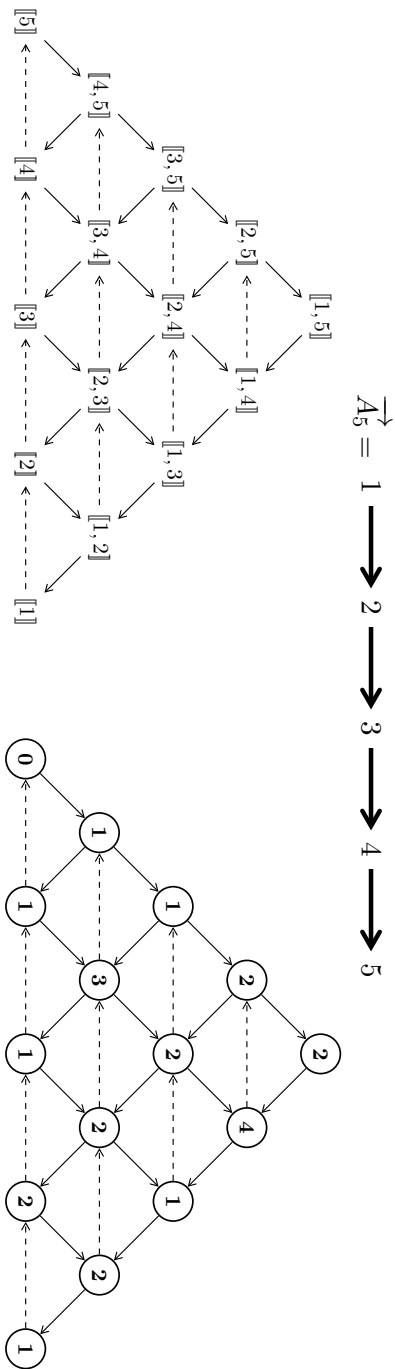


FIGURE 3.1. The quiver $\vec{A}_5$, its Auslander-Reiten quiver and a choice of a representation X seen as a *filling* of the AR-quiver. We label $\llbracket i, j \rrbracket$ the vertex corresponding to the isomorphism classes of the indecomposable representation $X_{\llbracket i, j \rrbracket}$ in $\text{rep}(\vec{A}_5)$. The dashed arrows correspond to the action of the Auslander-Reiten translation τ .

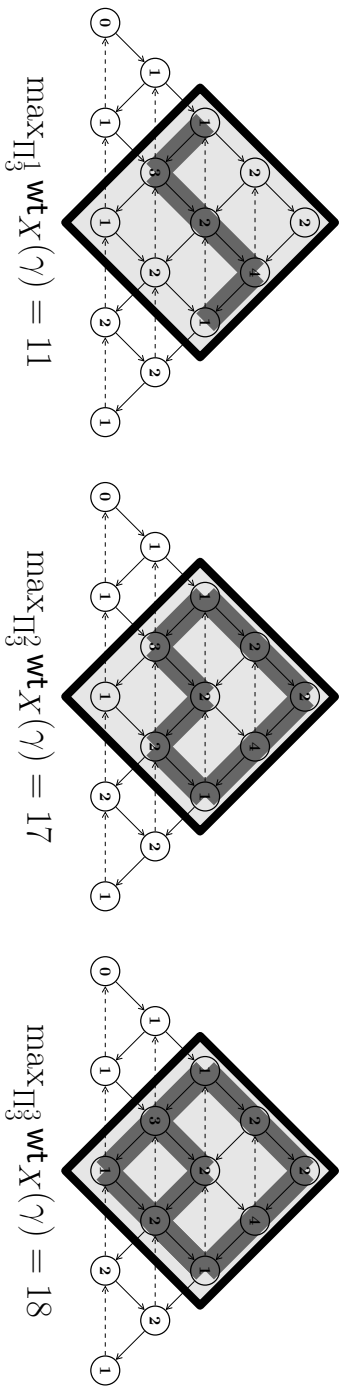


FIGURE 3.2. Explicit way to calculate $\max_{\Pi_3^e} \text{wt}_X(\gamma)$ for the representation X defined in Figure 3.1.

The *dominance order* on partitions of an integer n is defined as it follows: for any λ and μ partitions of n , $\lambda \trianglelefteq \mu$ if $\lambda_1 + \dots + \lambda_k \leq \mu_1 + \dots + \mu_k$ for each $k \geq 1$ where we add zero parts to λ and μ if necessary.

We extend this order to any n -tuple of partitions. Introduce first a notation: for $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ and $\boldsymbol{\pi} = (\pi^1, \dots, \pi^n)$ a n -tuple of integer partitions, we write $\boldsymbol{\pi} \vdash \mathbf{d}$ if $\pi^i \vdash d_i$ for $i \in \{1, \dots, n\}$. Now fix $\mathbf{d} \in \mathbb{N}^n$. For $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^n)$ and $\boldsymbol{\mu} = (\mu^1, \dots, \mu^n)$ such that $\boldsymbol{\lambda} \vdash \mathbf{d}$ and $\boldsymbol{\mu} \vdash \mathbf{d}$, we say that $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ if for all $i \in \{1, \dots, n\}$ $\lambda^i \trianglelefteq \mu^i$.

Before stating a precise result on the generic Jordan form data for any representation of Q , we recall a key result from Gansner.

Let Γ be a finite acyclic quiver. Label the vertices of Γ from 1 to $\#\Gamma_0$ such that, for all $i, j \in \{1, \dots, \#\Gamma_0\}$, if there is an arrow $i \rightarrow j$ in Γ , then $i < j$.

For $\ell \geq 1$, we write Π_Γ^ℓ the set of all ℓ -tuples of maximal paths in Γ . For $\gamma = (\gamma_1, \dots, \gamma_\ell) \in \Pi_\Gamma^\ell$, write $\text{Supp}(\gamma)$ the set of vertices passed through by some γ_i . We define $\Delta(\Gamma) = (\Delta_\ell(\Gamma))_{\ell \geq 1}$ for all $\ell \geq 1$ by:

- $\Delta_1(\Gamma) = \max_{\gamma \in \Pi_\Gamma^1} \#\text{Supp}(\gamma)$,
- for $\ell \geq 2$, $\Delta_\ell(\Gamma) = \max_{\gamma \in \Pi_\Gamma^\ell} \#\text{Supp}(\gamma) - \max_{\gamma \in \Pi_\Gamma^{\ell-1}} \#\text{Supp}(\gamma)$.

Note that $\Delta_\ell(\Gamma) \geq 0$ for all $\ell \geq 1$, and there exists $\ell_0 \geq 1$ such that $\Delta_{\ell_0}(\Gamma) = 0$.

We define a *generic matrix* of Γ as a $\#\Gamma_0 \times \#\Gamma_0$ matrix $M = (m_{i,j})$ where $m_{i,j} = 0$ whenever there is no arrow $i \rightarrow j$ in Γ , and the rest of its entries are complex numbers algebraically independent over $\mathbb{Q}$.

We now state the following result proven by Gansner [10, Theorem 2.1] and independently by Saks in his thesis [19, Theorem 6.3].

Theorem 3.5. *Let Γ be a finite acyclic quiver. Any generic matrix M of Γ is nilpotent. Moreover, we have $\text{JF}(M) = \Delta(\Gamma)$.*

One can notice that the construction of $\Delta(\Gamma)$ is closely similar to the one for calculating the Greene–Kleitman invariant at each vertex of Q . These similarities, along with some previous results from [13], allow us to state the following result.

Theorem 3.6. *Let Q be an A_n type quiver. Let Y be a finite-dimensional representation over an algebraically closed field $\mathbb{K}$. Then $\text{NEnd}(Y)$ is an irreducible algebraic variety. Furthermore, there is a maximum value of JF , with respect to $\trianglelefteq$, on $\text{NEnd}(Y)$ which is attained on a dense open set of $\text{NEnd}(Y)$, and this value is exactly $\text{GK}(Y)$.*

Proof. As a direct consequence of [13, Theorem 2.3] stated in a more considerable generality (instead of finite-dimensional representation Y over an A_n type quiver Q , they proved the same result for any finite-dimensional left module Y over a finite-dimensional $\mathbb{K}$ -algebra), we can already affirm that $\text{NEnd}(Y)$ is an irreducible algebraic variety, and the fact that JF admits a maximal value on $\text{NEnd}(Y)$ attained on a dense open set.

We must prove that this maximal value is $\text{GK}(Y)$. This result is a consequence of Theorem 3.5. We can in fact notice that $\text{GK}(Y) = (\Delta(\Gamma(q)))_{q \in Q_0}$ where $\Gamma(q)$ is the full subquiver of the Auslander–Reiten quiver of $\text{rep}(Q)$ in which we replace each vertex corresponding to the isomorphism class of X_K by a chain of length $m_K = \text{mult}(Y, X_K)$. This completes the proof. $\square$

By Theorem 3.6, we can define $\text{GenJF}(X)$, the *generic Jordan form data* of X , as this maximal value of JF on $\text{NEnd}(X)$. Keep in mind that $\text{GenJF}(X) = \text{GK}(X)$. We only change its name for representation-theoretic purposes.

Definition 3.7. A subcategory $\mathcal{C}$ of $\text{rep}(Q)$ is called *Jordan recoverable* if from a tuple of partitions λ there is at most a unique (up to isomorphism) $X \in \mathcal{C}$ such that $\text{GenJF}(X) = \lambda$.

The Jordan recoverable categories of $\text{rep}(Q)$ are precisely those for which GK is a complete invariant.

Example 3.8. Consider the A_3 type quiver Q in Figure 3.3.

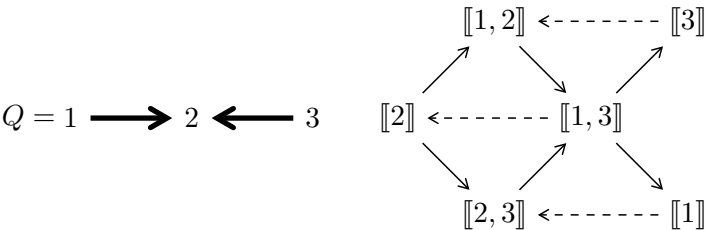


FIGURE 3.3. The A_3 type quiver considered (on the left) and its Auslander-Reiten quiver (on the right).

Here are some examples of Jordan recoverable $\text{rep}(Q)$ subcategories.

- The subcategory $\mathcal{C}_1 = \text{Cat}_Q(\{[1], [2], [3]\})$ is Jordan recoverable: the dimension vectors of the indecomposable representations which generate $\mathcal{C}_1$ are linearly independent. More explicitly, for $X \in \mathcal{C}_1$, there exists a unique triplet $(a, b, c) \in \mathbb{N}^3$ such that $X \cong X_{[1]}^a \oplus X_{[2]}^b \oplus X_{[3]}^c$. Following calculation of $\text{GK}(X)$ (see Figure 3.4), we get $\text{GenJF}(X) = ((a), (b), (c))$. This data determines $X \in \mathcal{C}$.

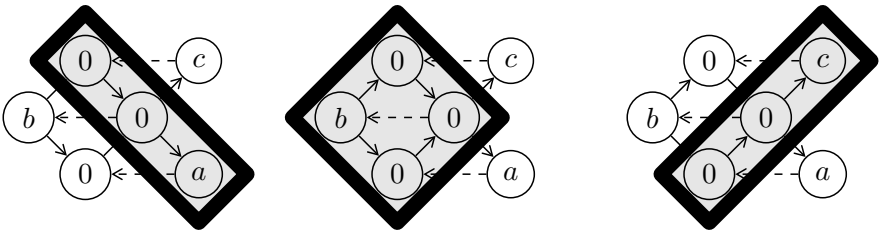
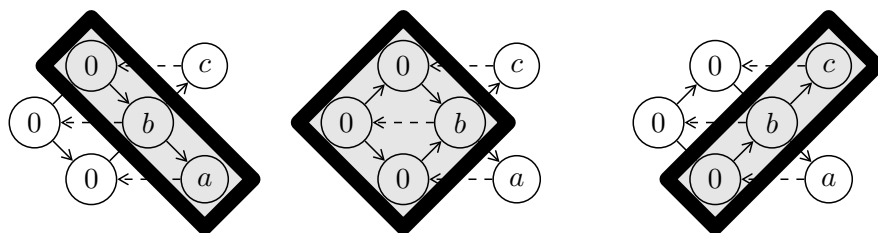


FIGURE 3.4. The rectangle corresponds to indecomposable representations that are in $\mathcal{C}_{Q,m}$ for $m = 1, 2, 3$ from left to right. We use it as in Figure 3.2 to get $\text{GenJF}(X)$ for $X \in \mathcal{C}_1$ in Example 3.8.

- The subcategory $\mathcal{C}_2 = \text{Cat}_Q(\{[1], [1, 3], [3]\})$ is Jordan recoverable. For $X \in \mathcal{C}$, there exists $(a, b, c) \in \mathbb{N}^3$ such that $X \cong X_{[1]}^a \oplus X_{[1,3]}^b \oplus X_{[3]}^c$. Following Figure 3.5, we get that $\text{GenJF}(X) = ((a + b), (b), (b + c))$. Note that we can recover $X \in \mathcal{C}$ from this data.

In general, contrary to the previous example, we must deal with many equations to prove that a subcategory $\mathcal{C} \subset \text{rep}(Q)$ is Jordan recoverable. We can ask whether there is a more general way to recover X from its generic Jordan form data.


 FIGURE 3.5. The way to get $\text{GenJF}(X)$ for $X \in \mathcal{C}_2$ in Example 3.8.

3.3. Generic representation with fixed Jordan form data. This section discusses two algebraic ways to get a specific canonical representation from fixed Jordan form data.

First, we recall the general setting in which we work. Let Q be an A_n type quiver, and $\mathbf{d} = (d_q)_{q \in Q_0} \in \mathbb{N}^n$. Write $\text{rep}(Q, \mathbf{d})$ for the set of $E \in \text{rep}(Q)$ such that $\dim(E) = \mathbf{d}$. One can identify this set as follows

$$\text{rep}(Q, \mathbf{d}) \cong \prod_{\alpha \in Q_1} \text{Hom}_{\mathbb{K}}(\mathbb{K}^{d_{s(\alpha)}}, \mathbb{K}^{d_{t(\alpha)}}) = \prod_{\alpha \in Q_1} \text{Mat}_{d_{s(\alpha)} \times d_{t(\alpha)}}(\mathbb{K})$$

by choosing a basis for each of the vector spaces. We endow $\text{rep}(Q, \mathbf{d})$ with the action of the algebraic group $\mathbf{GL}_{\mathbf{d}}(\mathbb{K}) = \prod_{q \in Q_0} \text{GL}_{d_q}(\mathbb{K})$ which changes the basis at each vertex. The orbits of this group action are the isomorphism classes of the representations in $\text{rep}(Q, \mathbf{d})$.

Now we present a method developed by [13]. For each $q \in Q_0$, fix a $\mathbb{K}$ -vector space V_q such that $\dim(V_q) = d_q$, and a nilpotent endomorphism $N_q : V_q \rightarrow V_q$. Let $\text{rep}(Q, N)$ be the set of representations $E \in \text{rep}(Q)$ such that, for all $q \in Q_0$, $E_q = V_q$, and $N = (N_q)_{q \in Q_0} \in \text{NEnd}(E)$.

Proposition 3.9 ([13, Section 2.3]). *Let Q be an A_n quiver, and $\mathbf{d} \in \mathbb{N}^n$. For any collection of $\mathbb{K}$ -vector spaces $(V_q)_{q \in Q_0}$ such that $\dim(V_q) = d_q$, and any collection of nilpotent endomorphisms $N = (N_q : V_q \rightarrow V_q)_{q \in Q_0}$, the set $\text{rep}(Q, N)$ is an irreducible variety, and there exists a dense open set $\Omega_N \subseteq \text{rep}(Q, N)$ within which the representations are all isomorphic. Moreover, for any collection of nilpotent endomorphisms $N' = (N'_q : V_q \rightarrow V_q)_{q \in Q_0}$ such that $\text{JF}(N) = \text{JF}(N')$, then the representations of Ω_N and those of $\Omega_{N'}$ are isomorphic.*

The proof of this result is mainly based on Kac's theorem [15, p. 85] and the fact that the indecomposable representations of a Dynkin quiver are characterized by their dimension vectors.

Therefore, for any n -tuple of integer partitions λ , we denote by $\text{GenRep}(\lambda)$ the representation $\text{GenRep}(N)$ for some n -tuple of nilpotent endomorphisms $N = (N_q : \mathbb{K}^{|\lambda_q|} \rightarrow \mathbb{K}^{|\lambda_q|})_{q \in Q_0}$ whose Jordan form data are parametrized by λ . We call it the *generic representation with Jordan form λ* .

In the following, we present a slightly different way to define the generic representation of Q with Jordan form λ .

Let $\lambda = (\lambda^q)_{q \in Q_0}$ be a n -tuple of integer partitions. Denote by $\text{rep}(Q, \lambda)$ the set of representations $Y = ((Y_q)_{q \in Q_0}, (Y_\alpha)_{\alpha \in Q_1})$ such that $\dim(Y) = (|\lambda^q|)_{q \in Q_0}$, and for which there exists a nilpotent endomorphism $N \in \text{NEnd}(Y)$ with $\text{JF}(N) = \lambda$.

Proposition 3.10. *Let Q be an A_n type quiver, and $\lambda = (\lambda^q)_{q \in Q_0}$ be a n -tuple of integer partitions. Then $\text{rep}(Q, \lambda)$ is an irreducible space. Moreover, there exists a dense open set $\mathcal{U} \subset \text{rep}(Q, \lambda)$ such that all the representations in $\mathcal{U}$ are isomorphic.*

Proof. Set $\mathbf{d} = (|\lambda^q|)_{q \in Q_0}$. For any $E \in \text{rep}(Q, \lambda)$ and $G \in \mathbf{GL}_{\mathbf{d}}(\mathbb{K})$, we have $G \cdot E \in \text{rep}(Q, \lambda)$. Moreover, the action of $\mathbf{GL}_{\mathbf{d}}(\mathbb{K})$ is transitive on nilpotent endomorphisms of the same Jordan form. Thus, for any collection $N = (N_q)_{q \in Q_0}$ of nilpotent endomorphisms whose Jordan forms are given by λ , $\text{rep}(Q, \lambda) = \mathbf{GL}_{\mathbf{d}}(\mathbb{K}) \cdot \text{rep}(Q, N)$. In the following, we fix such a collection of nilpotent endomorphisms $N = (N_q)_{q \in Q_0}$.

We set

$$\Phi_N : \begin{cases} \mathbf{GL}_{\mathbf{d}}(\mathbb{K}) \times \text{rep}(Q, N) & \longrightarrow \text{rep}(Q, \mathbf{d}) \\ (\phi, E) & \longmapsto \phi \cdot E. \end{cases}$$

Note that this is an algebraic morphism. The space $\mathbf{GL}_{\mathbf{d}}(\mathbb{K}) \times \text{rep}(Q, N)$ is irreducible as a product of irreducible spaces. So $\text{rep}(Q, \lambda) = \Phi_N(\mathbf{GL}_{\mathbf{d}}(\mathbb{K}) \times \text{rep}(Q, N))$ is an irreducible space.

By Proposition 3.9, there exists a dense open set $\Omega_N \subset \text{rep}(Q, N)$ within which the representations are all isomorphic. Set $\mathcal{U} = \Phi_N(\mathbf{GL}_{\mathbf{d}}(\mathbb{K}) \times \Omega_N)$. It is clear that $\mathcal{U}$ is dense in $\text{rep}(Q, \lambda)$. Moreover, as the image of an open set by an algebraic morphism is open in the closure of its image [16, Section I.8, Corollary 2], we have that $\mathcal{U}$ is open in $\text{rep}(Q, \lambda)$. $\square$

The previous result allows us to construct $\text{GenRep}(\lambda)$ by exhibiting a generic behavior among the representations in $\text{rep}(Q, \lambda)$.

Remark 3.11. For any Dynkin-type quiver Q , the two algebraic ways presented to define the generic representation with a fixed Jordan form coincide. Propositions 3.9 and 3.10 hold in this setting.

3.4. Canonical Jordan recoverability. Consider $\mathcal{C}$ a Jordan recoverable subcategory of $\text{rep}(Q)$. One can legitimately ask if GenRep gives an inverse of GenJF . However, Example 1.1 highlights an example of a Jordan recoverable category for which it does not work. Therefore, we need to refine the notion of Jordan recoverability.

Definition 3.12. A subcategory $\mathcal{C}$ of $\text{rep}(Q)$ is said to be *canonically Jordan recoverable* if, for any $X \in \mathcal{C}$, $\text{GenRep}(\text{GenJF}(X)) \cong X$.

We have an inverse of $\mathbf{GK}$ in such a category. Then, obviously, any canonically Jordan recoverable category is Jordan recoverable. However, there are Jordan recoverable categories that are not canonically Jordan recoverable.

Example 3.13. Let Q be the A_3 type quiver of Example 3.8.

- The category $\mathcal{C}_1$ is not canonically Jordan recoverable by following the explanations already given in Example 1.1.
- The category $\mathcal{C}_2$ is canonically Jordan recoverable. Let $\lambda = ((a+b), (b), (b+c))$ for a fixed triplet $(a, b, c) \in \mathbb{N}^3$. Consider $Y \in \text{rep}(Q, \lambda)$ and $N = (N_1, N_2, N_3) \in \mathbf{NEnd}(Y)$ such that $\mathbf{JF}(N) = \lambda$. There exists $u_1 \in Y_1$ such that $N_1^{a+b-1}(u_1) \neq 0$. Thus, by writing $u_i = N_1^{i-1}(u_1)$ for $i \in \{1, \dots, a+b\}$, we get that $(u_1, \dots, u_{a+b})$ is a basis of Y_1 , adapted to N . Similarly, we construct the bases $(v_1, \dots, v_b)$ and $(w_1, \dots, w_{b+c})$ of respectively Y_2 and Y_3 such that they are adapted to N (see Figure 3.6). We now have to describe Y_α and Y_β .

Thanks to the chosen bases, and square commutativity relations satisfied by N , we only have to describe the image of u_1 by Y_α to describe all Y_α , and the same goes for Y_β . Then $Y_\alpha(u_1)$ has to be a $\mathbb{K}$ -linear combination of $(v_1, \dots, v_b)$, say $(\alpha) : Y_\alpha(u_1) = k_1 v_1 + \dots + k_b v_b$ with $k_1, \dots, k_b \in \mathbb{K}$. Among all the choices we could

make, there exists a dense open set $\Omega_1 \subset \text{rep}(Q, \lambda)$ such that for $Y \in \Omega_1$, $Y_\alpha(u_1) \notin \text{Im}(N_2)$ (this dense open set can be seen as taking $k_1 \neq 0$ in (α)). Analogously, there exists a dense open set Ω_2 such that $Y_\beta(w_1) \notin \text{Im}(N_2)$. Therefore, for all Y in the dense open set $\Omega_1 \cap \Omega_2$, we get that $Y \cong X_{[1]}^a \oplus X_{[1,3]}^b \oplus X_{[3]}^c$. This proves our claim.

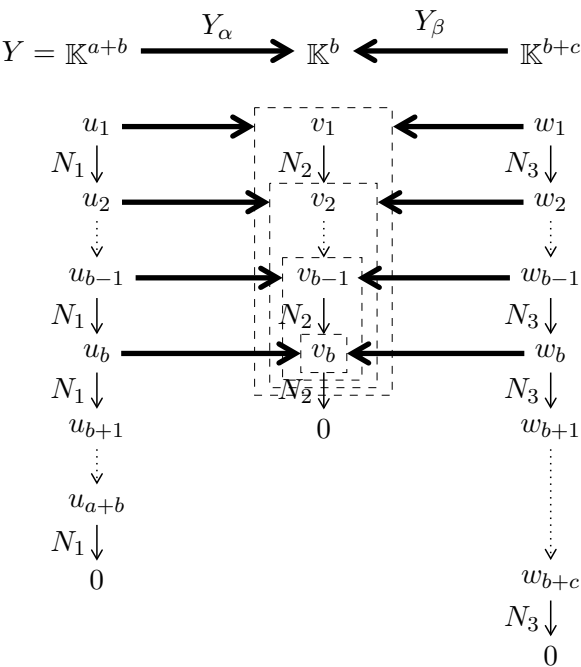


FIGURE 3.6. Illustration of the configuration described to study the canonical Jordan recoverability of $\mathcal{C}_2$. Note that $Y_\alpha(u_i) = 0$ and $Y_\beta(w_i) = 0$ for $i > b$ by square commutativity relations.

This paper describes all the canonically Jordan recoverable subcategories of $\text{rep}(Q)$ for any A_n type quiver.

4. STORABILITY

In this section, we introduce a relation among integer partitions, which we call *storability*, that corresponds to certain interlacing behaviors. We introduce diagonal transformations on n -tuples of integer partitions, which underlie the behaviors of the generic Jordan form data under the action of the reflection functors. Later on, in Section 5, we will use it to define operators on additive subcategories that preserve Jordan recoverability and canonical Jordan recoverability, under certain assumptions. These are relevant tools to prove Theorem 1.4.

4.1. Storable pairs.

Definition 4.1. Let λ and μ be two integer partitions. The pair (λ, μ) is *storable* if for all $i \in \mathbb{N}^*$, $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ (adding zero parts as needed). Such a pair is *strongly storable* if in addition $\lambda_1 = \mu_1$.

We represent and characterize storable pairs visually. Fill two rows of 45° rotated boxes with the entries of λ and μ as in Figure 4.1, adding infinitely many zeros to the right. Then (λ, μ) is storable if the entries weakly decrease from left to right.

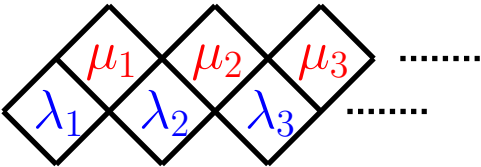


FIGURE 4.1. Illustration of storability of (λ, μ) .

We give two results that arise from the definition.

- Lemma 4.2.** *Let λ and μ be two integer partitions.*
- (1) *If (λ, μ) and (μ, λ) are both storable, then $\lambda = \mu$.*
 - (2) *If (λ, μ) is storable, then $\ell(\lambda) \in \{\ell(\mu), \ell(\mu) + 1\}$.*

4.2. Storable triplets.

Definition 4.3. Let λ, μ and ν be three integer partitions. The triplet (λ, μ, ν) is *storable* if the two following conditions are satisfied:

- either (λ, μ) or (μ, λ) is a storable pair;
- either (μ, ν) or (ν, μ) is a storable pair.

More precisely, we say that (λ, μ, ν) is:

- $(\boxplus\boxplus)$ $(\boxplus, \boxplus)$ -storable if (λ, μ) and (ν, μ) are storable pairs;
- $(\boxplus\boxminus)$ $(\boxplus, \boxminus)$ -storable if (λ, μ) and (μ, ν) are storable pairs;
- $(\boxminus\boxplus)$ $(\boxminus, \boxplus)$ -storable if (μ, λ) and (ν, μ) are storable pairs;
- $(\boxminus\boxminus)$ $(\boxminus, \boxminus)$ -storable if (μ, λ) and (μ, ν) are storable pairs.

Such a triplet is *strongly storable* whenever $\lambda_1 = \mu_1$ or $\mu_1 = \nu_1$.

In Figure 4.2, we illustrate the four storability configurations.

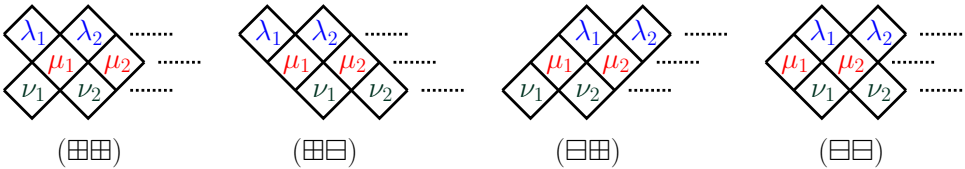


FIGURE 4.2. Illustration of the four storability configurations of (λ, μ, ν) .

Definition 4.4. Let λ, μ , and ν be three integer partitions. Assume that (λ, μ, ν) is a storable triplet. We define the *diagonal transformation of μ in (λ, μ, ν)* , denoted $\text{diag}(\lambda, \mu, \nu)$, to be the integer partition $\theta = (\theta_1, \theta_2, \dots)$ such that:

- if (λ, μ, ν) is $(\boxplus, \boxplus)$ -storable, then we define, for all $i \geq 1$,
- $$\theta_i = \begin{cases} \max(\lambda_1, \nu_1) & \text{if } i = 1, \\ \min(\lambda_{i-1}, \nu_{i-1}) + \max(\lambda_i, \nu_i) - \mu_{i-1} & \text{otherwise;} \end{cases}$$

- if (λ, μ, ν) is $(\boxplus, \boxminus)$ -storable, then we define, for all $i \geq 1$,

$$\theta_i = \begin{cases} \lambda_1 + \max(\lambda_2, \nu_1) - \mu_1 & \text{if } i = 1, \\ \min(\lambda_i, \nu_{i-1}) + \max(\lambda_{i+1}, \nu_i) - \mu_i & \text{otherwise;} \end{cases}$$

- if (λ, μ, ν) is $(\boxminus, \boxplus)$ -storable, then we define, for all $i \geq 1$,

$$\theta_i = \begin{cases} \nu_1 + \max(\lambda_1, \nu_2) - \mu_1 & \text{if } i = 1, \\ \min(\lambda_{i-1}, \nu_i) + \max(\lambda_i, \nu_{i+1}) - \mu_i & \text{otherwise;} \end{cases}$$

- if (λ, μ, ν) is $(\boxminus, \boxminus)$ -storable, then we define, for all $i \geq 1$,

$$\theta_i = \min(\lambda_i, \nu_i) + \max(\lambda_{i+1}, \nu_{i+1}) - \mu_{i+1}.$$

We can picture the diagonal operation as doing local operations for each square of μ in the diagram representing the storable triple (λ, μ, ν) (Figure 4.3).

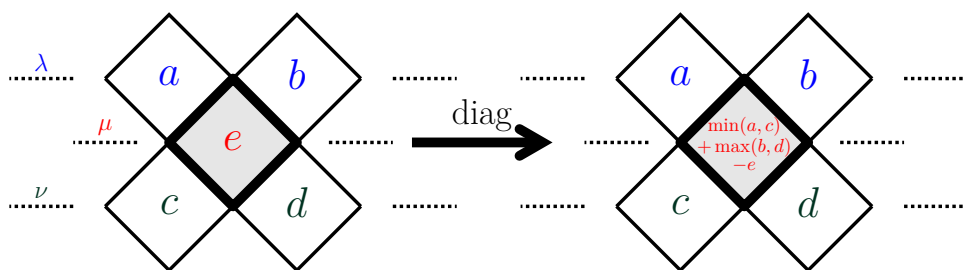


FIGURE 4.3. Illustration of the local operations to calculate $\text{diag}(\lambda, \mu, \nu)$.

Remark that λ and ν play symmetric roles : $\text{diag}(\lambda, \mu, \nu) = \text{diag}(\nu, \mu, \lambda)$. Here are some elementary statements we get for the diagonal transformation.

Lemma 4.5. *Let λ, μ and ν be three integer partitions. When it is well-defined, consider $\theta = \text{diag}(\lambda, \nu, \mu)$.*

- (1) *If (λ, μ) is a storable pair, then $\text{diag}(\lambda, \mu, \mu) = \lambda$.*
- (2) *If (λ, μ, ν) is $(\boxplus, \boxplus)$ -storable, then (λ, θ, ν) is strongly $(\boxminus, \boxminus)$ -storable.*
- (3) *If (λ, μ, ν) is $(\boxplus, \boxminus)$ -storable, then (λ, θ, ν) is $(\boxplus, \boxminus)$ -storable.*
- (4) *If (λ, μ, ν) is $(\boxminus, \boxplus)$ -storable, then (λ, θ, ν) is $(\boxminus, \boxplus)$ -storable.*
- (5) *If (λ, μ, ν) is $(\boxminus, \boxminus)$ -storable, then (λ, θ, ν) is $(\boxplus, \boxplus)$ -storable.*
- (6) *If (λ, μ, ν) is either $(\boxplus, \boxplus)$ -storable, $(\boxplus, \boxminus)$ -storable, $(\boxminus, \boxplus)$ -storable or strongly $(\boxminus, \boxminus)$ -storable, then $\text{diag}(\lambda, \theta, \nu) = \mu$.*

4.3. Rephrasing results of Garver–Patrias–Thomas. In [13], they use the following notion.

Definition 4.6. Let ρ, μ be two integer partitions, and $t \in \mathbb{Z}$. We say that ρ and μ are t -interlaced if:

- in the case $t \geq 0$,

$$\rho_1 \geq \rho_2 \geq \dots \geq \rho_t \geq \rho_{t+1} \geq \mu_1 \geq \rho_{t+2} \geq \rho_{t+3} \geq \mu_2 \geq \rho_{t+4} \geq \dots,$$

- in the case $t \leq 0$, $\rho_i = \mu_i$ for $1 \leq i \leq -t$ and

$$\rho_{-t+1} \geq \mu_{-t+1} \geq \rho_{-t+2} \geq \rho_{-t+3} \geq \mu_{-t+2} \geq \rho_{-t+4} \geq \dots$$

We can state a link between t -interlaced pairs and storable triplets. For two integer partitions λ, ν , we denote $\lambda + \nu$ the integer partition whose multiset of parts is composed of the parts of λ and ν .

Lemma 4.7. *Let λ, μ and ν be three integer partitions. The following assertions hold:*

- (i) *if (λ, μ, ν) is $(\boxplus, \boxplus)$ -storable, then $\lambda + \nu$ and μ are 1-interlaced;*
- (ii) *if (λ, μ, ν) is either $(\boxplus, \boxminus)$ -storable or $(\boxminus, \boxplus)$ -storable, then $\lambda + \nu$ and μ are 0-interlaced;*
- (iii) *if (λ, μ, ν) is strongly $(\boxminus, \boxminus)$ -storable, then $\lambda + \nu$ and μ are -1 -interlaced.*

Remark 4.8. Note that:

- if (λ, μ, ν) is $(\boxminus, \boxminus)$ -storable but not strongly $(\boxminus, \boxminus)$ -storable, then $\lambda + \nu$ and μ are not interlaced.
- if (λ, μ, ν) is such that $\lambda + \nu$ and μ are t -interlaced, then this does not imply that (λ, μ, ν) is a storable triplet.

Let q be a vertex of an A_n type quiver Q , $\mathbf{d} \in \mathbb{N}^n$ and $\boldsymbol{\pi} \vdash \mathbf{d}$. We extend the tuple of partitions $\boldsymbol{\pi}$ with $\pi^0 = \pi^{n+1} = (0)$. We will write that $\boldsymbol{\pi}$ is (strongly) $(\boxminus, \boxminus)$ -storable at q if $(\pi^{q-1}, \pi^q, \pi^{q+1})$ is a (strongly) $(\boxminus, \boxminus)$ -storable triplet. We use the same formulation for the three other storability configurations.

Let v be a source or a sink of Q . We define $\sigma_v(\boldsymbol{\pi})$ to be the n -tuple of partitions obtained from $\boldsymbol{\pi}$ by replacing π^v with $\text{diag}(\pi^{v-1}, \pi^v, \pi^{v+1})$.

Lemma 4.9. *Let v be a vertex of a quiver of A_n type. Let $\boldsymbol{\pi}$ be a n -tuple of integer partitions such that $\boldsymbol{\pi}$ is either $(\boxplus, \boxplus)$ -storable, $(\boxplus, \boxminus)$ -storable, $(\boxminus, \boxplus)$ -storable or strongly $(\boxminus, \boxminus)$ -storable at v . Consider $X = \text{GenRep}(\boldsymbol{\pi})$ and assume that $\text{GenJF}(X) = \boldsymbol{\pi}$. Then, if v is a source, $\text{GenJF}(\mathcal{R}_v^-(X)) = \sigma_v(\boldsymbol{\pi})$. Similarly, if v is a sink, $\text{GenJF}(\mathcal{R}_v^+(X)) = \sigma_v(\boldsymbol{\pi})$.*

Proof. Let $\boldsymbol{\pi}$ be as assumed. Then $\rho = \pi^{v+1} + \pi^{v-1}$ and $\mu = \pi^v$ are t -interlaced for $t \in \{1, 0, -1\}$ by Lemma 4.7. The desired result follows from [13, Theorem 3.12]. $\square$

Remark 4.10. [13, Theorem 3.12] asks for having $t \geq 0$. However, one can check that the proof of this result relies on [13, Lemma 3.6], which is true for $t \in \mathbb{Z}$. Thus, in reality, this theorem holds for $t < 0$ as well.

Theorem 4.11. *Let v be a vertex of an A_n type quiver Q . Let $\boldsymbol{\pi}$ be a n -tuple of integer partitions such that $\boldsymbol{\pi}$ is $(\boxplus, \boxplus)$ -storable, $(\boxplus, \boxminus)$ -storable, $(\boxminus, \boxplus)$ -storable, or strongly $(\boxminus, \boxminus)$ -storable at v . Assume that $X \cong \text{GenRep}(\boldsymbol{\pi})$. Then if v is a source, then $\mathcal{R}_v^-(X) \cong \text{GenRep}(\sigma_v(\boldsymbol{\pi}))$. Similarly, if v is a sink, $\mathcal{R}_v^+(X) \cong \text{GenRep}(\sigma_v(\boldsymbol{\pi}))$.*

Proof. Let $\boldsymbol{\pi}$ be as assumed. Then $\rho = \pi^{v+1} + \pi^{v-1}$ and $\mu = \pi^v$ are t -interlaced for $t \in \{1, 0, -1\}$ by Lemma 4.7. The result we wished for follows from [13, Theorem 3.10]. $\square$

5. OPERATIONS PRESERVING CANONICAL JORDAN RECOVERABILITY

In this section, we introduce elementary categorical operations: adding simple objects and applying reflection functors. Those operations allow one to build any additive subcategory of a Dynkin-type quiver. We show that, under specific assumptions, they preserve Jordan recoverability and canonical Jordan recoverability. Later on, in Section 7, we use those operations to build all the maximal canonically Jordan recoverable, based on combinatorial data, developed in Section 6, and an algorithm using those operations.

5.1. Adding a simple representation. We define the operation AddS_v on subcategories of $\text{rep}(Q)$ by $\text{AddS}_v(\mathcal{C}) = \text{add}(\mathcal{C}, S_v)$ for any subcategory $\mathcal{C}$ of $\text{rep}(Q)$. In general, this operation does not preserve the canonical Jordan recoverability property. This subsection shows it does so under a storability condition on generic Jordan forms of all $X \in \mathcal{C}$.

First, we are interested in preserving Jordan recoverability. Before stating the result, we need the following lemma.

Lemma 5.1. *Let v be a source or a sink of an A_n type quiver Q . Consider $a \in \mathbb{N}$ and $X \in \text{rep}(Q)$. Write $\pi = \text{GenJF}(X)$. Then $\text{GenJF}(S_v^a \oplus X) = \xi$ where $\xi^q = \pi^q$ if $q \neq v$ and $\xi^v = (\pi_1^v + a, \pi_2^v, \pi_3^v, \dots)$.*

Proof. The lemma is a direct consequence of the combinatorial way to calculate $\xi = \text{GenJF}(S_k^a \oplus X)$. Theorem 3.6 tells us that GenJF is given by the Greene–Kleitman invariant GK introduced in Section 3.1.

We first note that the calculation of ξ differs from the one for $\pi = \text{GenJF}(X)$ only at the vertex v . Moreover, as v is a source (respectively a sink), any maximal path in the Auslander–Reiten quiver of Q over indecomposable objects of $\mathcal{C}$ admitting v in their support will go through S_v , a consequence of the fact that S_v is at the end (respectively at the beginning) of any of those paths. Then $\xi_1^v = \pi_1^v + a$. For the other parts of ξ^v , as S_v will not reappear in the calculation, $\xi_i^v = \pi_i^v$ for $i \geq 2$. $\square$

Now we can give a sufficient assumption on Jordan recoverable subcategories $\mathcal{C}$ such that $\text{AddS}_v(\mathcal{C})$ is also Jordan recoverable.

Proposition 5.2. *Let Q be a quiver of A_n type, and v be a source or a sink of Q . Let $\mathcal{C} \subset \text{rep}(Q)$ be a Jordan recoverable category such that*

($\star$) *For any $X \in \mathcal{C}$, $\text{GenJF}(X)$ is strongly $(\boxminus, \boxminus)$ -storable at v .*

Then $\mathcal{D} = \text{AddS}_v(\mathcal{C})$ is Jordan recoverable.

Proof. Let v , Q and $\mathcal{C}$ be as assumed. Remark that S_v is not an indecomposable object of $\mathcal{C}$ by ($\star$). Consider $\mathcal{D} = \text{AddS}_v(\mathcal{C})$ and let us prove that $\mathcal{D}$ is Jordan recoverable.

Let $Y, Z \in \mathcal{D}$. We know that $Y \cong S_v^a \oplus Y'$ and $Z \cong S_v^b \oplus Z'$ with $Y', Z' \in \mathcal{C}$ and $a, b \in \mathbb{N}$. Suppose that $\text{GenJF}(Y) = \text{GenJF}(Z)$. If we take $\lambda = \text{GenJF}(Y')$ and $\mu = \text{GenJF}(Z')$, then, by Lemma 5.1, we get :

- (1) $\lambda^q = \mu^q$ for all $q \neq k$;
- (2) $\lambda_1^v + a = \mu_1^v + b$;
- (3) $\lambda_s^v = \mu_s^v$ for all $s > 1$.

By ($\star$) and (1), we know that $\lambda_1^v = \max(\lambda_1^{v-1}, \lambda_1^{v+1}) = \max(\mu_1^{v-1}, \mu_1^{v+1}) = \mu_1^v$. Hence $a = b$ and $\lambda = \mu$. Therefore, we get that $Y' \cong Z'$ using the fact that $\mathcal{C}$ is Jordan recoverable. We finally conclude that $Y \cong Z$ and thus $\mathcal{D}$ is Jordan recoverable. $\square$

We now show that we preserve canonical Jordan recoverability under the same assumption ($\star$).

Proposition 5.3. *Let Q be a quiver of A_n type, and v be a source or a sink of Q . Let $\mathcal{C} \subset \text{rep}(Q)$ be a canonically Jordan recoverable category satisfying ($\star$). Then $\mathcal{D} = \text{AddS}_v(\mathcal{C})$ is canonically Jordan recoverable.*

Proof. Consider $Y \in \mathcal{D} = \text{AddS}_k(\mathcal{C})$. By definition of $\mathcal{D}$, there exist $a \in \mathbb{N}$ and $Y' \in \mathcal{C}$ such that $Y \cong S_v^a \oplus Y'$. Write $\pi = \text{GenJF}(Y')$. We get $\xi = \text{GenJF}(Y)$ from π as described in Lemma 5.1.

From $(\star)$, we know that π is strongly $(\boxminus, \boxminus)$ -storable at v . So $\pi_1^v = \max(\pi_1^{v-1}, \pi_1^{v+1})$.

Let $Z \in \text{rep}(Q, \xi)$. Consider $N \in \text{NEnd}(Z)$ such that $\text{JF}(N) = \xi$. Assume, without loss of generality, that v is a source. Denote $\alpha : v-1 \leftarrow v$ and $\beta : v \rightarrow v+1$ the arrows incident to v . From the relation $\pi_1^v = \max(\pi_1^{v-1}, \pi_1^{v+1})$ and the definition of ξ , we get that $N^{\pi_1^v}(Z_v) \subseteq \text{Ker}(Z_\alpha) \cap \text{Ker}(Z_\beta)$. Thus, $\text{mult}(S_v, Z) \geq a$. Saying that $\text{mult}(S_v, Z) > a$ is equivalent to asking the induced morphism from the quotient $Z_v/N^{\pi_1^v}(Z_v)$ to $Z_\alpha \oplus Z_\beta$ to have a nontrivial kernel, which is a closed condition. As $Y \in \text{rep}(Q, \xi)$ and $\text{mult}(S_v, Y) = a$, there exists therefore a dense open set $\Theta \subset \text{rep}(Q, \xi)$ such that, for any $Z \in \Theta$, $\text{mult}(S_v, Z) = a$, which means there exists $Z' \in \text{rep}(Q)$ such that $Z \cong S_v^a \oplus Z'$ and Z' has no indecomposable summand isomorphic to S_v .

Thanks to Lemma 5.1, we know that $\text{GenJF}(Z') = \pi$. By canonical Jordan recoverability of $\mathcal{C}$, we know that there exists a dense open set $\Phi \subset \text{rep}(Q, \pi)$ such that for any $Z' \in \Phi$, $Z' \cong Y'$. Hence there exists a dense open set $\Omega \subset \text{rep}(Q, \xi)$ such that, for all $Z \in \Omega$, $Z \cong S_v^a \oplus Y' \cong Y$. We conclude that $\mathcal{D}$ is canonically Jordan recoverable. $\square$

5.2. Applying reflection functors. This subsection aims to show that under some general storability conditions, reflection functors preserve canonical Jordan recoverability.

First, we prove the following result.

Proposition 5.4. *Let v be a vertex of an A_n type quiver Q . Let $\mathcal{C} \subseteq \text{rep}(Q)$ be a Jordan recoverable category such that :*

(∇) *For any $X \in \mathcal{C}$, $\text{GenJF}(X)$ is either $(\boxplus, \boxplus)$ -storable, $(\boxminus, \boxplus)$ -storable, $(\boxplus, \boxminus)$ -storable or strongly $(\boxminus, \boxminus)$ -storable at v .*

If v is a source, then $\mathcal{R}_v^-(\mathcal{C})$ is a Jordan recoverable category of $\text{rep}(\sigma_v(Q))$. Similarly, if v is a sink, then $\mathcal{R}_v^+(\mathcal{C})$ is a Jordan recoverable category of $\text{rep}(\sigma_v(Q))$.

Proof. Assume that v is a source. Let $Y, Z \in \mathcal{R}_v^-(\mathcal{C})$. By (∇), we know that $S_v \notin \mathcal{C}$. Therefore there exists a unique representation $Y' \in \mathcal{C}$ (up to isomorphism) such that $\mathcal{R}_v^-(Y') \cong Y$. Similarly, there exists a unique $Z' \in \mathcal{C}$ such that $\mathcal{R}_v^-(Z') = Z$.

Consider $\lambda = \text{GenJF}(Y')$ et $\mu = \text{GenJF}(Z')$. So $\sigma_v(\lambda) = \text{GenJF}(Y)$ and $\sigma_v(\mu) = \text{GenJF}(Z)$. Now assume that $\theta = \sigma_v(\lambda) = \sigma_v(\mu)$. We claim that $Y \cong Z$. We know, by (∇) and Lemma 4.5, that θ is either $(\boxplus, \boxplus)$ -storable, $(\boxminus, \boxplus)$ -storable, $(\boxplus, \boxminus)$ -storable or strongly $(\boxminus, \boxminus)$ -storable at v . As a consequence of Lemma 4.5(6), we have $\lambda = \mu$. Using the fact that $\mathcal{C}$ is Jordan recoverable, we conclude that $Y' \cong Z'$. Thus, $Y \cong Z$, and we end up with the result we wished for.

The same goes analogously for $\mathcal{R}_v^+(\mathcal{C})$, whenever v is a sink. $\square$

Under the same assumption (∇), we can show that reflection functors also preserve canonical Jordan recoverability.

Proposition 5.5. *Let v be a source of an A_n -type quiver Q . Let $\mathcal{C} \subset \text{rep}(Q)$ be a canonically Jordan recoverable category satisfying (∇). Then $\mathcal{R}_v^-(\mathcal{C})$ is a canonically Jordan recoverable subcategory of $\text{rep}(\sigma_v(Q))$. Similarly, if v is a sink, $\mathcal{R}_v^+(\mathcal{C})$ is a canonically Jordan recoverable subcategory of $\text{rep}(\sigma_v(Q))$.*

Proof. Assume that v is a source. Let $Y \in \mathcal{R}_v^-(\mathcal{C})$. By (∇), $S_v \notin \mathcal{C}$. So there exists a unique representation $Y' \in \mathcal{C}$ (up to isomorphism) such that $\mathcal{R}_v^-(Y') \cong Y$.

Let $\pi = \text{GenJF}(Y')$. As $\mathcal{C}$ is canonically Jordan recoverable, we have that $\text{GenRep}(\pi) \cong Y'$. We can use Lemma 4.9 to get that $\text{GenJF}(Y) = \sigma_v(\pi)$. Moreover, Theorem 4.11 gives us that $\text{GenRep}(\sigma_v(\pi)) \cong \mathcal{R}_v^-(Y') \cong Y$. This completes the proof.

The same goes similarly for $\mathcal{R}_v^+(\mathcal{C})$, whenever v is a sink. $\square$

6. ADJACENCY-AVOIDING INTERVAL SETS

Recall the following definition.

Definition 6.1. Two intervals $K, L \in \mathcal{I}_n$ are *adjacent* if either $b(K) = e(L) + 1$ or $b(L) = e(K) + 1$. An interval set $\mathcal{J}$ is said to be *adjacency-avoiding* if there are no pairs of adjacent intervals in $\mathcal{J}$; meaning, in a more affirmative way, that for all $K, L \in \mathcal{J}$, we have either $K \cap L \neq \emptyset$, $b(K) \geq e(L) + 2$ or $b(L) \geq e(K) + 2$.

We saw in Section 1.2 that this notion is a crucial point to describe the canonically Jordan recoverable subcategories of $\text{rep}(Q)$ for any A_n type quiver Q . We aim to investigate the combinatorial behavior of this family of interval sets, with their representation-theoretic interest in mind.

More precisely, we give an explicit description of all the (maximal) adjacency-avoiding subsets of intervals, and we explore their behavior under the combinatorial lookalike operators of the ones in Section 5. In Section 7, while applying an algorithm using those operators, it allows us to keep track of the category we are constructing, and to check that they correspond to the stated categories in Theorem 1.4.

6.1. Interval sets from shifted bipartitions. In this section, we describe and characterize all maximal adjacency-avoiding subsets of $\mathcal{I}_n$.

Definition 6.2. Let $\mathbf{B}$ and $\mathbf{E}$ be two subsets of $\{1, \dots, n\}$. We define $\mathcal{J}(\mathbf{B}, \mathbf{E})$ to be the following subset of $\mathcal{I}_n$:

$$\mathcal{J}(\mathbf{B}, \mathbf{E}) = \{K \in \mathcal{I}_n \mid b(K) \in \mathbf{B} \text{ and } e(K) \in \mathbf{E}\}.$$

Example 6.3. Let $n \geq 1$. For $m \in \{1, \dots, n\}$ ($\mathbf{B} = [1, m], \mathbf{E} = [m, n]$), we get

$$\mathcal{J}(\mathbf{B}, \mathbf{E}) = \{K \in \mathcal{I}_n \mid m \in K\}.$$

Note that for a given pair of subsets $(\mathbf{B}, \mathbf{E})$, there could exist $b \in \mathbf{B}$ such that for all $e \in \mathbf{E}$, $e < b$: this implies that there are no intervals K in $\mathcal{J}(\mathbf{B}, \mathbf{E})$ such that $b(K) = b$. Hence $\mathcal{J}(\mathbf{B}, \mathbf{E}) = \mathcal{J}(\mathbf{B} \setminus \{b\}, \mathbf{E})$. We consider the following notion, as we want to completely characterize these interval subsets by pairs of subsets $(\mathbf{B}, \mathbf{E})$. We say that $(\mathbf{B}, \mathbf{E})$ is a *effective pair of subsets* if it satisfies the two assertions below:

- for all $b \in \mathbf{B}$ there exists $e \in \mathbf{E}$ such that $b \leq e$,
- for all $e \in \mathbf{E}$, there exists $b \in \mathbf{B}$ such that $b \leq e$.

Lemma 6.4. For all $(\mathbf{C}, \mathbf{F})$ pairs of subsets of $\{1, \dots, n\}$, there exists a unique effective pair of subsets $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$ such that $\mathcal{J}(\mathbf{C}, \mathbf{F}) = \mathcal{J}(\mathbf{B}, \mathbf{E})$.

For all $\mathbf{A} \subset \{1, \dots, n\}$, we denote by $\mathbf{A}[1]$ the *shift of $\mathbf{A}$* defined by $\mathbf{A}[1] = \{a + 1 \mid a \in \mathbf{A}\}$.

Definition 6.5. Let $(\mathbf{B}, \mathbf{E})$ be a pair of subsets of $\{1, \dots, n\}$. We say that $(\mathbf{B}, \mathbf{E})$ is a *shifted disjoint pair* if $\mathbf{B} \cap \mathbf{E}[1] = \emptyset$.

Proposition 6.6. Let $(\mathbf{B}, \mathbf{E})$ be an effective shifted disjoint pair of subsets in $\{1, \dots, n\}$. Then $\mathcal{J}(\mathbf{B}, \mathbf{E})$ is adjacency-avoiding.

Proof. Let $K, L \in \mathcal{J}(\mathbf{B}, \mathbf{E})$. We want to prove that they are not adjacent. If they intersect, we are done. Otherwise, without loss of generality, assume that $b(L) > e(K)$. By definition, $b(L) \in \mathbf{B}$ and $e(K) \in \mathbf{E}$. We know that $\mathbf{B} \cap \mathbf{E}[1] = \emptyset$. This means that $e(K) + 1 \notin \mathbf{B}$ and so $b(L) \geq e(K) + 2$. Therefore, K and L are not adjacent. $\square$

Definition 6.7. A pair of subsets $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$ is a *shifted interval bipartition* if $\mathbf{B} \cup \mathbf{E}[1] \in \mathcal{I}_{n+1} \cup \{\emptyset\}$, and $\mathbf{B} \cap \mathbf{E}[1] = \emptyset$. Moreover, such a pair is said to be *complete* if $\mathbf{B} \cup \mathbf{E}[1] = \{1, \dots, n+1\}$.

Remark 6.8. Some remarks:

- Note that if $(\mathbf{B}, \mathbf{E})$ is a pair of subsets of $\{1, \dots, n\}$ such that $\mathbf{B} \cup \mathbf{E}[1] = \{1, \dots, n+1\}$, then $(\mathbf{B}, \mathbf{E})$ is effective since necessarily $1 \in \mathbf{B}$ and $n \in \mathbf{E}$.
- Define rev the *reverse map* on $\{1, \dots, n\}$, by $\text{rev}(i) = n+1-i$. Write $\mathbf{A}^{\text{rev}} = \text{rev}(\mathbf{A})$ for all $\mathbf{A} \subseteq \{1, \dots, n\}$. Now if $(\mathbf{B}, \mathbf{E})$ is an effective shifted interval bipartition, then $(\mathbf{E}^{\text{rev}}, \mathbf{B}^{\text{rev}})$ is too.

We will show that the complete shifted bipartitions describe all maximal (with respect to inclusion) adjacency-avoiding interval sets.

Lemma 6.9. Let $(\mathbf{B}, \mathbf{E})$ be a complete shifted bipartition of $\{1, \dots, n\}$. Then for all $K \in \mathcal{I}_n \setminus \mathcal{J}(\mathbf{B}, \mathbf{E})$, either $b(K) - 1 \in \mathbf{E}$ or $e(K) + 1 \in \mathbf{B}$.

Proof. Let $K \in \mathcal{I}_n \setminus \mathcal{J}(\mathbf{B}, \mathbf{E})$. We have either $b(K) \notin \mathbf{B}$ or $e(K) \notin \mathbf{E}$. We know that $\mathbf{B} \cup \mathbf{E}[1] = \{1, \dots, n+1\}$ and $\mathbf{B} \cap \mathbf{E}[1] = \emptyset$. If $b(K) \notin \mathbf{B}$ then $b(K) \in \mathbf{E}[1]$ and therefore $b(K) - 1 \in \mathbf{E}$. Otherwise $e(K) \notin \mathbf{E}$ and then $e(K) + 1 \in \mathbf{B}$. $\square$

Proposition 6.10. Let $(\mathbf{B}, \mathbf{E})$ be a complete shifted bipartition of $\{1, \dots, n\}$. Then $\mathcal{J}(\mathbf{B}, \mathbf{E})$ is a maximal (for inclusion) adjacency-avoiding subset of $\mathcal{I}_n$.

Proof. By Proposition 6.6, we already know that $\mathcal{J} = \mathcal{J}(\mathbf{B}, \mathbf{E})$ is adjacency-avoiding.

Let $K \in \mathcal{I}_n \setminus \mathcal{J}$. Then either $b(K) - 1 \in \mathbf{E}$ or $e(K) + 1 \in \mathbf{B}$ by Lemma 6.9. In the first case, by taking $\llbracket 1, b(K) - 1 \rrbracket$ or, in the second case, by taking $\llbracket e(K) + 1, n \rrbracket$, we conclude that $\mathcal{J} \cup \{K\}$ is not adjacency-avoiding. $\square$

Lemma 6.11. Consider $\mathcal{J}$ a maximal (for inclusion) adjacency-avoiding subset of intervals of $\mathcal{I}_n$. Then:

- $\llbracket 1; n \rrbracket \in \mathcal{J}$;
- if $K, L \in \mathcal{J}$ with $b(K) \leq e(L)$, then $\llbracket b(K), e(L) \rrbracket \in \mathcal{J}$;
- if $K, L \in \mathcal{J}$ and if $K \cap L \neq \emptyset$, then $K \cap L \in \mathcal{J}$;
- if $K \in \mathcal{J}$, then there exists $m \in K$ such that $\llbracket m \rrbracket \in \mathcal{J}$.

Proof. Let $\mathcal{J}$ be as assumed.

- No interval $K \in \mathcal{I}_n$ is adjacent to $\llbracket 1; n \rrbracket$. Hence $\mathcal{J} \cup \{\llbracket 1; n \rrbracket\}$ is adjacency-avoiding. By maximality of $\mathcal{J}$, $\llbracket 1; n \rrbracket \in \mathcal{J}$.
- Let $K, L \in \mathcal{J}$ such that $b(K) \leq e(L)$. There is no interval $T \in \mathcal{J}$ adjacent to $\llbracket b(K), e(L) \rrbracket$; otherwise, such a T would have to be adjacent to either K or L . Hence $\mathcal{J} \cup \{\llbracket b(K), e(L) \rrbracket\}$ is adjacency-avoiding. By maximality of $\mathcal{J}$, we have $\llbracket b(K), e(L) \rrbracket \in \mathcal{J}$.
- Let $K, L \in \mathcal{J}$ such that $K \cap L \neq \emptyset$. Without loss of generality, we may assume that $b(L) \leq e(K)$. Therefore $b(L) \leq b(K) \leq e(L) \leq e(K)$ and $K \cap L = \llbracket b(K), e(L) \rrbracket$. By (ii), we conclude that $K \cap L \in \mathcal{J}$.
- Let $K \in \mathcal{J}$. Let us consider a minimal interval $T \subseteq K$ such that $T \in \mathcal{J}$.

Let $U \in \mathcal{I}_n$ such that $\emptyset \neq U \subseteq T$. If there exists $L \in \mathcal{J}$ such that L is adjacent to U , then either L is adjacent to T , which is impossible by hypothesis on $\mathcal{J}$, or $L \cap T \neq \emptyset$. Using (iii), we get that $T \supseteq L \cap T \in \mathcal{J}$. By minimality of T in $\mathcal{J}$, we assert that $L \cap T = T$ and thus $L \cap U = U \neq \emptyset$, contradicting the fact that L and U are adjacent. Thus, for all $L \in \mathcal{J}$, L and U are not adjacent.

We obtain that $\mathcal{J} \cup \{U\}$ is adjacency-avoiding. By maximality of $\mathcal{J}$, we get that $T = U$. The only case where any nonempty subset of T is equal to U is when $b(T) = e(T)$. We deduce the desired result. $\square$

Proposition 6.12. *All the maximal adjacency-avoiding subsets of $\mathcal{I}_n$ can be written as $\mathcal{J}(\mathbf{B}, \mathbf{E})$ where $(\mathbf{B}, \mathbf{E})$ is a complete shifted bipartition of $\{1, \dots, n\}$.*

Proof. Let $\mathcal{J}$ be a maximal adjacency-avoiding subset of $\mathcal{I}_n$. By Lemma 6.11(i), we know that $\llbracket 1; n \rrbracket \in \mathcal{J}$. Applying Lemma 6.11(iv) to $\llbracket 1; n \rrbracket$, we get that there exists $m \in \llbracket 1; n \rrbracket$ such that $\llbracket m \rrbracket \in \mathcal{J}$. Knowing that there is at least one $m \in \{1, \dots, n\}$ such that $\llbracket m \rrbracket \in \mathcal{J}$, assume that there are $p \in \mathbb{N}^*$ of those. We order and denote them by $1 \leq m_1 < m_2 < \dots < m_p \leq n$. Note obviously that we cannot have $m_{s+1} = m_s + 1$ for any $s \in \{1, \dots, p-1\}$.

For $s \in \{1, \dots, p-1\}$, let a_s be the maximal index a such that $m_s \leq a < m_{s+1}$ and such that $\llbracket m_s; a \rrbracket \in \mathcal{J}$, and b_s be the minimal index b such that $m_s < b \leq m_{s+1}$ and such that $\llbracket b; m_{s+1} \rrbracket \in \mathcal{J}$.

We show that, for all $s \in \{1, \dots, p-1\}$, $b_s \geq a_s + 2$. By contradiction:

- if $b_s = a_s + 1$ then $\mathcal{J}$ is not adjacency-avoiding;
- if $b_s \leq a_s$, then $\llbracket b_s, a_s \rrbracket \in \mathcal{J}$ by Lemma 6.11(iii), and this implies by Lemma 6.11(iv) that we should have a $m' \in \llbracket b_s, a_s \rrbracket$ such that $\llbracket m' \rrbracket \in \mathcal{J}$. However, by construction, $m_s < m' < m_{s+1}$.

By taking

- $\mathbf{B} = \llbracket 1, m_1 \rrbracket \cup \llbracket a_1 + 2, m_2 \rrbracket \cup \dots \cup \llbracket a_{p-1} + 2, m_p \rrbracket$ and,
- $\mathbf{E} = \llbracket m_1, a_1 \rrbracket \cup \llbracket m_2, a_2 \rrbracket \cup \dots \cup \llbracket m_p, n \rrbracket$,

we can easily check that $(\mathbf{B}, \mathbf{E})$ is a complete shifted bipartition of $\{1, \dots, n\}$. Moreover, by construction of $\mathcal{J}(\mathbf{B}, \mathbf{E})$ and Lemma 6.11(ii), we can assert that $\mathcal{J}(\mathbf{B}, \mathbf{E}) \supseteq \mathcal{J}$. Thus $\mathcal{J}(\mathbf{B}, \mathbf{E}) = \mathcal{J}$ by Proposition 6.10. $\square$

6.2. Interval reflections.

Definition 6.13. Let $v \in \{1, \dots, n\}$. The *interval reflection at v* , denoted refl_v is a function on $\mathcal{I}_n \setminus \{\llbracket v \rrbracket\}$ defined as follows:

$$\forall K \in \mathcal{I}_n \setminus \{\llbracket v \rrbracket\}, \text{refl}_v(K) = \begin{cases} K \cup \{v\} & \text{if } v \notin K \text{ and } K \cup \{v\} \in \mathcal{I}_n, \\ K \setminus \{v\} & \text{if } v \in K \text{ and } K \setminus \{v\} \in \mathcal{I}_n, \\ K & \text{otherwise.} \end{cases}$$

Remark 6.14. The interval reflection at v is an involution on $\mathcal{I}_n \setminus \{\llbracket v \rrbracket\}$.

For all $\mathcal{J} \subseteq \mathcal{I}_n$, we denote $\text{refl}_v(\mathcal{J})$ the interval subset made of all the intervals $\text{refl}_v(K)$ for $K \in \mathcal{J} \setminus \{\llbracket v \rrbracket\}$. Here is a direct consequence of the definition of refl_v and Proposition 2.5.

Corollary 6.15. *Let Q be an A_n type quiver and $v \in Q_0$ be either a source or a sink of Q . Consider $\mathcal{J} \subset \mathcal{I}_n$. Then the reflection functor at v applied to $\text{Cat}_Q(\mathcal{J})$ yields $\text{Cat}_{\sigma_v(Q)}(\text{refl}_v(\mathcal{J}))$.*

The result below shows that the adjacency-avoiding property is stable under refl_v for all $v \in \{1, \dots, n\}$.

Proposition 6.16. *Let $\mathcal{J}$ be an adjacency-avoiding subset of $\mathcal{I}_n$. Then $\text{refl}_v(\mathcal{J})$ is adjacency-avoiding. Moreover, if $\llbracket v \rrbracket \notin \mathcal{J}$, then $\mathcal{J}$ is adjacency-avoiding if and only if $\text{refl}_v(\mathcal{J})$ is too.*

Proof. Let $\mathcal{J}$ be an adjacency-avoiding interval subset. Suppose that $\text{refl}_v(\mathcal{J})$ is not adjacency-avoiding. We thus have two adjacent intervals T and U in $\text{refl}_v(\mathcal{J})$. Let us say that $b(U) = e(T) + 1$ without loss of generality.

By definition, let $K, L \in \mathcal{J} \setminus \{\llbracket v \rrbracket\}$ such that $\text{refl}_v(K) = T$ and $\text{refl}_v(L) = U$. By involution, we get $K = \text{refl}_v(T)$ and $L = \text{refl}_v(U)$. Now,

- if $v \notin \{e(T), b(U)\}$, then $b(L) = b(U) = e(T) + 1 = e(K) + 1$;
- if $v = e(T)$, then $b(L) = b(U) - 1 = e(T) = e(K) + 1$;
- if $v = b(U)$, then $b(L) = b(U) + 1 = e(T) + 2 = e(K) + 1$.

In all cases, we find that K and L are adjacent, which is a contradiction. So $\text{refl}_v(\mathcal{J})$ is adjacency-avoiding.

If $\llbracket v \rrbracket \notin \mathcal{J}$, then $\text{refl}_v(\text{refl}_v(\mathcal{J})) = \mathcal{J}$ and we are done. $\square$

It seems natural to ask how the maximal adjacency-avoiding interval subsets behave under this action. The example below must motivate us to define an action on effective shifted interval bipartitions by showing that we can describe the image under refl_v in terms of another effective shifted interval bipartition.

Example 6.17. Let $n = 6$, $\mathbf{B} = \{2, 4\}$ and $\mathbf{E} = \{2, 4, 5\}$. We have $\mathcal{J} = \mathcal{J}(\mathbf{B}, \mathbf{E}) = \{\llbracket 2 \rrbracket, \llbracket 2, 4 \rrbracket, \llbracket 2, 5 \rrbracket, \llbracket 4 \rrbracket, \llbracket 4, 5 \rrbracket\}$. Then we get

$$\text{refl}_3(\mathcal{J}) = \{\llbracket 2, 3 \rrbracket, \llbracket 2, 4 \rrbracket, \llbracket 2, 5 \rrbracket, \llbracket 3, 4 \rrbracket, \llbracket 3, 5 \rrbracket\} = \mathcal{J}(\mathbf{B}', \mathbf{E}') \setminus \{\llbracket 3 \rrbracket\},$$

with $\mathbf{B}' = \{2, 3\} = (\mathbf{B} \cup \{3\}) \setminus \{4\}$ and $\mathbf{E}' = \{3, 4, 5\} = (\mathbf{E} \cup \{3\}) \setminus \{2\}$.

Before defining the *toggle action*, let us introduce the *completion of an effective shifted interval bipartition via extended shifted bipartitions*.

Definition 6.18. An *extended shifted bipartition* of $\{1, \dots, n\}$ is a pair $(\mathbf{C}, \mathbf{F})$ such that $\mathbf{C} \subseteq \{1, \dots, n+1\}$, $\mathbf{F} \subseteq \{0, \dots, n\}$ and $\{\mathbf{C}, \mathbf{F}[1]\}$ is a bipartition of $\{1, \dots, n+1\}$.

Proposition 6.19. *For any pair $(\mathbf{B}, \mathbf{E})$, with $\mathbf{B} \neq \emptyset \neq \mathbf{E}$, forming an interval shifted bipartition of $\{1, \dots, n\}$, there is a unique extended shifted bipartition $(\bar{\mathbf{B}}, \bar{\mathbf{E}})$ of $\{1, \dots, n\}$ such that $\mathcal{J}(\mathbf{B}, \mathbf{E}) = \mathcal{J}(\bar{\mathbf{B}}, \bar{\mathbf{E}})$. We call $(\bar{\mathbf{B}}, \bar{\mathbf{E}})$ the completion of $(\mathbf{B}, \mathbf{E})$.*

Remark 6.20. Note that if either $\mathbf{B} = \emptyset$ and $n \notin \mathbf{E}$, or $\mathbf{E} = \emptyset$ and $1 \notin \mathbf{B}$, then the completion is not unique. For instance, $(\emptyset, \emptyset)$ admits $n+2$ different completions. As this pair will remain important throughout this section, we introduce notation for its completions. Let, for $0 \leq m \leq n+1$, $\bar{\emptyset}_m = (\{m+1, \dots, n+1\}, \{0, \dots, m-1\})$ be the m^{th} completion of $(\emptyset, \emptyset)$.

Proof of Proposition 6.19. Let $(\mathbf{B}, \mathbf{E})$ as assumed. Then let:

- $\bar{\mathbf{B}} = \mathbf{B} \cup \{i \mid i > \max(\mathbf{B} \cup \mathbf{E}[1])\}$;
- $\bar{\mathbf{E}} = \mathbf{E} \cup \{i-1 \mid i < \min(\mathbf{B} \cup \mathbf{E}[1])\}$.

It is easy to check that $(\bar{\mathbf{B}}, \bar{\mathbf{E}})$ is an extended shifted bipartition of $\{1, \dots, n\}$ such that $\mathcal{J}(\mathbf{B}, \mathbf{E}) = \mathcal{J}(\bar{\mathbf{B}}, \bar{\mathbf{E}})$. It is also clear that this extended shifted bipartition is the unique one satisfying the desired properties. $\square$

Corollary 6.21. *The map $(\mathbf{B}, \mathbf{E}) \mapsto (\bar{\mathbf{B}}, \bar{\mathbf{E}})$ gives a bijection from pairs $(\mathbf{B}, \mathbf{E})$ of nonempty subsets of $\{1, \dots, n\}$ forming effective shifted interval bipartitions, and pairs $(\mathbf{C}, \mathbf{F})$ forming extended shifted bipartitions of $\{1, \dots, n\}$ such that $\mathcal{J}(\mathbf{C}, \mathbf{F}) \neq \emptyset$.*

Proof. This result is a direct consequence of Lemma 6.4 and Proposition 6.19. $\square$

For any pair $(\mathbf{C}, \mathbf{F})$ such that $\mathbf{C} \subseteq \{1, \dots, n+1\}$ and $\mathbf{F} \subseteq \{0, \dots, n\}$, denote by $\text{eff}(\mathbf{C}, \mathbf{F})$ the unique effective shifted interval bipartition of $\{1, \dots, n\}$ such that

$$\mathcal{J}(\mathbf{C}, \mathbf{F}) = \mathcal{J}(\text{eff}(\mathbf{C}, \mathbf{F})).$$

In the next example, we extend our observations from Example 6.17 to all other reflections we can perform. We will see in Proposition 6.29 that this is exactly the case.

Example 6.22. Let $n = 6$, $\mathbf{B} = \{2, 4\}$ and $\mathbf{E} = \{2, 4, 5\}$. So $\bar{\mathbf{B}} = \{2, 4, 7\}$ and $\bar{\mathbf{E}} = \{0, 2, 4, 5\}$. Let $\mathcal{J} = \mathcal{J}(\mathbf{B}, \mathbf{E}) = \{\llbracket 2 \rrbracket, \llbracket 2, 4 \rrbracket, \llbracket 2, 5 \rrbracket, \llbracket 4 \rrbracket, \llbracket 4, 5 \rrbracket\}$. Then,

- $\text{refl}_1(\mathcal{J}) = \{\llbracket 1, 2 \rrbracket, \llbracket 1, 4 \rrbracket, \llbracket 1, 5 \rrbracket, \llbracket 4 \rrbracket, \llbracket 4, 5 \rrbracket\} = \mathcal{J}(\mathbf{B}', \mathbf{E}')$ with:
 - $\bar{\mathbf{B}}' = \{1, 4, 7\} = (\bar{\mathbf{B}} \cup \{1\}) \setminus \{2\}$,
 - $\bar{\mathbf{E}}' = \{1, 2, 4, 5\} = (\bar{\mathbf{E}} \cup \{1\}) \setminus \{0\}$;
- $\text{refl}_2(\mathcal{J}) = \{\llbracket 3, 4 \rrbracket, \llbracket 3, 5 \rrbracket, \llbracket 4 \rrbracket, \llbracket 4, 5 \rrbracket\} = \mathcal{J}(\mathbf{B}', \mathbf{E}')$ with:
 - $\bar{\mathbf{B}}' = \{3, 4, 7\} = (\bar{\mathbf{B}} \cup \{3\}) \setminus \{2\}$,
 - $\bar{\mathbf{E}}' = \{0, 1, 4, 5\} = (\bar{\mathbf{E}} \cup \{1\}) \setminus \{2\}$;
- $\text{refl}_3(\mathcal{J}) = \{\llbracket 2, 3 \rrbracket, \llbracket 2, 4 \rrbracket, \llbracket 2, 5 \rrbracket, \llbracket 3, 4 \rrbracket, \llbracket 3, 5 \rrbracket\} = \mathcal{J}(\mathbf{B}', \mathbf{E}')$ with:
 - $\bar{\mathbf{B}}' = \{2, 3, 7\} = (\bar{\mathbf{B}} \cup \{3\}) \setminus \{4\}$,
 - $\bar{\mathbf{E}}' = \{0, 3, 4, 5\} = (\bar{\mathbf{E}} \cup \{3\}) \setminus \{2\}$.

Definition 6.23. Let $(\mathbf{C}, \mathbf{F})$ be an extended shifted bipartition of $\{1, \dots, n\}$. Let $v \in \{1, \dots, n\}$. We define $\text{tog}_v(\mathbf{C}, \mathbf{F})$ the toggle at v of $(\mathbf{C}, \mathbf{F})$ as the pair $(\mathbf{C}', \mathbf{F}')$ where:

$$\bullet \mathbf{C}' = \begin{cases} (\mathbf{C} \cup \{v\}) \setminus \{v+1\} & \text{if } v \notin \mathbf{C} \cup \mathbf{F}, \\ (\mathbf{C} \cup \{v+1\}) \setminus \{v\} & \text{if } v \in \mathbf{C} \cap \mathbf{F}, \\ \mathbf{C} & \text{otherwise;} \end{cases}$$

$$\bullet \mathbf{F}' = \begin{cases} (\mathbf{F} \cup \{v\}) \setminus \{v-1\} & \text{if } v \notin \mathbf{C} \cup \mathbf{F}, \\ (\mathbf{F} \cup \{v-1\}) \setminus \{v\} & \text{if } v \in \mathbf{C} \cap \mathbf{F}, \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

Remark 6.24. Note that, by construction, $\text{rev}(\text{tog}_v(\mathbf{C}, \mathbf{F})) = \text{tog}_{n+1-v}(\mathbf{F}^{\text{rev}}, \mathbf{C}^{\text{rev}})$.

The following interpretation of the application of tog_v follows directly from Definition 6.23.

Lemma 6.25. Let $(\mathbf{C}, \mathbf{F})$ be an extended shifted bipartition of $\{1, \dots, n\}$. Consider $(\mathbf{C}', \mathbf{F}') = \text{tog}_v(\mathbf{C}, \mathbf{F})$. Then $\mathbf{C}'$ is defined from $\mathbf{C}$ by replacing v by $v-1$ and by replacing $v-1$ by v , and $\mathbf{F}'$ is defined from $\mathbf{F}$ by replacing v by $v+1$ and by replacing $v+1$ by v .

The following proposition follows immediately from Lemma 6.25.

Proposition 6.26. Let $(\mathbf{C}, \mathbf{F})$ be an extended shifted bipartition of $\{1, \dots, n\}$. For all $v \in \{1, \dots, n\}$, $\text{tog}_v(\mathbf{C}, \mathbf{F})$ is also an extended shifted bipartition of $\{1, \dots, n\}$.

Thus, the toggles on the extended shifted bipartitions of $\{1, \dots, n\}$ induce toggle operations on the effective shifted interval bipartitions.

Definition 6.27. Let $(\mathbf{B}, \mathbf{E})$ be an effective shifted interval bipartition of $\{1, \dots, n\}$, and $v \in \{1, \dots, v\}$. We define $\text{tog}_v(\mathbf{B}, \mathbf{E})$ the toggle at v of $(\mathbf{B}, \mathbf{E})$ by:

$$\text{tog}_v(\mathbf{B}, \mathbf{E}) = \begin{cases} \text{eff}(\text{tog}_v(\bar{\varnothing}_v^n)) & \text{if } (\mathbf{B}, \mathbf{E}) = (\varnothing, \varnothing); \\ \text{eff}(\text{tog}_v(\text{compl}_n(\mathbf{B}, \mathbf{E}))) & \text{otherwise.} \end{cases}$$

Remark 6.28. We can describe tog_v explicitly on effective shifted interval bipartitions of $\{1, \dots, n\}$: we have $\text{tog}_v(\mathbf{B}, \mathbf{E}) = (\mathbf{B}', \mathbf{E}')$ where:

$$\bullet \mathbf{B}' = \begin{cases} (\mathbf{B} \cup \{v\}) \setminus \{v+1\} & \text{if } v \notin \mathbf{B} \cup \mathbf{E} \text{ and } v+1 \in \mathbf{B}, \\ \mathbf{B} \cup \{v\} & \text{if } v \notin \mathbf{B} \cup \mathbf{E}, v+1 \notin \mathbf{B} \text{ and either } v-1 \in \mathbf{E}, \\ & \text{or } \mathbf{B} = \emptyset, \\ (\mathbf{B} \cup \{v+1\}) \setminus \{v\} & \text{if } v \in \mathbf{B} \cap \mathbf{E} \text{ and either } v+1 \in \mathbf{E}, \\ & \text{or } v+2 \in \mathbf{B}, \\ \mathbf{B} \setminus \{v\} & \text{if } v \in \mathbf{B} \cap \mathbf{E}, v+1 \notin \mathbf{E} \text{ and } v+2 \notin \mathbf{B}, \\ \mathbf{B} & \text{otherwise;} \end{cases}$$

$$\bullet \mathbf{E}' = \begin{cases} (\mathbf{E} \cup \{v\}) \setminus \{v-1\} & \text{if } v \notin \mathbf{B} \cup \mathbf{E} \text{ and } v-1 \in \mathbf{E}, \\ \mathbf{E} \cup \{v\} & \text{if } v \notin \mathbf{B} \cup \mathbf{E}, v-1 \notin \mathbf{E} \text{ and either } v+1 \in \mathbf{B}, \\ & \text{or } \mathbf{E} = \emptyset, \\ (\mathbf{E} \cup \{v-1\}) \setminus \{v\} & \text{if } v \in \mathbf{B} \cap \mathbf{E} \text{ and either } v-1 \in \mathbf{B}, \\ & \text{or } v-2 \in \mathbf{E}, \\ \mathbf{E} \setminus \{v\} & \text{if } v \in \mathbf{B} \cap \mathbf{E}, v-1 \notin \mathbf{B} \text{ and } v-2 \notin \mathbf{E}, \\ \mathbf{E} & \text{otherwise.} \end{cases}$$

However, Definition 6.27 is easier to handle than this explicit description.

Proposition 6.29. Let $(\mathbf{C}, \mathbf{F})$ be an extended shifted bipartition of $\{1, \dots, n\}$, and $v \in \{1, \dots, n\}$. Write $\mathcal{J} = \mathcal{J}(\mathbf{C}, \mathbf{F})$ and $\mathcal{T} = \mathcal{J}(\text{tog}_v(\mathbf{C}, \mathbf{F}))$. We get the following results:

- (1) if $v \notin \mathbf{C} \cup \mathbf{F}$, then $\mathcal{T} = \text{refl}_v(\mathcal{J}) \cup \{\llbracket v \rrbracket\}$.
- (2) otherwise, $\mathcal{T} = \text{refl}_v(\mathcal{J})$.

Proof. Let $(\mathbf{C}, \mathbf{F})$ and v be as assumed. Write $(\mathbf{C}', \mathbf{F}') = \text{tog}_v(\mathbf{C}, \mathbf{F})$. It is obvious that $\llbracket v \rrbracket \in \mathcal{J}(\mathbf{C}', \mathbf{F}')$ if and only if $\mathbf{C} \cap \{v, v+1\} = \{v+1\}$ and $\mathbf{F} \cap \{v, v-1\} = \{v-1\}$, which is equivalent to $v \notin \mathbf{C} \cup \mathbf{F}$ as claimed.

Let $K' \in \mathcal{I}_n$. By definition, $K' \in \mathcal{J}(\mathbf{C}', \mathbf{F}')$ whenever $b(K') \in \mathbf{C}'$ and $e(K') \in \mathbf{F}'$. As we already treated this case, assume that $K' \neq \llbracket v \rrbracket$. By Lemma 6.25, the conditions $b(K') \in \mathbf{C}'$ and $e(K') \in \mathbf{F}'$ is equivalent to saying that $K \in \mathcal{J}(\mathbf{C}, \mathbf{F})$ and $K' = \text{refl}_v(K)$. This completes the proof. $\square$

Corollary 6.30. Let $(\mathbf{B}, \mathbf{E})$ be an effective shifted interval partition of $\{1, \dots, n\}$ and $v \in \{1, \dots, n\}$. Write $\mathcal{J} = \mathcal{J}(\mathbf{B}, \mathbf{E})$ and $\mathcal{T} = \mathcal{J}(\text{tog}_v(\mathbf{B}, \mathbf{E}))$. We get the following results:

- (1) if $v \notin \mathbf{B} \cup \mathbf{E}$, and, $v-1 \in \mathbf{E}$ or $v+1 \in \mathbf{B}$, then $\mathcal{T} = \text{refl}_v(\mathcal{J}) \cup \{\llbracket v \rrbracket\}$.
- (2) otherwise, $\mathcal{T} = \text{refl}_v(\mathcal{J})$.

Proof. This results from Corollary 6.21, Definition 6.27, and Proposition 6.29. $\square$

Remark 6.31. Note that the condition $v-1 \in \mathbf{E}$ or $v+1 \in \mathbf{B}$ makes sure that there exists an interval in $\mathcal{J}(\mathbf{B}, \mathbf{E})$ adjacent to $\llbracket v \rrbracket$, and so $\mathcal{J}(\mathbf{B}, \mathbf{E})$ will not be fixed under refl_v .

Corollary 6.32. For any effective shifted interval bipartitions $(\mathbf{B}, \mathbf{E})$, there exists a unique effective shifted interval bipartition $(\mathbf{B}', \mathbf{E}')$ such that either $\mathcal{J}(\mathbf{B}, \mathbf{E}) = \text{refl}_v(\mathcal{J}(\mathbf{B}', \mathbf{E}'))$ if $v \notin \mathbf{B} \cap \mathbf{E}$ or $\mathcal{J}(\mathbf{B}, \mathbf{E}) = \text{refl}_v(\mathcal{J}(\mathbf{B}', \mathbf{E}')) \cup \{\llbracket v \rrbracket\}$ otherwise.

Proof. We use the fact that refl_v is an involution on interval subsets which do not contain $\llbracket v \rrbracket$ and Corollary 6.30. Therefore $(\mathbf{B}', \mathbf{E}') = \text{tog}_v(\mathbf{B}, \mathbf{E})$. $\square$

Remark 6.33. A similar statement can be made for extended shifted bipartitions of $\{1, \dots, n\}$. In what follows, we will focus only on the effective shifted-interval bipartitions.

7. PROOF OF THE MAIN RESULT

In this section, we prove Theorem 1.4. To do so, we will prove that for any complete shifted bipartition $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$, and any quiver Q of A_n type, $\text{Cat}_Q(\mathcal{J}(\mathbf{B}, \mathbf{E}))$ is canonically Jordan recoverable. We first prove this claim for Q linearly oriented and then generalize it to any A_n type quiver.

7.1. The linearly oriented case. We introduce an algorithm as a sequence of operations seen in Section 5 which builds the category $\mathcal{C}_{\overrightarrow{A_n}}(\mathbf{B}, \mathbf{E}) = \text{Cat}_{\overrightarrow{A_n}}(\mathcal{J}(\mathbf{B}, \mathbf{E}))$ for any given complete shifted bipartition $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$. Thanks to it, we will prove that this category is canonically Jordan recoverable.

Algorithm 7.1 (Let $n \geq 1$ and $Q = \overrightarrow{A_n}$).

- (0) **Input:** a complete shifted bipartition $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$.
- (1) Set:
 - $Q_{1,0} = \overrightarrow{A_n}$;
 - $\mathcal{C}^{1,0} = \{0\} \subseteq \text{rep}(Q_{1,0})$;
 - $\mathcal{D}^{1,0} = \{0\} \subseteq \text{rep}(Q_{1,0})$.
- (2) For $i \in \{1, \dots, n+1\}$:
 - (a) For $j \in \{0, \dots, n-i\}$, we put
 - $Q_{i,j+1} = \sigma_{n-j}(Q_{i,j})$;
 - $\mathcal{D}^{i,j+1} = \mathcal{R}_{n-j}^-(\mathcal{C}^{i,j})$;
 and,
 - $\mathcal{C}^{i,j+1} = \begin{cases} \text{AddS}_{n-j}(\mathcal{D}^{i,j+1}) & \text{if } i \in \mathbf{B} \text{ and } i+j \in \mathbf{E}, \\ \mathcal{D}^{i,j+1} & \text{otherwise.} \end{cases}$
 - (b) Set:
 - $Q_{i+1,0} = Q_{i,n-i+1}$;
 and,
 - $\mathcal{C}^{i+1,0} = \mathcal{C}^{i,n-i+1}$.
- (3) **Output:** $Q_{n+1,0}$ and $\mathcal{C}^{n+1,0} \subseteq \text{rep}(Q_{n+1,0})$.

In the following, we set $\mathcal{C}^{i,j} = \mathcal{C}_{Q_{i,j}}^{i,j}(\mathbf{B}, \mathbf{E})$ and $\mathcal{D}^{i,j} = \mathcal{D}_{Q_{i,j}}^{i,j}(\mathbf{B}, \mathbf{E})$ for $1 \leq i \leq n+1$ and $0 \leq j \leq n-i+1$.

We can rephrase Algorithm 7.1 by giving a similar algorithm for interval sets corresponding to $\mathcal{C}^{i,j}$ and $\mathcal{D}^{i,j}$.

Algorithm 7.2 (Let $n \geq 1$ and $Q = \overrightarrow{A_n}$).

- (0) **Input:** a complete shifted bipartition $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$.
- (1) Set
 - $\mathcal{J}^{1,0} = \emptyset$;
 and
 - $\mathcal{T}^{1,0} = \emptyset$.

- (2) For $i \in \{1, \dots, n+1\}$:
- (a) For $j \in \{0, \dots, n-i\}$, we put
- $\mathcal{T}^{i,j+1} = \text{refl}_{n-j}(\mathcal{J}^{i,j})$;
- and
- $\mathcal{J}^{i,j+1} = \begin{cases} \mathcal{T}^{i,j+1} \cup \{\llbracket n-j \rrbracket\} & \text{if } i \in \mathbf{B} \text{ and } i+j \in \mathbf{E}, \\ \mathcal{T}^{i,j+1} & \text{otherwise.} \end{cases}$
- (b) Set $\mathcal{J}^{i+1,0} = \mathcal{J}^{i,n-i+1}$.
- (3) **Output:** $\mathcal{J}^{n+1,0} \subseteq \mathcal{I}_n$.

In the following, we set $\mathcal{J}^{i,j} = \mathcal{J}^{i,j}(\mathbf{B}, \mathbf{E})$ and $\mathcal{T}^{i,j} = \mathcal{T}^{i,j}(\mathbf{B}, \mathbf{E})$ for $1 \leq i \leq n+1$ and $0 \leq j \leq n-i+1$.

By construction, we have that $\text{Cat}_{Q_{i,j}}(\mathcal{J}^{i,j}) = \mathcal{C}^{i,j}$ and $\text{Cat}_{Q_{i,j}}(\mathcal{T}^{i,j}) = \mathcal{D}^{i,j}$.

Proposition 7.3. *The Algorithm 7.1 returns $\mathcal{C}^{n+1,0} = \mathcal{C}_{A_n}^{\rightarrow}(\mathbf{B}, \mathbf{E})$.*

Proof. First, we can check, by a simple induction, that the quiver $Q_{p,0}$, for $p \geq 2$, is as follows.

$$1 \longrightarrow 2 \longrightarrow \dots \longrightarrow p-1 \longleftarrow p \longleftarrow \dots \longleftarrow n$$

Thus $Q_{n+1,0} = \overrightarrow{A_n}$ as claimed. As there is a bijection from subcategories of any given A_n type quiver Q to interval subsets, we can work with the sequence $\mathcal{J}^{i,j}$ and prove that $\mathcal{J}^{n+1,0} = \mathcal{J}(\mathbf{B}, \mathbf{E})$. It will imply the result we wished for.

To do so, we will first study the evolution of $\llbracket n-j \rrbracket$ appearing in $\mathcal{J}^{i,j+1}$ following the algorithm until arriving at $\mathcal{J}^{n+1,0}$. Following the sequence of reflections from $\mathcal{J}^{i,j+1}$ to $\mathcal{J}^{i+1,0} = \mathcal{J}^{i,n-i+1}$ we get

$$\text{refl}_i \circ \text{refl}_{i+1} \circ \dots \circ \text{refl}_{n-j-1}(\llbracket n-j \rrbracket) = \llbracket i, n-j \rrbracket.$$

In the sequence of reflections applied to get $\mathcal{J}^{i+2,0}$ from $\mathcal{J}^{i+1,0}$, the only reflection that affects $\llbracket i, n-j \rrbracket$ is refl_{n-j+1} . So $\llbracket n-j \rrbracket$ in $\mathcal{J}^{i,j+1}$ corresponds to $\llbracket i, n-j+1 \rrbracket$ in $\mathcal{J}^{i+2,0}$.

By the same argument, we get that $\llbracket n-j \rrbracket$ in $\mathcal{J}^{i,j+1}$ corresponds to $\llbracket i, n \rrbracket$ in $\mathcal{J}^{i+j+1,0}$. Here, following the sequence of reflections applied to get $\mathcal{J}^{i+j+2,0} = \mathcal{J}^{i+j+1,n-i-j+2}$ from $\mathcal{J}^{i+j+1,0}$, the interval $\llbracket i, n \rrbracket$ becomes $\llbracket i, i+j \rrbracket$ in $\mathcal{J}^{i+j+2,0}$. As the remainder of the sequence of reflections we still have to apply to get $\mathcal{J}^{n+1,0}$ from $\mathcal{J}^{i+j+2,0}$ does not affect anymore $\llbracket i, i+j \rrbracket$, because they only touch the vertices $q \geq i+j+2$, we conclude that $\llbracket n-j \rrbracket$ in $\mathcal{J}^{i,j+1}$ corresponds to $\llbracket i, i+j \rrbracket \in \mathcal{J}^{n+1,0}$.

To end the proof, we only have to notice that during the construction, we add in our interval subset $\llbracket n-j \rrbracket \in \mathcal{J}^{i,j+1}$ if and only if we have $i \in \mathbf{B}$ and $i+j \in \mathbf{E}$. $\square$

Theorem 7.4. *Let $n \geq 1$. For all effective shifted interval bipartitions $(\mathbf{B}, \mathbf{E})$ of the set $\{1, \dots, n\}$, $\mathcal{C}_{A_n}^{\rightarrow}(\mathbf{B}, \mathbf{E})$ is canonically Jordan recoverable. Moreover, for all $X \in \mathcal{C}_{A_n}^{\rightarrow}(\mathbf{B}, \mathbf{E})$:*

- (a) $\text{GenJF}(X)$ is $(\boxplus, \boxplus)$ -storable at q for $q \notin \mathbf{B} \cup \mathbf{E}$;
- (b) $\text{GenJF}(X)$ is $(\boxplus, \boxminus)$ -storable at q for $q \in \mathbf{E} \setminus \mathbf{B}$;
- (c) $\text{GenJF}(X)$ is $(\boxminus, \boxplus)$ -storable at q for $q \in \mathbf{B} \setminus \mathbf{E}$;
- (d) $\text{GenJF}(X)$ is $(\boxminus, \boxminus)$ -storable at q for $q \in \mathbf{B} \cap \mathbf{E}$;
- (e) $\text{GenJF}(X)^q = (0)$ whenever $q < \min(\mathbf{B})$ or $q > \max(\mathbf{E})$.

Proof of Theorem 7.4. Assume first that $(\mathbf{B}, \mathbf{E})$ is a complete shifted bipartition of $\{1, \dots, n\}$. In this case, we show the claimed result by an induction proof of the following claim, for all $i \in \{1, \dots, n+1\}$ and $j \in \{0, \dots, n-i+1\}$:

- $(\mathbf{H}_{i,j})$ The category $\mathcal{C}^{i,j}$ is canonically Jordan recoverable and for all $X \in \mathcal{C}^{i,j}$:
- (a) $\text{GenJF}(X)$ is $(\boxplus, \boxplus)$ -storable at q for $q \in \{1, \dots, n\}$ such that either:
 - (1) $q < i-1$ and $q \notin \mathbf{B} \cup \mathbf{E}$, or;
 - (2) $q = n-j$ if $i \in \mathbf{B}$ and $n-j \neq i-1$.
 - (b) $\text{GenJF}(X)$ is $(\boxplus, \boxminus)$ -storable at q for $q \in \{1, \dots, n\}$ such that either:
 - (1) $q < i-1$ and $q \in \mathbf{E} \setminus \mathbf{B}$, or;
 - (2) $q = i-1$, if $i-1 \notin \mathbf{B}$, or;
 - (3) $q > i-1$ and $q \notin \{n-j, n-j+1\}$, or;
 - (4) $q \in \{n-j, n-j+1\}$ if $i \notin \mathbf{B}$.
 - (c) $\text{GenJF}(X)$ is $(\boxminus, \boxplus)$ -storable at q for $q \in \{1, \dots, n\}$ such that $q < i-1$ and $q \in \mathbf{B} \setminus \mathbf{E}$;
 - (d) $\text{GenJF}(X)$ is $(\boxminus, \boxminus)$ -storable at q for $q \in \{1, \dots, i-1\}$ such that either:
 - (1) $q < i-1$ and $q \in \mathbf{B} \cap \mathbf{E}$, or;
 - (2) $q = i-1$, if $i-1 \in \mathbf{B}$, or;
 - (3) $q = n-j+1$ if $i \in \mathbf{B}$.
 - (e) $\text{GenJF}(X)^q = \text{GenJF}(X)^{q+1}$ for $q \in \{0, \dots, n\}$ such that either:
 - (1) $n-q+i \leq n-j$ and $n-q+i \in \mathbf{B}$, or;
 - (2) $n-q+i+1 > n-j$ and $n-q+i+1 \in \mathbf{B}$.

Recall that we extended the generic Jordan form data of any $X \in \text{rep}(Q_{i,j})$ by writing $\text{GenJF}(X)^0 = \text{GenJF}(X)^{n+1} = (0)$.

Note, by Proposition 7.3, that the claim $(\mathbf{H}_{n+1,0})$ corresponds to the wished-for result.

For $i = 1$ and $j = 0$, $\mathcal{C}^{1,0} = \{0\}$ is canonically Jordan recoverable, and $\text{GenJF}(0) = ((0))_{1 \leq q \leq n}$ satisfies all the storability conditions and equalities we ask for.

Now assume that for a fixed $i \in \{1, \dots, n+1\}$, and $j \in \{0, \dots, n-i+1\}$, $\mathcal{C}^{i,j}$ satisfies $(\mathbf{H}_{i,j})$.

We will show that either $\mathcal{C}^{i,j+1}$ satisfies $(\mathbf{H}_{i,j+1})$ if $j < n-i+1$, or $\mathcal{C}^{i+1,0}$ satisfies $(\mathbf{H}_{i+1,0})$ otherwise. We can already say that the only vertices q where the storability conditions change are $q = n-j-1$, $q = n-j$, and $q = n-j+1$.

We have several cases to treat:

- The case $j = n-i+1$ is trivial by the fact that $\mathcal{C}^{i,n-i+1} = \mathcal{C}^{i+1,0}$ by Algorithm 7.1. We only need to check that $(i, n-i+1)$ and $(i+1, 0)$ yield the same storability conditions.
- Assume that $i \notin \mathbf{B}$ and $j < n-i+1$. Following Algorithm 7.1, to get $\mathcal{C}^{i,j+1}$ from $\mathcal{C}^{i,j}$, we have to apply the reflection functor $\mathcal{R}_{n-j}^-$. By $(\mathbf{H}_{i,j})$, we know that $\mathcal{C}^{i,j}$ is canonically Jordan recoverable, and, for all $X \in \mathcal{C}^{i,j}$, $\text{GenJF}(X)$ is $(\boxplus, \boxminus)$ -storable at $n-j$. As a consequence of Proposition 5.5, $\mathcal{C}^{i,j+1} = \mathcal{R}_{n-j}^-(\mathcal{C}^{i,j})$ is canonically Jordan recoverable. Moreover, by Lemma 4.9, and Lemma 4.5(3), for all $X \in \mathcal{C}^{i,j}$, $\text{GenJF}(\mathcal{R}_{n-j}^-(X))$ is $(\boxplus, \boxplus)$ -storable at $n-j$. Thus, the storability conditions satisfied by representations in $\mathcal{C}^{i,j}$ and those satisfied by representations in $\mathcal{C}^{i,j+1}$ are the same. If $\text{GenJF}(X)^{n-j} = \text{GenJF}(X)^{n-j-1}$, then $\text{GenJF}(\mathcal{R}_{n-j}^-(X))^{n-j} = \text{GenJF}(\mathcal{R}_{n-j}^-(X))^{n-j+1}$ by Lemma 4.5(1).
- Assume that $i \in \mathbf{B}$, $j < n-i+1$ and $i+j \notin \mathbf{E}$. To go from $\mathcal{C}^{i,j}$ to $\mathcal{C}^{i,j+1}$, we only have to apply $\mathcal{R}_{n-j}^-$. By induction, we have that $\text{GenJF}(X)$ is $(\boxplus, \boxplus)$ -storable at

$n - j$ for all $X \in \mathcal{C}^{i,j}$. Then, by Proposition 5.5, we get that $\mathcal{C}^{i,j+1}$ is canonically Jordan recoverable and, by Lemma 4.9, $\text{GenJF}(\mathcal{R}_{n-j}^-(X))$ is strongly $(\boxplus, \boxminus)$ -storable at $n - j$. We can also check easily that $\text{GenJF}(\mathcal{R}_{n-j}^-(X))$ is $(\boxplus, \boxplus)$ -storable at $n - j - 1$ and $(\boxplus, \boxminus)$ -storable at $n - j + 1$ if $n - j - 1 > i - 1$ by the diagonal transformation at $n - j$. We can also remark that if $\text{GenJF}(X)^{n-j} = \text{GenJF}(X)^{n-j-1}$, then $\text{GenJF}(\mathcal{R}_{n-j}^-(X))^{n-j} = \text{GenJF}(\mathcal{R}_{n-j}^-(X))^{n-j+1}$ by Lemma 4.5 (1).

- Assume that $i \in \mathbf{B}$, $j < n - i + 1$ and $i + j \in \mathbf{E}$. To go from $\mathcal{C}^{i,j}$ to $\mathcal{C}^{i,j+1}$, we need to use $\mathcal{R}_{n-j}^-$ followed by AddS_{n-j} . By induction, we have that $\text{GenJF}(X)$ is $(\boxplus, \boxplus)$ -storable at $n - j$ for all $X \in \mathcal{C}^{i,j}$. By Proposition 5.5, $\mathcal{D}^{i,j+1} = \mathcal{R}_{n-j}^-(\mathcal{C}^{i,j})$ is canonically Jordan recoverable and, by Lemma 4.9, $\text{GenJF}(\mathcal{R}_{n-j}^-(X))$ strongly $(\boxplus, \boxminus)$ -storable at $n - j$. By Proposition 5.3, $\mathcal{C}^{i,j+1} = \text{AddS}_{n-j}(\mathcal{D}^{i,j+1})$ is also canonically Jordan recoverable, and by Lemma 5.1, $\text{GenJF}(Z)$ is $(\boxplus, \boxplus)$ -storable at $n - j$ for all $Z \in \mathcal{C}^{i,j+1}$. We also have that $\text{GenJF}(Z)$ is $(\boxplus, \boxplus)$ -storable at $n - j - 1$, if $n - j - 1 > i - 1$ and $(\boxplus, \boxminus)$ -storable at $n - j + 1$.

This completes the induction proof in this case.

Therefore $(\mathbf{H}_{i,j})$ is true for all $i \in \{1, \dots, n + 1\}$ and $j \in \{0, \dots, n - i + 1\}$, and we get the wished-for result for all complete shifted bipartition $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$.

Now assume that $(\mathbf{B}, \mathbf{E})$ is an arbitrary effective shifted interval bipartition of $\{1, \dots, n\}$. We can see the category $\mathcal{C}_{A_n}^{\rightarrow}(\mathbf{B}, \mathbf{E})$ has a subcategory in $\mathcal{C}_{A_m}^{\rightarrow}(\mathbf{B}^\circ, \mathbf{E}^\circ)$ where $m = \max(\mathbf{E}) - \min(\mathbf{B}) + 1$, $\mathbf{B}^\circ = \{i - \min(\mathbf{B}) + 1 \mid i \in \mathbf{B}\}$ and $\mathbf{E}^\circ = \{j - \min(\mathbf{B}) + 1 \mid j \in \mathbf{E}\}$. As $(\mathbf{B}^\circ, \mathbf{E}^\circ)$ is a complete shifted bipartition of $\{1, \dots, m\}$, we get that $\mathcal{C}_{A_n}^{\rightarrow}(\mathbf{B}, \mathbf{E})$ is canonically Jordan recoverable and the storable condition of $\text{GenJF}(X)$ holds for all $X \in \mathcal{C}_{A_n}^{\rightarrow}(\mathbf{B}, \mathbf{E})$. Note that, for such a representation X , it is obvious that $\text{GenJF}(X)^q = (0)$ for $q \notin \{\min(\mathbf{B}), \dots, \max(\mathbf{E})\}$.

This completes the proof of Theorem 7.4. $\square$

7.2. For other orientations. In this subsection, we show that Theorem 1.4 holds for all quivers of A_n type. To do so, we will first prove a result similar to Theorem 7.4 available for any A_n type quiver and then use it to give the final proof.

Theorem 7.5. *Let $n > 0$ and Q be an A_n type quiver. For all effective shifted interval bipartitions $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$, $\mathcal{C}_Q(\mathbf{B}, \mathbf{E})$ is canonically Jordan recoverable. Moreover, for all $X \in \mathcal{C}_Q(\mathbf{B}, \mathbf{E})$:*

- $\text{GenJF}(X)$ is $(\boxplus, \boxplus)$ -storable at q for $q \notin \mathbf{B} \cup \mathbf{E}$;
- $\text{GenJF}(X)$ is $(\boxplus, \boxminus)$ -storable at q for $q \in \mathbf{E} \setminus \mathbf{B}$;
- $\text{GenJF}(X)$ is $(\boxminus, \boxplus)$ -storable at q for $q \in \mathbf{B} \setminus \mathbf{E}$;
- $\text{GenJF}(X)$ is $(\boxminus, \boxminus)$ -storable at q for $q \in \mathbf{B} \cap \mathbf{E}$;
- $\text{GenJF}(X)^q = (0)$ whenever $q < \min(\mathbf{B})$ or $q > \max(\mathbf{E})$.

Proof. Note that we can go from $\overrightarrow{A_n}$ to any A_n type quiver by a sequence of mutations only done at sources. Using that fact, we will prove our desired result by induction.

First, we know that for $Q = \overrightarrow{A_n}$, the claim is valid by Theorem 7.4.

Assume now that for a fixed A_n type quiver Q , the same is true. Let v be a source of Q and put $\Xi = \sigma_v(Q)$. We will prove that the same goes for Ξ .

Let $(\mathbf{B}, \mathbf{E})$ be an effective shifted interval bipartition of $\{1, \dots, n\}$. We want to prove that $\mathcal{C}_\Xi(\mathbf{B}, \mathbf{E})$ is canonically Jordan recoverable and for all $X \in \mathcal{C}_\Xi(\mathbf{B}, \mathbf{E})$, $\text{GenJF}(X)$ satisfies the announced storability conditions. By Corollary 6.32, we know that there

exists $(\mathbf{B}', \mathbf{E}') = \text{tog}_v(\mathbf{B}, \mathbf{E})$ an effective shifted interval bipartition of $\{1, \dots, n\}$ such that either $\mathcal{J}(\mathbf{B}, \mathbf{E}) = \text{refl}_v(\mathcal{J}(\mathbf{B}', \mathbf{E}'))$ if $v \notin \mathbf{B} \cap \mathbf{E}$, or $\mathcal{J}(\mathbf{B}, \mathbf{E}) = \text{refl}_v(\mathcal{J}(\mathbf{B}', \mathbf{E}')) \cup \{\llbracket v \rrbracket\}$ otherwise.

- If $v \notin \mathbf{B} \cap \mathbf{E}$, then $\mathcal{C}_{\Xi}(\mathbf{B}, \mathbf{E}) = \mathcal{R}_v^-(\mathcal{C}_Q(\mathbf{B}', \mathbf{E}'))$. By induction, we know that $\mathcal{C}_Q(\mathbf{B}', \mathbf{E}')$ is canonically Jordan recoverable, and for all $X \in \mathcal{C}_Q(\mathbf{B}', \mathbf{E}')$, we have that $\text{GenJF}(X)$ is either $(\boxplus, \boxminus)$ -storable, $(\boxminus, \boxplus)$ -storable, or $(\boxminus, \boxminus)$ -storable at v .

In the first two cases, we conclude by Proposition 5.3 and Lemma 4.7.

In the last case, this means $v \in \mathbf{B}' \cap \mathbf{E}'$. By the induction hypothesis, for all $Y \in \mathcal{C}_Q(\mathbf{B}', \mathbf{E}')$, $\text{GenJF}(Y)$ is $(\boxminus, \boxminus)$ -storable at v . By considering $\mathcal{D}_Q^v(\mathbf{B}', \mathbf{E}')$ the subcategory of $\mathcal{C}_Q(\mathbf{B}', \mathbf{E}')$ generated by modules without S_v in its summands, using Lemma 5.1, we obtain that, for all $Z \in \mathcal{D}_Q^v(\mathbf{B}', \mathbf{E}')$, $\text{GenJF}(Z)$ is strongly $(\boxminus, \boxminus)$ -storable at v . By Proposition 5.5, $\mathcal{C}_{\Xi}(\mathbf{B}, \mathbf{E}) = \mathcal{R}_v^-(\mathcal{D}_Q^v(\mathbf{B}', \mathbf{E}'))$ is canonically Jordan recoverable and, by Lemma 4.5(5), for all $X \in \mathcal{C}_{\Xi}(\mathbf{B}, \mathbf{E})$, $\text{GenJF}(X)$ is $(\boxplus, \boxplus)$ -storable at v .

The only other storability conditions that change from $\mathcal{C}_Q(\mathbf{B}', \mathbf{E}')$ to $\mathcal{C}_{\Xi}(\mathbf{B}, \mathbf{E})$ are at $v - 1$ and at $v + 1$. If $v - 1 \in \mathbf{E}' \setminus \mathbf{B}'$, then the $(\boxminus, \boxplus)$ -storability condition satisfied by $\text{GenJF}(Z)$, for $Z \in \mathcal{C}_Q(\mathbf{B}', \mathbf{E}')$, becomes a $(\boxminus, \boxminus)$ -storability condition satisfied by $\text{GenJF}(X)$ for $X \in \mathcal{C}_{\Xi}(\mathbf{B}, \mathbf{E})$. This corresponds with $v - 1 \in \mathbf{B} \cap \mathbf{E}$. We can treat similarly the case where we go from $(\boxplus, \boxplus)$ -storability condition to $(\boxplus, \boxminus)$ -storability condition at $v - 1$, if $v - 1 \in \mathbf{B}' \cap \mathbf{E}'$. A similar, symmetric argument yields the same result at $v + 1$, considering the two possible changes in storability conditions.

This completes the proof of the induction step in this case.

- If $v \in \mathbf{B} \cap \mathbf{E}$, $\mathcal{C}_{\Xi}(\mathbf{B}, \mathbf{E}) = \text{AddS}_v(\mathcal{R}_v^-(\mathcal{C}_Q(\mathbf{B}', \mathbf{E}')))$. Using the definition of $(\mathbf{B}', \mathbf{E}')$, we have that $v \notin \mathbf{B}' \cup \mathbf{E}'$ and $v - 1 \in \mathbf{E}'$ or $v + 1 \in \mathbf{B}'$. Therefore, the induction hypothesis allows us to state that for all $X \in \mathcal{C}_Q(\mathbf{B}', \mathbf{E}')$, $\text{GenJF}(X)$ is $(\boxplus, \boxplus)$ -storable at k .

Following Theorem 4.11 and Lemma 4.5, we get that $\mathcal{R}_v^-(\mathcal{C}_Q(\mathbf{B}', \mathbf{E}'))$ is canonically Jordan recoverable and for any representation Y in this category, $\text{GenJF}(Y)$ is strongly $(\boxminus, \boxminus)$ -storable at v . Hence by Proposition 5.3, we get that $\mathcal{C}_{\Xi}(\mathbf{B}, \mathbf{E})$ is canonically Jordan recoverable and for any representation Z in it we have that $\text{GenJF}(Z)$ is $(\boxminus, \boxminus)$ -storable at v .

By analogous arguments to those given in the previous point, we can deduce that the storability conditions satisfied by $\text{GenJF}(X)$ for all $X \in \mathcal{C}_{\Xi}(\mathbf{B}, \mathbf{E})$ are the ones we claimed.

This completes the proof of the induction step in this case

Thus, we have proved the induction step, and so we have proved the wished-for result. $\square$

We can now prove the main result of this paper.

Proof of Theorem 1.4. Fix Q a quiver of A_n type. Let $\mathcal{C}$ be a subcategory of $\text{rep}(Q)$. By Proposition 1.2, we already know that if $\mathcal{C}$ is canonically Jordan recoverable, then $\text{Int}(\mathcal{C})$ is adjacency-avoiding.

Now assume that $\text{Int}(\mathcal{C})$ is adjacency-avoiding. Using Proposition 6.12, there exists a complete shifted bipartition $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$ such that $\text{Int}(\mathcal{C}) \subset \mathcal{J}(\mathbf{B}, \mathbf{E})$, and $\mathcal{C}$ is therefore a subcategory of $\mathcal{C}_Q(\mathbf{B}, \mathbf{E})$. By Theorem 7.5, we know that $\mathcal{C}_Q(\mathbf{B}, \mathbf{E})$ is canonically Jordan recoverable. Thus, so is $\mathcal{C}$. $\square$

8. TO GO FURTHER

We could ask ourselves some questions based on this work.

Question 8.1. *Can we translate the adjacency-avoiding property for intervals into another algebraic property for subcategories of $\text{rep}(Q)$ for any A_n type quiver Q ?*

First, the following simple lemma allows us to translate the adjacency property into the algebraic world.

Lemma 8.2. *Let Q be an A_n type quiver. Let $K, L \in \mathcal{I}_n$. Then K and L are adjacent if and only if there is a short exact sequence with end terms X_K and X_L in some order and an indecomposable middle term.*

The following theorem follows as a direct consequence of Theorem 1.4 and Lemma 8.2.

Theorem 8.3. *Let Q be a quiver of A_n type. Fix a collection of indecomposable representations $\mathcal{X} \subseteq \text{Ind}(Q)$. Then $\text{add}(\mathcal{X})$ is canonically Jordan recoverable if and only if for all $X, Y \in \mathcal{X}$ and for all short exact sequences*

$$0 \longrightarrow X \longrightarrow E \longrightarrow Y \longrightarrow 0,$$

the representation E is not indecomposable.

Thus, for any $T \in \text{rep}(Q)$ a tilting representation, $\text{add}(T)$ is canonically Jordan recoverable. But the interaction between canonically Jordan recoverable subcategories and tilting representations does not end here. We formulate an exact statement as the next conjecture.

Let Q be an A_n type quiver, and T a tilting representation of Q . Write $T = T_1 \oplus \cdots \oplus T_n$ for the decomposition of T . For all $i \in \{1, \dots, n\}$, we define the *mutation of T by T_i* , denoted $\mu_{T_i}(T)$, to be, if it is possible, the unique tilting representation (up to isomorphism) isomorphic to $T_1 \oplus \cdots \oplus T'_i \oplus \cdots \oplus T_n$ such that $T'_i \not\cong T_i$, otherwise $\mu_{T_i}(T) = T$. Riedtmann and Schofield proved that T'_i can be obtained from $U_i = T_1 \oplus \cdots \oplus T_{i-1} \oplus T_{i+1} \oplus T_n$ and T_i as either the kernel or the cokernel of a minimal $\text{add}(U_i)$ -approximation (see [17]).

Following this result, in our case, we can divide the non-trivial mutations into two kinds:

- The 1-term mutations in the two following cases:
 - there exists $i \neq k \in \{1, \dots, n\}$ such that $T_i \rightarrow T_k$ is a minimal left $\text{add}(U_i)$ -approximation, and $T'_i = \text{Coker}(T_i \rightarrow T_k)$;
 - there exists $i \neq k \in \{1, \dots, n\}$ such that $T_k \rightarrow T_i$ is a minimal right $\text{add}(U_i)$ -approximation, and $T'_i = \text{Ker}(T_k \rightarrow T_i)$.
- The 2-terms mutations in the two following cases:
 - there exist $\ell, k \in \{1, \dots, n\}$ such that $\ell \neq k, \ell \neq i \neq k$ and $T_i \rightarrow T_k \oplus T_\ell$ is a minimal left $\text{add}(U_i)$ -approximation, and $T'_i = \text{Coker}(T_i \rightarrow T_k \oplus T_\ell)$.
 - there exist $\ell, k \in \{1, \dots, n\}$ such that $\ell \neq k, \ell \neq i \neq k$ and $T_k \oplus T_\ell \rightarrow T_i$ is a minimal right $\text{add}(U_i)$ -approximation, and $T'_i = \text{Ker}(T_k \oplus T_\ell \rightarrow T_i)$.

Note that T_i can admit both a left and a right $\text{add}(U_i)$ -approximation, but only one of them defines the summand T'_i used to construct $\mu_{T_i}(T)$.

Now we can state our conjecture.

Conjecture 8.4 (Let Q be an A_n type quiver).

- (a) For any tilting representation $T \in \text{rep}(Q)$, there exists a unique maximal canonically Jordan recoverable subcategory $\mathcal{C}$ such that $T \in \mathcal{C}$;

- (b) For any maximal canonically Jordan recoverable subcategory $\mathcal{C}$ of $\text{rep}(Q)$, and for any tilting representation $T \in \mathcal{C}$, $\mathcal{C}$ is additively generated by indecomposable summands of tilting representations that can be obtained by a (finite) sequence of 2-term mutations from T .

This result could open the way to algebraically characterizing the canonically Jordan recoverable subcategories for at least Dynkin quivers.

Remark 8.5. In Theorem 8.3, we can see a kind of complementarity with the notion of *maximal almost rigid* modules [2]: they are defined as modules $M = \bigoplus_{i=1}^s M(i)$ where $M(i) \in \text{Ind}(Q)$ give a maximal collection of indecomposable representations such that for all $1 \leq i, j \leq s$ and all nonsplit short exact sequence

$$0 \longrightarrow M(i) \longrightarrow E \longrightarrow M(j) \longrightarrow 0,$$

the representation E is indecomposable.

This conjecture is attacked using exact structure interpretations in [8], offering, by the way, a deeper interaction between combinatorial and categorical tools.

Question 8.6. *Can we hope to characterize the Jordan recoverable categories of $\text{rep}(Q)$ for Q of A_n type?*

Let $\mathcal{J} \subset \mathcal{I}_n$ and $L \in \mathcal{I}_n$. A $\mathcal{J}$ -partition of L is a partition $\{T_1, \dots, T_p\}$ of L such that for all $i \in \{1, \dots, p\}$, we have $T_i \in \mathcal{J}$.

Proposition 8.7. *Let $\mathcal{J} \subseteq \mathcal{I}_n$ such that there exists an interval $L \in \mathcal{I}_n$ admitting two distinct $\mathcal{J}$ -partitions. Then $\text{Cat}_Q(\mathcal{J})$ is not a Jordan recoverable category of $\text{rep}(Q)$.*

Proof. This result is a consequence of Lemma 1.3. □

As with the adjacency-avoiding interval subsets and the canonical Jordan recoverable categories, the following notion seems to play a significant role in determining all Jordan recoverable categories.

Definition 8.8. Let $\mathcal{J} \subset \mathcal{I}_n$. We say that $\mathcal{J}$ is *double interval partition-avoiding* if any $L \in \mathcal{I}_n$ admits at most one $\mathcal{J}$ -partition.

Example 8.9. For all $n \in \mathbb{N}^*$, $\mathcal{J} = \{\llbracket i \rrbracket \mid i \in \{1, \dots, n\}\}$ is double interval partition-avoiding. Note also that this is a maximal one (with respect to inclusion).

Conjecture 8.10. Let Q be an A_n type quiver. A subcategory $\mathcal{C} \subset \text{rep}(Q)$ is Jordan recoverable if and only if $\text{Int}(\mathcal{C})$ is double interval partition-avoiding.

We hope to prove this result in the near future.

Question 8.11. *Can we hope to extend the definition of the Greene–Kleitman invariant on representations of string quivers?*

Let us define a string quiver.

Definition 8.12. A *string quiver* is a pair (Q, R) where Q is a finite connected quiver and R is a set of monomial relations of degree 2 such that:

- all the vertices in Q admit at most two ingoing arrows and at most two outgoing arrows;
- for any $\alpha, \beta, \gamma \in Q_1$ such that $\beta\alpha$ and $\gamma\alpha$ are paths of Q , then

$$\{\beta\alpha, \gamma\alpha\} \cap R \neq \emptyset;$$

- for any $\alpha, \beta, \gamma \in Q_1$ such that $\alpha\beta$ and $\alpha\gamma$ are paths of Q , then

$$\{\alpha\beta, \alpha\gamma\} \cap R \neq \emptyset.$$

A representation of a string quiver (Q, R) is a representation $X \in \text{rep}(Q)$ such that, for any $\alpha, \beta \in Q_1$ with $\beta\alpha \in R$, we have $X_\beta X_\alpha = 0$. We denote by $\text{rep}(Q, R)$ the finite-dimensional representations of (Q, R) . A *string algebra* is a quotient algebra $\mathbb{K}Q/I$ where (Q, R) is a string quiver and $I = \langle R \rangle$. Recall that $\text{rep}(Q, R)$ is equivalent to the category of finitely generated (left) $\mathbb{K}Q/I$ -modules.

Note that, at least, if the Auslander–Reiten quiver of a string quiver (Q, R) is acyclic, then we can define a similar Greene–Kleitman invariant. We can first ask how much we can extend this invariant in a larger case than the one we explore in this article.

Garver, Patrias, and Thomas proved that we can define generic Jordan form data for any finite-dimensional module of any algebra [13]. We can therefore ask in which circumstances the two invariants coincide.

Question 8.13. *May we expect to extend Theorem 1.4 for gentle, locally gentle, or string algebras?*

Recall that the idea of considering adjacency-avoiding interval subsets comes from previous work [5] for gentle algebras. It seems reasonable to extend this result to gentle and even string algebras.

The reader is invited to explore these problems and related issues.

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