Annals of Representation Theory

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Volume 3, issue 2 (2026), p. 299-334

https://doi.org/10.5802/art.42

Communicated by Volodymyr Mazorchuk.

Annals of Representation Theory is published by the Norwegian University of Science and Technology and is a member of the Centre Mersenne for Open Scientific Publishing

e-ISSN: 2704-2081

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ABSTRACT. We study a family of categories of $\mathfrak{gl}(\infty)$-modules depending on the choice of a Levy-type subalgebra. More precisely for a fixed choice of Cartan, Borel and Levi-type subalgebras $\mathfrak{h}, \mathfrak{b}$ and $\mathfrak{l}$ with $\mathfrak{l} \cong \mathfrak{gl}(\infty)^n$ for some $n \in \mathbb{N}$, we define $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$ to be the category of $\mathfrak{h}$-semisimple, $\mathfrak{n}$-torsion modules that satisfy a certain "large annihilator condition" when seen as $\mathfrak{l}$-modules.

We see these categories as various analogues of the BGG category $\mathcal{O}$ of $\mathfrak{gl}(n, \mathbb{C})$. Our main result is that the category $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$ is a highest weight category in the sense of Cline, Parshall and Scott. We compute the simple multiplicities of standard objects and the standard multiplicities in injective objects explicitly, prove a version of BGG reciprocity, and present the irreducible blocks of $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$.

1. Introduction

The algebra $\mathfrak{gl}(\infty)$ is the most basic example of a complex locally-finite Lie algebra, i.e. a limit of finite-dimensional complex Lie algebras. Its representation theory has been a very active area of study for the last ten years, see for example [7, 12, 13, 18, 19, 20]. Beyond its intrinsic interest the representation theory of $\mathfrak{gl}(\infty)$ has connections to two other areas of Lie theory: the stable representation theory of the family $\mathfrak{gl}(d, \mathbb{C})$, and the classical representation theory of the Lie superalgebra $\mathfrak{gl}(n|m)$.

The relation between representations of $\mathfrak{gl}(\infty)$ and $\mathfrak{gl}(n|m)$ goes back to Brundan [1], who discovered a categorical action of the quantized enveloping algebra of $\mathfrak{gl}(\infty)$ on category $\mathcal{O}_{\mathfrak{gl}(n|m)}$. It follows that the complexified Grothendieck group of this category is a $\mathfrak{gl}(\infty)$-module, and by studying its structure Brundan was able to compute the characters of atypical finite dimensional modules. This approach was expanded by Brundan and Stroppel in [3], and by Brundan, Losev and Webster in [2]. Also, Hoyt, Penkov and Serganova studied the $\mathfrak{gl}(\infty)$-module structure of the integral block $\mathcal{O}^{\mathbb{Z}}_{\mathfrak{gl}(n|m)}$ in [13];

Manuscript received 2025-06-02, revised 2026-02-24 and accepted 2026-07-13.
Keywords. Infinite dimensional Lie algebras, representations, $\mathfrak{gl}(\infty)$, large annihilator condition.
2020 *Mathematics Subject Classification.* 17B10, 17B65.

This paper was written while the author was a postdoctoral researcher at Jacobs University under the DFG grant 980/8-1. The author is a CONICET researcher and received funding from BID PICT 0099/2019.

this paper is one of the main inspirations for the present article. Recently Serganova has extended this work to a categorification of Fock-modules of $\mathfrak{sl}(\infty)$ through the Deligne categories $\mathcal{V}_t$ in [22].

Let us now turn to the connections of $\mathfrak{gl}(\infty)$ with stable representation theory. Informally this refers to the study of sequences of representations V_d of $\mathfrak{gl}(d,\mathbb{C})$, compatible in a suitable sense, as d goes to infinity. For example, set $W_d = \mathbb{C}^d$ and $V_d(p,q) = W_d^{\otimes p} \otimes (W_d^*)^{\otimes q}$. For each $d \geq 1$ the decomposition of this module into its simple components is a consequence of Schur–Weyl duality, and we can ask whether this decomposition is in some way compatible with the obvious inclusion maps $V_d(p,q) \hookrightarrow V_{d+1}(p,q)$. One way to solve this problem is to regard the limit $T^{p,q} = \varinjlim_d V_d(p,q)$ as a $\mathfrak{gl}(\infty)$-module and study its decomposition, as done by Penkov and Styrkas in [20]. A more categorical approach to this problem appeared independently in the articles by Sam and Snowden [21] and by Dan-Cohen, Penkov and Serganova in [7]. The former studies the category $\mathrm{Rep}(\mathsf{GL}(\infty))$ of stable algebraic representation theory of the family of groups $\mathsf{GL}_d(\mathbb{C})$, and the latter is focused on the category $\mathbb{T}^0_{\mathfrak{gl}(\infty)}$ of modules that arise as subquotients of the modules $T^{p,q}$. As mentioned in the introduction to [21], these categories are equivalent.

By [7], objects in the category $\mathbb{T}^0_{\mathfrak{gl}(\infty)}$ are the integrable finite length $\mathfrak{gl}(\infty)$-modules that satisfy the so-called large annihilator condition (LAC from now on). This category is analogous in many ways to the category of integrable finite dimensional representations of $\mathfrak{gl}(n,\mathbb{C})$, and constitutes the backbone of the representation theory of $\mathfrak{gl}(\infty)$. Since this category is well understood by now, it is natural to look for analogues of category $\mathcal{O}$ for $\mathfrak{gl}(\infty)$. Several alternatives have been proposed such as those by Nampaisarn [15], Coulembier and Penkov [5], and Penkov and Serganova [19].

The fact that there is no one obvious analogue of $\mathcal{O}$ is due to two reasons. First, Cartan and Borel subalgebras of $\mathfrak{gl}(\infty)$ behave in a much more complicated way than in the finite dimensional case. For instance, different choices of Borel subalgebras can produce non-equivalent categories of highest weight modules. The second problem is that the enveloping algebra of $\mathfrak{gl}(\infty)$ is not left-Noetherian, so its finitely generated modules do not form an abelian category. The first problem can be solved (or rather swept under the rug) by fixing once and for all some well-chosen Cartan and Borel subalgebras; this is the road followed in this paper. The second however does not have an obvious solution.

In [19] Penkov and Serganova introduced the LAC as an alternative "smallness" condition. They study the category $\mathcal{OLA}$ consisting of $\mathfrak{h}$-semisimple and $\mathfrak{n}$-nilpotent modules satisfying the LAC, and in particular show that it is a highest weight category. On the other hand, the modules arising from the categorifications found in [13, 22] only satisfy the LAC over a Levi-type subalgebra $\mathfrak{l} \subset \mathfrak{gl}(\infty)$. Given the importance of the work relating representations of $\mathfrak{gl}(\infty)$ to Lie superalgebras, this motivates the study of analogues of category $\mathcal{O}$ where the condition of being finitely generated is replaced by this weaker version of the LAC. Thus in this paper we study the category $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$ of $\mathfrak{h}$-semisimple, $\mathfrak{n}$-torsion modules satisfying the LAC with respect to such subalgebras $\mathfrak{l}$ (here $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$).

As mentioned above our main result is that $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$ is a highest weight category. Let us give some details on the result. We introduce the notion of eligible weights, which are precisely those that can appear in a module satisfying the LAC. Elligible weights are endowed with an interval-finite order that has maximal but not minimal elements. Simple objects of $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$ are precisely the simple highest weight modules whose highest weight is eligible. While $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$ does not contain Verma modules or dual Verma modules, we obtain standard objects by taking the largest submodule of a dual Verma module lying in

$\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$. These standard modules are quite big and have infinite length; we show that their simple multiplicities are given in terms of weight multiplicities of a (huge) representation of $\mathfrak{gl}(\infty)$. Finally, injective envelopes exist by a standard categorical argument, but it takes some work to show that they have finite standard filtrations, and that they satisfy a form of BGG reciprocity. The proof we give relates our category to a different analogue of category $\mathcal{O}$ for $\mathfrak{gl}(\infty)$, studied by Nampaisarn. We finish this overview by pointing out that $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{gl}(\infty)$ does not have enough projectives, and hence no costandard modules, and thus no obvious analogue for tilting modules.

The article is structured as follows. Section 2 contains some general notation. In Section 3 we introduce several presentations of $\mathfrak{gl}(\infty)$, each of which highlights some particular features and subalgebras. Section 4 deals with various matters related to representation theory of Lie algebras, and some specifics regarding $\mathfrak{gl}(\infty)$. In Section 5 we discuss some basic categories of representations of $\mathfrak{gl}(\infty)$ in its various incarnations. We begin our study of $\mathcal{O}_{\mathsf{LA}}$ in Section 6, where we classify its simple objects, prove some general categorical properties, and show that simple multiplicities of a general object can be computed in terms of simple multiplicities for category $\overline{\mathcal{O}}_{\mathfrak{s}}$. In Section 7 we prove the existence of standard modules and compute their simple multiplicities. This last result depends on a long technical computation given in an appendix to the section. Finally, in Section 8 we wrap up the proof that $\mathcal{O}_{\mathsf{LA}}$ is a highest weight category with an analysis of injective modules. We also prove an analogue of BGG reciprocity and give a decomposition of $\mathcal{O}_{\mathsf{LA}}$ into irreducible blocks.

The usual zoo of Cartan, Borel, parabolic, and Levi subalgebras, along with their nilpotent ideals and opposite subalgebras, is augmented in each case by objects peculiar to locally finite algebras, such as exhaustions and subalgebras spanned by finite-root spaces. To help keep track of these wild fauna most of these subalgebras are introduced at once in Section 3.6, along with a visual device to describe them.

2. Generalities

2.1. **Notation.** For each $r \in \mathbb{N}$ we set $[\![r]\!] = \{1, 2, \dots, r\}$. Throughout we denote by $\mathbb{Z}^\times$ the set of nonzero integers endowed with the following, not quite usual order

$$1 \prec 2 \prec 3 \prec \cdots \prec -3 \prec -2 \prec -1.$$

The symbol $\delta_{i,j}$ denotes the Kronecker delta.

We denote by Part the set of all partitions, and we identify each partition with its Young diagram. Given $\lambda, \mu, \nu \in \mathsf{Part}$ we will denote the corresponding Schur functor by $\mathbb{S}_\lambda$, and by $c^\nu_{\lambda,\mu}$ the corresponding Littlewood–Richardson coefficient.

Vector spaces and unadorned tensor products are always taken over $\mathbb{C}$ unless explicitly stated. Given a vector space V we denote its algebraic dual by V^*. Given $d \in \mathbb{Z}_{>0}$ we will denote by either V^d or dV the direct sum of d copies of V. We denote by $\mathbf{S}^d(V)$ the d^{th} symmetric power of V, by $\mathbf{S}^\bullet(V) = \bigoplus_{d \geq 0} \mathbf{S}^d(V)$ its symmetric algebra, and set $\mathbf{S}^{\leq d}(V) = \bigoplus_{e \leq d} \mathbf{S}^e(V)$. We will often use that given a second vector space W we have a natural isomorphism

$$\mathbf{S}^\bullet(V \otimes W) \cong \bigoplus_{\lambda \in \mathsf{Part}} \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W).$$

In particular if V, W are modules over a Lie algebra $\mathfrak{g}$ then this isomorphism is a $\mathfrak{g}$-module isomorphism.

2.2. **Locally finite Lie algebras.** A locally finite Lie algebra $\mathfrak{g}$ is one where any finite subset of elements is contained in a finite dimensional subalgebra. If $\mathfrak{g}$ is countable dimensional then this is equivalent to the existence of a chain of finite dimensional subalgebras $\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots$ such that $\mathfrak{g} = \bigcup_{r \geq 0} \mathfrak{g}_r$. Any such chain is called an *exhaustion* of $\mathfrak{g}$. The algebra $\mathfrak{g}$ is locally reductive, nilpotent, etc. if it has an exhaustion such that each $\mathfrak{g}_r$ is of the corresponding type. We say $\mathfrak{g}$ is locally root-reductive if each $\mathfrak{g}_r$ can be taken to be reductive, there is a fixed choice of Cartan subalgebras $\mathfrak{h}_r \subset \mathfrak{g}_r$ satisfying $\mathfrak{h}_r = \mathfrak{h}_{r+1} \cap \mathfrak{g}_r$, and for this choice of Cartan subalgebras the inclusion $\mathfrak{g}_r \subset \mathfrak{g}_{r+1}$ sends root spaces to root spaces.

We denote by $\mathfrak{gl}(\infty)$ the direct limit of the sequence of Lie algebras

$$\mathfrak{gl}(1, \mathbb{C}) \hookrightarrow \mathfrak{gl}(2, \mathbb{C}) \hookrightarrow \ldots \mathfrak{gl}(n, \mathbb{C}) \hookrightarrow \ldots$$

where each map is given by inclusion in the upper left corner. This restricts to inclusions of $\mathfrak{sl}(n, \mathbb{C})$ into $\mathfrak{sl}(n+1, \mathbb{C})$, and we denote the limit of these algebras by $\mathfrak{sl}(\infty)$, which is clearly a subalgebra of $\mathfrak{gl}(\infty)$. Both are locally finite and root-reductive Lie algebras.

Fix a locally finite, countable-dimensional Lie algebra $\mathfrak{g}$ and an exhaustion $\mathfrak{g}_r$. Suppose also that we have a sequence (M_r, j_r) where every M_r is a $\mathfrak{g}_r$-module (which makes every M_s with $s \geq r$ a $\mathfrak{g}_r$-module), and every $j_r : M_r \to M_{r+1}$ is a morphism of $\mathfrak{g}_r$-modules. The limit vector space $M = \varinjlim M_r$ has the structure of a $\mathfrak{g}_r$-module for each r, and these structures are compatible in the obvious sense, so M is a $\mathfrak{g}$-module. We refer to M_r as an exhaustion of M. Any $\mathfrak{g}$-module M has an exhaustion: if X is a generating set of M, we can take as M_r the $\mathfrak{g}_r$-submodule generated by X.

We will write $\mathfrak{g} = \mathfrak{b} \ni \mathfrak{a}$ if $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{a}$ is an ideal that is also a vector space complement for $\mathfrak{b}$. Suppose we have a locally finite Lie algebra $\mathfrak{g}$ with fixed splitting Cartan subalgebra $\mathfrak{h}$ (i.e. such that $\mathfrak{g}$ is a semisimple $\mathfrak{h}$-module through the adjoint action), and Borel (i.e. maximal locally solvable) subalgebra $\mathfrak{b}$ of the form $\mathfrak{h} \ni \mathfrak{n}$ for some subalgebra $\mathfrak{n}$. Given any weight $\lambda \in \mathfrak{h}^*$, we can define Verma modules as usual by $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$. We will denote the Verma module by $M_{\mathfrak{g}}(\lambda)$ or simply $M(\lambda)$ when $\mathfrak{g}$ is clear from the context. We will also denote by $L_{\mathfrak{g}}(\lambda) = L(\lambda)$ the corresponding unique simple quotient of $M_{\mathfrak{g}}(\lambda)$, which exists by the usual argument.

More generally, concepts such as Cartan subalgebras, root systems, Borel and parabolic subalgebras relative to these systems, weight modules, etc. are defined for locally finite Lie algebras. In the context of this paper these objects will behave as in the finite dimensional case, but there are some subtle differences which we will point out when relevant. For details we refer the reader to the monograph [17] and the references therein.

3. The many faces of $\mathfrak{gl}(\infty)$

In this section we will review some standard facts about the Lie algebra $\mathfrak{gl}(\infty)$ and introduce several different ways to describe it. While this amounts to choosing different exhaustions, we will take a different approach to these various "avatars" of $\mathfrak{gl}(\infty)$, which brings to the fore some non-obvious structures in this Lie algebra.

3.1. **The Lie algebra $\mathfrak{g}(\mathbb{V})$.** Let I be any infinite denumerable set and let $\mathbb{V} = \langle v_i \mid i \in I \rangle$ and $\mathbb{V}_* = \langle v^i \mid i \in I \rangle$ be two complex vector spaces with bases indexed by I. We endow these spaces with a perfect pairing $\operatorname{tr} : \mathbb{V}_* \otimes \mathbb{V} \to \mathbb{C}$ given by $v^i \otimes v_j \mapsto \delta_{i,j}$.

The vector space $\mathbb{V} \otimes \mathbb{V}_*$ is a nonunital associative algebra with product given by $v \otimes w \cdot v' \otimes w' = \operatorname{tr}(w \otimes v') v \otimes w'$ and extended by bilinearity. We denote the Lie algebra associated to this algebra by $\mathfrak{g}(\mathbb{V}, \mathbb{V}_*, \mathrm{tr})$, or simply by $\mathfrak{g}(\mathbb{V})$. For $i, j \in I$ we set $E_{i,j} = v_i \otimes v^j$.

Take for example $I = \mathbb{Z}_{>0}$ and for $r \in \mathbb{Z}_{>0}$ set $\mathfrak{g}_r = \langle E_{i,j} \mid i, j \in [\![r]\!]\rangle$. Then $\mathfrak{g}_r$ is a subalgebra of $\mathfrak{g}(\mathbb{V})$ isomorphic to $\mathfrak{gl}(r, \mathbb{C})$. The inclusions $\mathfrak{g}_r \subset \mathfrak{g}_{r+1}$ correspond to the injective Lie algebra morphisms $\mathfrak{gl}(r, \mathbb{C}) \hookrightarrow \mathfrak{gl}(r+1, \mathbb{C})$ given by embedding a $r \times r$ matrix into the upper left corner of a $r+1 \times r+1$ matrix with its last row and column filled with zeroes. Thus

$$\mathfrak{g}(\mathbb{V}) = \bigcup_{r \geq 1} \mathfrak{g}_r \cong \varinjlim \mathfrak{gl}(r, \mathbb{C}) \cong \mathfrak{gl}(\infty).$$

The choice of I does not change the isomorphism type of this algebra, so in general every algebra of the form $\mathfrak{g}(\mathbb{V}, \mathbb{V}_*, \mathrm{tr})$ is isomorphic to $\mathfrak{gl}(\infty)$.

Notice that $\mathbb{V}$ and $\mathbb{V}_*$ are $\mathfrak{g}(\mathbb{V})$-modules with $E_{i,j} v_k = \delta_{j,k} v_i$ and $E_{i,j} v^k = -\delta_{k,i} v^j$. A simple comparison shows that $\mathbb{V} = \varinjlim V_r$, where V_r is the natural representation of $\mathfrak{gl}(r, \mathbb{C})$ and the maps $V_r \to V_{r+1}$ are uniquely determined up to isomorphism by the fact that they are $\mathfrak{gl}(r, \mathbb{C})$-linear. In a similar fashion, $\mathbb{V}_* \cong \varinjlim V_r^*$.

3.2. **Cartan subalgebra and root decomposition.** The definition and study of Cartan subalgebras of $\mathfrak{gl}(\infty)$ is an interesting and subtle topic, which was taken up in [16]. General Cartan subalgebras can behave quite differently from Cartan subalgebras of finite dimensional reductive Lie algebras, for example they may not produce a root decomposition of $\mathfrak{gl}(\infty)$. We will sidestep this problem by choosing one particularly well-behaved Cartan subalgebra as "the" Cartan subalgebra of $\mathfrak{g}(\mathbb{V})$, namely the obvious maximal commutative subalgebra $\mathfrak{h}(\mathbb{V}) = \langle E_{i,i} \mid i \in I\rangle$, and freely borrow notions from the theory of finite root systems that work for this particular choice.

The action of the Cartan subalgebra $\mathfrak{h}(\mathbb{V})$ on $\mathfrak{g}(\mathbb{V})$ is semisimple. Denoting by ε_i the unique functional of $\mathfrak{h}(\mathbb{V})^*$ such that $\varepsilon_i(E_{j,j}) = \delta_{i,j}$, the root system of $\mathfrak{g}(\mathbb{V})$ with respect to $\mathfrak{h}(\mathbb{V})$ is

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid i, j \in I\}$$

with the component of weight $\varepsilon_i - \varepsilon_j$ spanned by $E_{i,j}$, and hence 1-dimensional. Also $\mathfrak{g}(\mathbb{V})_0 = \mathfrak{h}(\mathbb{V})$, which is of course infinite dimensional.

3.3. **Positive roots, simple roots and Borel subalgebras.** If we fix a total order $\prec$ on I (as we did implicitly when setting $I = \mathbb{Z}_{>0}$) we can see an element of $\mathfrak{g}(\mathbb{V})$ as an infinite matrix whose rows and columns are indexed by the set I. A total order also allows us to define a set of positive roots

$$\Delta_I^+ = \{\varepsilon_i - \varepsilon_j \mid i \prec j\}$$

and a corresponding Borel subalgebra $\mathfrak{b}_I = \mathfrak{h}(\mathbb{V}) \oplus \bigoplus_{\alpha \in \Delta_I^+} \mathfrak{g}(\mathbb{V})_\alpha$.

Again, Borel subalgebras of $\mathfrak{gl}(\infty)$ behave quite differently than Borel subalgebras of $\mathfrak{gl}(r, \mathbb{C})$, and their behaviour is tied to the order type of I. One striking difference is the following: simple roots can be defined as usual, namely as those positive roots which can not be written as the sum of two positive roots, but it may happen that the linear span of the set of simple roots span is a proper subspace of the linear span of all roots. In the extreme case $I = \mathbb{Q}$ there are *no* simple roots. In this article we will only consider a few well-behaved examples, the reader interested in the general theory of Borel subalgebras of $\mathfrak{gl}(\infty)$ can consult [6, 8, 10].

Up to order isomorphism, the only cases where the span of the simple roots coincides with the span of all roots are $I = \mathbb{Z}_{>0}, \mathbb{Z}_{<0}$ and $\mathbb{Z}$ with their usual orders. These are called the *Dynkin* cases, and the corresponding Borel subalgebras are said to be of Dynkin type. Taking $I = \mathbb{Z}^\times$ the simple roots are

$$\{\varepsilon_i - \varepsilon_{i+1} \mid i \in \mathbb{Z}^\times \setminus \{-1\}\}.$$

Notice the root $\varepsilon_1 - \varepsilon_{-1}$ is not in the span of the simple roots, so this is a non-Dynkin case. We get a basis of the root space by adding this extra root to the set of simple roots. A root is called *finite* if it is in the span of simple roots, otherwise it is called *infinite*. By definition we are in a Dynkin case if and only if every root is finite.

3.4. **Further subalgebras and visual representation.** For the rest of this article we will assume that the bases of $\mathbb{V}$ and $\mathbb{V}_*$ are indexed by $\mathbb{Z}^\times$. Thus elements of $\mathfrak{g}(\mathbb{V})$ can be seen as infinite matrices with rows and columns indexed by $\mathbb{Z}^\times$ and finitely many nonzero entries. The choice of $\mathbb{Z}^\times$ as index set means that it makes sense to speak of the k-to-last row or column of an infinite matrix $g \in \mathfrak{g}(\mathbb{V})$ for any $k \in \mathbb{Z}_{>0}$.

We denote by $\mathfrak{b}(\mathbb{V})$ the non-Dynkin Borel subalgebra corresponding to the choice $I = \mathbb{Z}^\times$. We also denote by $\mathfrak{n}(\mathbb{V})$ the commutator subalgebra $[\mathfrak{b}(\mathbb{V}), \mathfrak{b}(\mathbb{V})]$, so $\mathfrak{b}(\mathbb{V}) = \mathfrak{h}(\mathbb{V}) \ltimes \mathfrak{n}(\mathbb{V})$. Notice that $\mathfrak{n}(\mathbb{V})$ is not nilpotent but only locally nilpotent.

We now introduce some useful subalgebras of $\mathfrak{g}(\mathbb{V})$. Fix $r \in \mathbb{Z}_{>0}$ and set

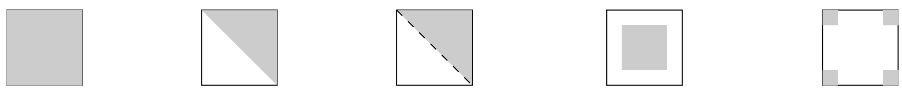
$$\mathfrak{g}(\mathbb{V})_r = \langle E_{i,j} \mid i, j \in \pm[\![r]\!]\rangle$$

This finite dimensional subalgebra of $\mathfrak{g}(\mathbb{V})$ is isomorphic to $\mathfrak{gl}(2r, \mathbb{C})$, and the $\mathfrak{g}(\mathbb{V})_r$ form an exhaustion of $\mathfrak{g}(\mathbb{V})$. We also set

$$\mathbb{V}[r] = \langle v_i \mid r+1 \preceq i \preceq -r-1\rangle \subset \mathbb{V}$$
$$\mathbb{V}_*[r] = \langle v^i \mid r+1 \preceq i \preceq -r-1\rangle \subset \mathbb{V}_*.$$

The map tr restricts to a non-degenerate pairing between these two subspaces and we write $\mathfrak{g}(\mathbb{V})[r] = \mathfrak{g}(\mathbb{V}[r], \mathbb{V}_*[r], \mathrm{tr})$. This Lie algebra is isomorphic to $\mathfrak{g}(\mathbb{V})$, and it is the centraliser of $\mathfrak{g}(\mathbb{V})_r$ inside $\mathfrak{g}(\mathbb{V})$.

We will meet many more subalgebras of $\mathfrak{g}(\mathbb{V})$, so we now introduce a visual aid to recall them. We will represent elements of $\mathfrak{g}(\mathbb{V})$ as squares, and we will represent a subalgebra by shading in gray the region where nonzero entries can be found, while unshaded areas will always be filled with zeroes. The following examples should help clarify this idea.



$\mathfrak{g}(\mathbb{V})$ $\qquad$ $\mathfrak{b}(\mathbb{V})$ $\qquad$ $\mathfrak{n}(\mathbb{V})$ $\qquad$ $\mathfrak{g}(\mathbb{V})[r]$ $\qquad$ $\mathfrak{g}(\mathbb{V})_r$

Notice that with these representations each corner of the square contains finitely many entries, while the centre concentrates infinitely many of them.

3.5. **The Lie algebra $\mathfrak{g}(\mathbb{V}^n)$.** Fix $n \in \mathbb{N}$. We denote by e_i the i^{th} vector of the canonical basis of $\mathbb{C}^n$, and by e^i the i^{th} vector in the dual canonical basis of $(\mathbb{C}^n)^*$. Set $\mathbb{V}^n = \mathbb{V} \otimes \mathbb{C}^n, \mathbb{V}^n_* = \mathbb{V}_* \otimes (\mathbb{C}^n)^*$ and set

$$\mathrm{tr}^n : \mathbb{V}^n_* \otimes \mathbb{V}^n \longrightarrow \mathbb{C}$$
$$(v^i \otimes e^k) \otimes (v_j \otimes e_l) \longmapsto \delta_{i,j}\delta_{k,l}.$$

We can form the Lie algebra $\mathfrak{g}(\mathbb{V}^n) = \mathfrak{g}(\mathbb{V}^n, \mathbb{V}_*^n, \mathrm{tr}^n)$ as before. Since $\mathbb{V}$ and $\mathbb{V}_*$ have bases indexed by $\mathbb{Z}^\times$, the vector spaces $\mathbb{V}^n$ and $\mathbb{V}_*^n$ have bases indexed by $\mathbb{Z}^\times \times [\![n]\!]$. Since the isomorphism type of $\mathfrak{g}(\mathbb{V})$ is independent of the indexing set as long as it is infinite and denumerable, any bijection between $\mathbb{Z}^\times$ and $\mathbb{Z}^\times \times [\![n]\!]$ will induce an isomorphism $\mathfrak{g}(\mathbb{V}^n) \cong \mathfrak{g}(\mathbb{V})$. Notice that under any such isomorphism $\mathfrak{h}(\mathbb{V}^n)$ is mapped to $\mathfrak{h}(\mathbb{V})$.

On the other hand there is an obvious isomorphism of vector spaces $\mathfrak{g}(\mathbb{V}^n) \cong \mathfrak{g}(\mathbb{V}) \otimes M_n(\mathbb{C})$. Thus we can see elements of $\mathfrak{g}(\mathbb{V}^n)$ as $n \times n$ block matrices, with each block an infinite matrix from $\mathfrak{g}(\mathbb{V})$. We set for each $k \in [\![n]\!], r \in \mathbb{Z}_{>0}$

$$\begin{aligned} \mathbb{V}^{(k)} &= \mathbb{V} \otimes \langle e_k \rangle, & \mathbb{V}_*^{(k)} &= \mathbb{V}_* \otimes \langle e^k \rangle; \\ \mathbb{V}^{(k)}[r] &= \mathbb{V}[r] \otimes \langle e_k \rangle, & \mathbb{V}_*^{(k)}[r] &= \mathbb{V}_*[r] \otimes \langle e^k \rangle. \end{aligned}$$

so $\mathfrak{g}(\mathbb{V}^n) = \bigoplus_{k,l \in [\![n]\!]} \mathbb{V}^{(k)} \otimes \mathbb{V}_*^{(l)}$.

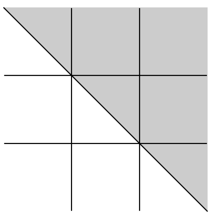
We highlight this particular avatar of $\mathfrak{gl}(\infty)$ since it reveals some internal structure which is not obvious in its usual presentation. The first example is the Borel subalgebra $\mathfrak{b}(\mathbb{V}^n)$, which is awkward to handle as a subalgebra of $\mathfrak{gl}(\infty)$ with the usual presentation. We also obtain a non-obvious exhaustion by setting $\mathfrak{g}_k(\mathbb{V}^n) = \mathfrak{g}_k(\mathbb{V}) \otimes M_n(\mathbb{C})$ for each $k \in \mathbb{Z}_{>0}$.

Another feature of $\mathfrak{gl}(\infty)$ which only comes to the fore when looking at $\mathfrak{g}(\mathbb{V}^n)$ is the existence of a $\mathbb{Z}^n$-grading, inherited from the weight grading of $M_n(\mathbb{C})$ as $\mathfrak{gl}(n, \mathbb{C})$-module. Thus for $k, l \in [\![n]\!]$ the direct summand $\mathbb{V}^{(k)} \otimes \mathbb{V}_*^{(l)}$ is contained in the homogeneous component of degree $e_k - e_l$. This grading highlights some interesting subalgebras of block diagonal and block upper-triangular matrices, namely

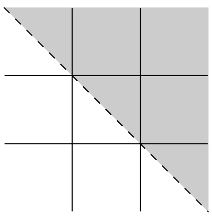
$$\mathfrak{l}(\mathbb{V}^n) = \bigoplus_{k=1}^{n} \mathbb{V}^{(k)} \otimes \mathbb{V}_*^{(k)}; \qquad \mathfrak{u}(\mathbb{V}^n) = \bigoplus_{k<l} \mathbb{V}^{(k)} \otimes \mathbb{V}_*^{(l)}; \qquad \mathfrak{p}(\mathbb{V}^n) = \bigoplus_{k \le l} \mathbb{V}^{(k)} \otimes \mathbb{V}_*^{(l)},$$

which are the subalgebras corresponding to 0, strictly positive, and non-negative $\mathfrak{gl}(n, \mathbb{C})$-weights, respectively. Notice the obvious Levi-type decomposition $\mathfrak{p} = \mathfrak{l} \ni \mathfrak{u}$, where $\mathfrak{l}$ is not reductive but rather locally reductive, and $\mathfrak{u}$ is $n-1$-step nilpotent.

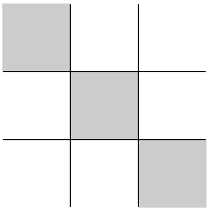
3.6. **Subalgebras of $\mathfrak{g}(\mathbb{V}^n)$ and their visual representation.** As before, we will use some visual aids to describe the various subalgebras of $\mathfrak{g}(\mathbb{V}^n)$. We will take $n = 3$ for these visual representations, and hence represent a subalgebra of $\mathfrak{g}(\mathbb{V}^n)$ by shading regions in a three-by-three grid of squares. We refer to these as the *pictures* of the subalgebras. The following examples should clarify the idea.



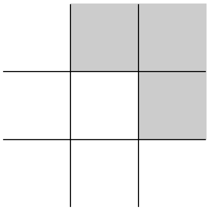
$\mathfrak{b}(\mathbb{V}^n)$



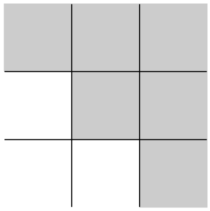
$\mathfrak{n}(\mathbb{V}^n)$



$\mathfrak{l}(\mathbb{V}^n)$



$\mathfrak{u}(\mathbb{V}^n)$

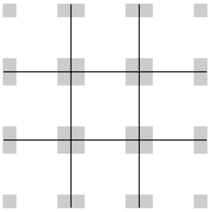


$\mathfrak{p}(\mathbb{V}^n) = \mathfrak{l}(\mathbb{V}^n) \ni \mathfrak{u}(\mathbb{V}^n)$

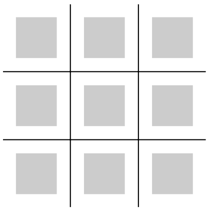
For each $r \in \mathbb{Z}_{>0}$ we set $\mathfrak{g}(\mathbb{V}^n)_r = \mathfrak{g}(\mathbb{V})_r \otimes M_n(\mathbb{C})$, and also

$$\mathfrak{g}(\mathbb{V}^n)[r] = \bigoplus_{k,l \in [\![r]\!]} \mathbb{V}^{(k)}[r] \otimes \mathbb{V}_*^{(l)}[r] \qquad \mathfrak{l}(\mathbb{V}^n)[r] = \bigoplus_{k \in [\![r]\!]} \mathbb{V}^{(k)}[r] \otimes \mathbb{V}_*^{(k)}[r].$$

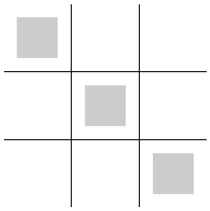
Here are the pictures of these subalgebras.



$\mathfrak{g}(\mathbb{V}^n)_r$



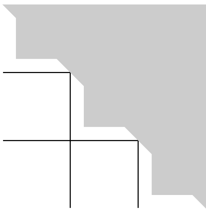
$\mathfrak{g}(\mathbb{V}^n)[r]$



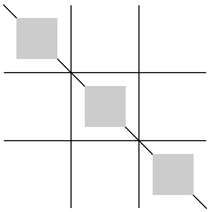
$\mathfrak{l}[r]$

For the rest of this section we will write $\mathfrak{g} = \mathfrak{g}(\mathbb{V}^n), \mathfrak{l} = \mathfrak{l}(\mathbb{V}^n)$, etc.

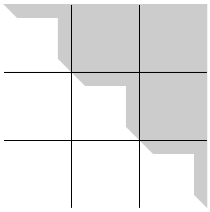
We also introduce the parabolic subalgebra $\mathfrak{p}(r) = \mathfrak{l}[r] + \mathfrak{b}$. This algebra has a Levi-type decomposition $\mathfrak{p}(k) = \mathfrak{l}[r]^+ \ni \mathfrak{u}(r)$, where $\mathfrak{l}[r]^+ = \mathfrak{l}[r] + \mathfrak{h}$ and $\mathfrak{u}(r)$ is the unique ideal giving this decomposition.



$\mathfrak{p}(r)$

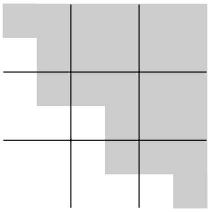


$\mathfrak{l}[r]^+$

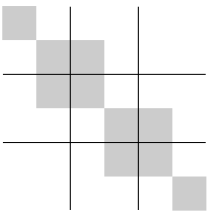


$\mathfrak{u}(r)$

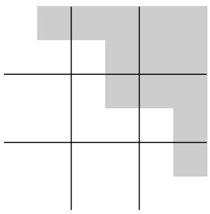
Other families of algebras we will study are related to the finite roots of $\mathfrak{g}$. We denote by $\mathfrak{s} = \mathfrak{s}(\mathbb{V}^n)$ the subalgebra spanned by all finite root spaces of $\mathfrak{g}$, and by $\mathfrak{q} = \mathfrak{q}(\mathbb{V}^n)$ the parabolic subalgebra $\mathfrak{s} + \mathfrak{b}$. This subalgebra has a Levi-type decomposition $\mathfrak{q} = \mathfrak{s} \ni \mathfrak{m}$, where $\mathfrak{m}$ is the subspace spanned by all root-spaces corresponding to positive infinite roots.



$\mathfrak{q}$



$\mathfrak{s}$



$\mathfrak{m}$

We will also occasionally need the exhaustion of $\mathfrak{s}$ given by $\mathfrak{s}_r = \mathfrak{s} \cap \mathfrak{g}_r$, whose picture is left for the interested reader.

3.7. **Transpose automorphism.** Denote by $E_{i,j}^{(k,l)}$ the element $E_{i,j} \otimes e_k \otimes e^l$. The Lie algebra $\mathfrak{g}(\mathbb{V}^n)$ has an anti-automorphism τ, analogous to transposition in $\mathfrak{gl}(r, \mathbb{C})$, given by $\tau(E_{i,j}^{(k,l)}) = E_{j,i}^{(l,k)}$. Given a subalgebra $\mathfrak{k} \subset \mathfrak{g}(\mathbb{V}^n)$ its image by τ will be denoted by $\bar{\mathfrak{k}}$. Thus we have further subalgebras $\bar{\mathfrak{b}}, \bar{\mathfrak{p}}, \bar{\mathfrak{u}}$, etc. Notice that the picture of the image of a subalgebra by τ is the reflection of the picture of the subalgebra by the main diagonal. We will denote by T the anti-automorphism of $U(\mathfrak{g}(\mathbb{V}^n))$ induced by τ, and given $u \in U(\mathfrak{g}(\mathbb{V}^n))$ we write $\bar{u}$ for $T(u)$.

3.8. **Roots, weights and eligible weights.** We denote by $\varepsilon_i^{(k)}$ the functional in $\mathfrak{h}(\mathbb{V}^n)^*$ given by $\varepsilon_i^{(k)}(E_{j,j}^{(l,l)}) = \delta_{i,j}\delta_{k,l}$. By a slight abuse of notation we can represent any element of $\mathfrak{h}(\mathbb{V}^n)^*$ as an infinite sum of the form $\sum_{i,k} a_i^{(k)} \varepsilon_i^{(k)}$ with $a_i^{(k)} \in \mathbb{C}$. We denote by $\omega^{(k)}$ the functional $\omega^{(k)} = \sum_{i \in \mathbb{Z}^\times} \varepsilon_i^{(k)}$. We set $\mathfrak{h}(\mathbb{V}^n)^\circ$ to be the span of $\{\varepsilon_i^{(k)}, \omega^{(k)} \mid i \in \mathbb{Z}^\times, k \in [\![n]\!]\}$, and refer to its elements as *eligible* weights. A weight will be called r-eligible if it is a linear combination of the $\omega^{(k)}$ and the $\varepsilon_i^{(k)}$ with $i \in \pm[\![r]\!]$. As we will see in Section 4, r-eligible weights parametrise the simple finite-dimensional representations of $\mathfrak{l}[r]^+$, which are all 1-dimensional. The importance of these representations will be discussed in that same section.

As usual, Λ denotes the $\mathbb{Z}$-span of the roots of $\mathfrak{g}(\mathbb{V}^n)$ inside $\mathfrak{h}(\mathbb{V}^n)^*$. It is a free $\mathbb{Z}$-module spanned by the families of roots

$$\begin{aligned}
&\varepsilon_i^{(k)} - \varepsilon_{i+1}^{(k)} && k \in [\![n]\!], i \in \mathbb{Z}^\times \setminus \{-1\};\\
&\varepsilon_{-1}^{(k)} - \varepsilon_1^{(k+1)} && k \in [\![n-1]\!];\\
&\varepsilon_{-1}^{(k)} - \varepsilon_1^{(k)} && k \in [\![n]\!].
\end{aligned}$$

The first two families consist of finite roots and any finite root is a linear combination of them, while the roots in the last family are infinite roots.

We denote by $\Lambda_{\mathbb{C}}^+$ the $\mathbb{C}$-span of the $\varepsilon_i^{(k)}$. There is a pairing $(-,-) : \mathfrak{h}(\mathbb{V}^n)^* \otimes \Lambda_{\mathbb{C}}^+ \to \mathbb{C}$ with $(\lambda, \varepsilon_i^{(k)}) = \lambda(E_{i,i}^{(k,k)})$. This allows us to define for any root α the reflection

$$\begin{aligned}
s_\alpha : \mathfrak{h}(\mathbb{V}^n)^* &\longrightarrow \mathfrak{h}(\mathbb{V}^n)^*\\
\lambda &\longmapsto \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\lambda.
\end{aligned}$$

The group generated by these reflections is the Weyl group of $\mathfrak{g}(\mathbb{V}^n)$, which we denote by $\mathcal{W}$.

Set $\mathfrak{h}_r = \mathfrak{h}(\mathbb{V}^n) \cap \mathfrak{g}(\mathbb{V}^n)_r$ and for $\lambda \in \mathfrak{h}(\mathbb{V}^n)$ denote by $\lambda|_r$ its restriction to $\mathfrak{h}_r$. Set also $\mathcal{W}_r$ to be the group generated by the reflections s_α with α a root of $\mathfrak{g}(\mathbb{V}^n)_r$. Clearly $\mathcal{W}$ is the union of the $\mathcal{W}_r$, and hence the direct limit of the Weyl groups of the Lie algebras in the exhaustion $\{\mathfrak{g}(\mathbb{V}^n)_r\}_{r>0}$. Furthermore, if $\sigma \in \mathcal{W}_r$ and $\lambda \in \mathfrak{h}(\mathbb{V}^n)$ then $\sigma(\lambda)|_r = \sigma(\lambda|_r)$. The group $\mathcal{W}$ is isomorphic to the group of bijections of $\mathbb{Z}^\times \times [\![n]\!]$ that leave a cofinite set fixed, and its action on $\mathfrak{h}(\mathbb{V}^n)^*$ is given by the corresponding permutation of the $\varepsilon_i^{(k)}$, suitably extended.

There is also an analogue of the dot action of the Weyl group for $\mathfrak{h}(\mathbb{V}^n)^*$. Set $\rho = \sum_{i,k} -i\varepsilon_i^{(k)}$, and set for every $\sigma \in \mathcal{W}$ and every $\lambda \in \mathfrak{h}(\mathbb{V}^n)^*$

$$\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho.$$

Both the usual and dot action send eligible weights to eligible weights but the dot action of the Weyl group of $\mathfrak{g}(\mathbb{V}^n)$ does not restrict to the dot action of $\mathfrak{g}(\mathbb{V}^n)_r$. However, if we denote by $\mathcal{W}(\mathfrak{s})$ and $\mathcal{W}(\mathfrak{s}_r)$ the groups generated by reflections associated to finite roots of $\mathfrak{g}(\mathbb{V}^n)$ and $\mathfrak{g}(\mathbb{V}^n)_r$ respectively, then $\mathcal{W}(\mathfrak{s})$ is the union of the $\mathcal{W}(\mathfrak{s}_r)$ and the dot action of $\sigma \in \mathcal{W}(\mathfrak{s}_r)$ on $\mathfrak{h}(\mathbb{V}^n)^*$ does satisfy that $(\sigma \cdot \lambda)|_r = \sigma \cdot (\lambda|_r)$.

As usual, we will write $\mu \preceq \lambda$ if $\lambda - \mu$ is a sum of positive roots. We also define a finer order $<_{\mathsf{fin}}$ as follows: $\mu <_{\mathsf{fin}} \lambda$ if and only if $\mu \preceq \lambda$ and $\mu = \sigma \cdot \lambda$ for some $\sigma \in \mathcal{W}(\mathfrak{s})$. It follows that if $\mu <_{\mathsf{fin}} \lambda$ then $\lambda - \mu$ is a sum of positive finite roots. The following lemma will be used several times in the sequel. Recall that two $\mathfrak{gl}(n, \mathbb{C})$ weights are said to be linked if they are in the same orbit of the dot-action of the corresponding Weyl group [14, Section 1.8].

Lemma 3.1. *Let λ, μ be r-eligible weights, with $\lambda \succeq \mu$. If $\lambda|_s$ and $\mu|_s$ are linked for all $s \geq r$ then $\mu = \sigma \cdot \lambda$, with $\sigma \in \mathcal{W}(\mathfrak{s}_{r+1})$. Thus $\lambda - \mu$ is a sum of positive finite roots of $\mathfrak{g}(\mathbb{V}^n)_r$, and in particular $\mu <_{\mathsf{fin}} \lambda$.*

Proof. Denote by σ_s the element in the Weyl group of $\mathfrak{g}(\mathbb{V}^n)_s$ such that $\sigma_s \cdot \lambda|_s = \mu|_s$, and by ρ_s the half-sum of the positive roots of $\mathfrak{g}_s$. If α is a simple root of $\mathfrak{g}(\mathbb{V}^n)_s$ that is also a root of $\mathfrak{l}[r]$ then $(\mu, \alpha) = 0$, so σ_s can only involve reflections s_β with β a root of $\mathfrak{g}(\mathbb{V}^n)_{r+1}$. It follows then that

$$0 = (\lambda|_s - \sigma_s(\mu|_s), \alpha) = (\sigma_s(\rho_s) - \rho_s, \alpha)$$

for all s and this is only possible if σ_s only involves reflections through simple roots, i.e., if $\sigma_s \in \mathcal{W}(\mathfrak{s}_{r+1})$. □

The statement, along with this proof, originally appeared as part of the proof of [19, Lemma 4.10].

4. Representation theory

Throughout this section $\mathfrak{g}$ is a locally finite Lie algebra with a fixed splitting Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{k}$ is a root-subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$.

4.1. **Socle filtrations.** Let M be a $\mathfrak{g}$-module. Recall that the socle of M is the largest semisimple $\mathfrak{g}$-submodule contained in M and is denoted by $\operatorname{soc} M$. The socle filtration of M is the filtration defined inductively as follows: $\operatorname{soc}^{(0)} M = \operatorname{soc} M$, and $\operatorname{soc}^{(n+1)} M$ is the preimage of $\operatorname{soc}(M/\operatorname{soc}^{(n)} M)$ by the quotient map $M \to M/\operatorname{soc}^{(n)} M$. The *layers* of the filtration are the semisimple modules $\overline{\operatorname{soc}}^{(n)} M = \operatorname{soc}^{(n)} M/\operatorname{soc}^{(n-1)} M$.

4.2. **Categories of weight modules.** Given a $\mathfrak{g}$-module M and $\lambda \in \mathfrak{h}^*$ we denote by M_λ the subspace of $\mathfrak{h}$-eigenvectors of eigenvalue λ, and refer to the set of all λ with $M_\lambda \neq 0$ as the *support* of M. We say M is a *weight* module if $M = \bigoplus_\lambda M_\lambda$. We denote by $(\mathfrak{g}, \mathfrak{h})$-Mod the full subcategory of $\mathfrak{g}$-Mod whose objects are weight modules.

The inclusion of $(\mathfrak{g}, \mathfrak{h})$-Mod in $\mathfrak{g}$-Mod is an exact functor with right adjoint $\Gamma_{\mathfrak{h}}$: $\mathfrak{g}$-Mod $\to (\mathfrak{g}, \mathfrak{h})$-Mod that assigns to each module M the largest $\mathfrak{h}$-semisimple submodule it contains. By standard homological algebra this functor is left exact, and sends injective objects to injective objects and direct limits to direct limits. In particular $(\mathfrak{g}, \mathfrak{h})$-Mod has enough injective objects. Given two weight modules M, N we denote by $\mathrm{Hom}_{\mathfrak{g},\mathfrak{h}}(M, N)$ the space of morphisms in the category of weight modules, and by $\mathrm{Ext}^{\bullet}_{\mathfrak{g},\mathfrak{h}}(N, M)$ the corresponding derived functors.

Given two $\mathfrak{g}$-modules M, N the space $\mathrm{Hom}_{\mathfrak{g}}(M, N)$ is a $\mathfrak{h}$-module in a natural way, and we denote by $\underline{\mathrm{Hom}}_{\mathfrak{g}}(M, N)$ the subspace spanned by its $\mathfrak{h}$-semisimple vectors. If M is a weight module then a map $\varphi : M \to N$ is semisimple of weight λ if and only if $\varphi(M_\mu) \subset N_{\lambda+\mu}$ for any weight μ.

4.3. **Induction and coinduction for weight modules.** Recall that $\mathfrak{k}$ denotes a root subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. The obvious restriction functor $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{k}} : (\mathfrak{g}, \mathfrak{h})\text{-Mod} \to (\mathfrak{k}, \mathfrak{h})\text{-Mod}$ has a left adjoint, given by the usual induction functor $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{k}} : (\mathfrak{k}, \mathfrak{h})\text{-Mod} \to (\mathfrak{g}, \mathfrak{h})\text{-Mod}$; both functors are exact. We often write just N instead of $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{k}} N$.

Restriction also has a right adjoint $\underline{\mathrm{Coind}}^{\mathfrak{g}}_{\mathfrak{k}} : (\mathfrak{k}, \mathfrak{h})\text{-Mod} \to (\mathfrak{g}, \mathfrak{h})\text{-Mod}$ given by

$$\underline{\mathrm{Coind}}^{\mathfrak{g}}_{\mathfrak{k}} M = \underline{\mathrm{Hom}}_{\mathfrak{k}}(U(\mathfrak{g}), M) = \Gamma_{\mathfrak{h}}(\mathrm{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), M))$$

and called the *semisimple coinduction* of M to $\mathfrak{g}$. A morphism $\varphi : U(\mathfrak{g}) \to M$ lies in $(\underline{\mathrm{Coind}}^{\mathfrak{g}}_{\mathfrak{k}} M)_\mu$ if and only if it is $\mathfrak{k}$-linear and maps the weight space $U(\mathfrak{g})_\lambda$ to $M_{\lambda+\mu}$. In particular the semisimple coinduction of M only depends on $\Gamma_{\mathfrak{h}}(M)$.

Unlike usual coinduction, semisimple coinduction is not exact. However it is left exact, sends injective objects to injective objects and direct limits to direct limits. This follows from the fact that it is right adjoint to an exact functor.

4.4. **Semisimple duals.** Suppose we have fixed an antiautomorphism τ of $\mathfrak{g}$ which restricts to the identity over $\mathfrak{h}$ (in all our examples $\mathfrak{g}$ will be $\mathfrak{gl}(r, \mathbb{C})$ or $\mathfrak{gl}(\infty)$, and τ will be the transposition map). Denote by T the antiautomorphism of $U(\mathfrak{g})$ induced by τ and by $\bar{\mathfrak{k}}$ the image of $\mathfrak{k}$ by τ. If M is a $\mathfrak{k}$-module with structure map $\rho : \mathfrak{k} \to \mathrm{End}(M)$ we can define a $\bar{\mathfrak{k}}$-module ${}^T M$ whose underlying vector space is M and whose structure map is $\rho \circ \tau : \bar{\mathfrak{k}} \to \mathrm{End}(M)$. This assignation is functorial and commutes with the functor $\Gamma_{\mathfrak{h}}$. The *semisimple dual* of a $\mathfrak{k}$-module M is the $\bar{\mathfrak{k}}$-module $M^\vee = {}^T\Gamma_{\mathfrak{h}}(M^*) = \Gamma_{\mathfrak{h}}({}^T M^*)$. The definition is analogous to the duality operator of category $\mathcal{O}$ for finite dimensional reductive Lie algebras, since $(M^\vee)_\lambda = (M_\lambda)^*$, and given $f \in M^\vee_\lambda$ we have $(u \cdot f)(m) = f(T(u)m)$ for all $u \in U(\bar{\mathfrak{k}})$ and $m \in M$. The following result relates semisimple duals with induction and semisimple coinduction functors.

Proposition 4.1. *Let $\mathfrak{k} \subset \mathfrak{g}$ be a subalgebra containing $\mathfrak{h}$, and let M be a weight $\mathfrak{k}$-module. There is a natural isomorphism of $\mathfrak{g}$-modules*

$$(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{k}} M)^\vee \cong \underline{\mathrm{Coind}}^{\mathfrak{g}}_{\bar{\mathfrak{k}}} M^\vee.$$

In particular semisimple coinduction is exact over the image of the semisimple dual functor.

Proof. By [11, Section 5.5.4, Proposition] there exists a natural isomorphism $(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{k}} M)^* \cong \mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{k}} M^*$. If we apply $\Gamma_{\mathfrak{h}}$ and then twist by the automorphism T we get

$$(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{k}} M)^\vee \cong \Gamma_{\mathfrak{h}}\Big({}^T \mathrm{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), M^*)\Big).$$

The automorphism T defines an equivalence between the categories of weight $\mathfrak{k}$ and $\bar{\mathfrak{k}}$-modules, so there is a natural isomorphism $\mathrm{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), M^*) \cong \mathrm{Hom}_{\bar{\mathfrak{k}}}({}^T U(\mathfrak{g}), {}^T M^*)$. Twisting the left $\mathfrak{g}$-action on this module by T corresponds to twisting the right $\mathfrak{g}$ action of $U(\mathfrak{g})$

by T, so

$$^T\operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), M^*) \cong \operatorname{Hom}_{\bar{\mathfrak{k}}}\Big({}^TU(\mathfrak{g})^T, {}^TM^*\Big) \cong \operatorname{Hom}_{\bar{\mathfrak{k}}}(U(\mathfrak{g}), {}^TM^*),$$

where the last isomorphism comes from the fact that $T : U(\mathfrak{g}) \to {}^TU(\mathfrak{g})^T$ defines an isomorphism of $U(\bar{\mathfrak{k}})$-$U(\mathfrak{g})$ bimodules. Thus

$$^T\Gamma_{\mathfrak{h}}(\operatorname{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), M^*)) \cong \underline{\operatorname{Hom}}_{\mathfrak{k}}\Big(U(\mathfrak{g}), \Gamma_{\mathfrak{h}}({}^TM^*)\Big) = \underline{\operatorname{Coind}}_{\mathfrak{k}}^{\mathfrak{g}} M^{\vee}.$$

This gives the natural isomorphism. Since induction and semisimple duality are exact functors, the last part of the statement follows. □

4.5. **Torsion and inflation.** Let M be a $\mathfrak{g}$-module. We say that $m \in M$ is $\mathfrak{k}$-*integrable* if for any $g \in \mathfrak{k}$ the dimension of the span of $\{g^r m \mid r \geq 0\}$ is finite, and that m is $\mathfrak{k}$-*locally nilpotent* if $g^r m = 0$ for all g, with $r \gg 0$ depending on g. We say that$m$ is $\mathfrak{k}$-*torsion* if $g^r m = 0$ and r can be chosen independently of g. We say that M is $\mathfrak{k}$-integrable, resp. locally nilpotent, torsion, if each of its elements is $\mathfrak{k}$-integrable, resp. locally nilpotent, torsion.

If $\mathfrak{k}$ is an ideal then the subspace of $\mathfrak{k}$-torsion vectors of a $\mathfrak{g}$-module M is again a $\mathfrak{g}$-module, which we denote by $\Gamma_{\mathfrak{k}}(M)$. This is a left exact functor, as can be easily checked. Denoting by J_r the ideal generated by $\mathfrak{k}^r$ inside $U(\mathfrak{g})$, we see there exists a natural isomorphism

$$\Gamma_{\mathfrak{k}} \cong \varinjlim \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g})/J_r, -).$$

Now suppose $\mathfrak{g} = \mathfrak{k} \ni \mathfrak{r}$. The projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \cong \mathfrak{k}$ induces a functor

$$\mathcal{I}_{\mathfrak{k}}^{\mathfrak{g}} : (\mathfrak{k}, \mathfrak{h})\text{-Mod} \longrightarrow (\mathfrak{g}, \mathfrak{h})\text{-Mod}.$$

We refer to this as the *inflation* functor. It can be identified with $\underline{\operatorname{Hom}}_{\mathfrak{g}}(U(\mathfrak{k}), -)$ where $U(\mathfrak{k})$ is seen as a right $\mathfrak{g}$-module, which implies it is exact and has a left adjoint $M \mapsto U(\mathfrak{k}) \otimes_{\mathfrak{g}} M \cong M/U(\mathfrak{g})\mathfrak{r}M$. We will write M for $\mathcal{I}_{\mathfrak{k}}^{\mathfrak{g}}(M)$ when the context makes clear that we are seeing M as a $\mathfrak{g}$-module.

4.6. **Weights, gradings and parabolic subalgebras.** Given any morphism of abelian groups $\varphi : \mathfrak{h}^* \to A$, we can turn a semisimple $\mathfrak{h}$-module into an A-graded vector space by setting $M_a = \bigoplus_{\varphi(\lambda)=a} M_\lambda$ for each $a \in A$. We will denote this A-graded vector space by M^φ, though we will often omit the superscript when it is clear from context. The assignation $M \mapsto M^\varphi$ is functorial, and in particular turns $\mathfrak{g}$ into an A-graded Lie algebra, and any weight module into an A-graded $\mathfrak{g}$-module.

Suppose now that $A = \mathbb{C}^n$ and that $\varphi(\alpha) \in \mathbb{Z}^n$ for each root. For $\chi, \xi \in \mathbb{C}^n$ we write $\chi \succeq \xi$ if $\chi - \xi \in \mathbb{Z}^n_{\geq 0}$. We get a decomposition $\mathfrak{g} = \mathfrak{g}^\varphi_{\prec 0} \oplus \mathfrak{g}^\varphi_0 \oplus \mathfrak{g}^\varphi_{\succ 0}$. If we have defined a set of positive roots Δ^+ along with a corresponding Borel subalgebra $\mathfrak{b}$, and if $\varphi(\alpha) \succ 0$ for every positive root, then the subalgebra $\mathfrak{g}^\varphi_{\succeq 0}$ is a parabolic subalgebra of $\mathfrak{g}$.

Example. Fix $n \in \mathbb{Z}_{>0}$ and set $\mathfrak{g} = \mathfrak{g}(\mathbb{V}^n)$.

(1) Consider the $\mathbb{C}$-linear map $\varphi : \mathfrak{h}(\mathbb{V}^n)^\circ \to \mathbb{C}^n$ given by $\varphi(\varepsilon_i^{(k)}) = \varphi(\omega^{(k)}) = e_k$, and extend to $\mathfrak{h}^*$ arbitrarily. Since roots are mapped to vectors in $\mathbb{Z}^n$ this map induces a $\mathbb{Z}^n$-grading on $\mathfrak{gl}(\infty)$, which coincides with the one introduced in Section 3.5. The corresponding parabolic subalgebra is $\mathfrak{g}^\varphi_{\succeq 0} = \mathfrak{p}$ with $\mathfrak{g}^\varphi_0 = \mathfrak{l}$ and $\mathfrak{g}^\varphi_{\succ 0} = \mathfrak{u}$.

(2) Fix $r > 0$ and denote by ρ_r the sum of all positive roots of $\mathfrak{s}_r$. Set $\theta_r : \mathfrak{h}^\circ \to \mathbb{Z}$ to be the map $\alpha \mapsto (\alpha, \rho_r) + (\varphi(\alpha), \rho_n)$, where we are seeing ρ_n as a vector of $\mathbb{C}^n$ in the usual way. The corresponding parabolic subalgebra is $\mathfrak{p}(r)$ with $\mathfrak{g}^{\theta_r}_0 = \mathfrak{l}[r]^+$ and $\mathfrak{g}^{\theta_r}_{>0} = \mathfrak{u}(r)$.

(3) Let $\psi : \mathfrak{h}(\mathbb{V}^n)^\circ \to \mathbb{C}$ be the map $\psi(\varepsilon_i^{(k)}) = \frac{1}{2}(n + 1 - 2k + \mathsf{sg}(i))$ (here $\mathsf{sg}(i)$ is 1 if i is a positive integer and -1 if it is negative) and $\psi(\omega^{(k)}) = 0$. This formula has the nice property that if α is a finite root then $\psi(\alpha) = 0$, while the infinite roots $\varepsilon_{-1}^{(k)} - \varepsilon_1^{(k)}$ are mapped by ψ to 1. The corresponding parabolic subalgebra is $\mathfrak{q}$ with $\mathfrak{g}_0^\psi = \mathfrak{s}$ and $\mathfrak{g}_{>0}^\psi = \mathfrak{m}$.

These maps and the induced gradings will reappear throughout the rest of the article, so we fix the notations φ, θ_r and ψ for them.

4.7. **Local composition series.** In general it is hard to establish a priori whether a $\mathfrak{gl}(\infty)$-module has a composition series. For this reason we introduce the following notion taken from [9, Proposition 3.2].

Definition 4.2. Let M be a $\mathfrak{g}$-module and let $\lambda \in \mathfrak{h}^*$. We say that M has a *local composition series at* λ if there exist a finite filtration

$$M = F_0M \supset F_1M \supset \cdots \supset F_tM = \{0\}$$

and a finite set $J \subset \{0, 1, \ldots, t-1\}$ such that

(i) if $j \in J$ then $F_jM/F_{j+1}M \cong L(\lambda_j)$ for some $\lambda_j \succeq \lambda$;
(ii) if $j \notin J$ and $\mu \succeq \lambda$ then $(F_{j-1}M/F_jM)_\mu = 0$.

We say that M has local composition series, or LCS for short, if it has a local composition series at all $\lambda \in \mathfrak{h}^*$.

It follows from the definition that a local composition series at λ induces a local composition series at λ' for any $\lambda' \succeq \lambda$, possibly with a different set J. A standard argument shows that if M has two local composition series at λ then the multiplicities of any simple object $L(\mu)$ in each series coincide. If M has LCS then we denote by $[M : L(\mu)]$ this common multiplicity. The class of modules having LCS is closed under submodules, quotients and extensions, and multiplicity is additive for extensions.

4.8. **Finite dimensional representations of $\mathfrak{gl}(\infty)$.** For each $a \in \mathbb{C}$ the algebra $\mathfrak{gl}(\infty)$ has a one-dimensional representation $\mathbb{C}_a$, where $g \in \mathfrak{gl}(\infty)$ acts by multiplication by $a\,\mathrm{tr}(g)$; in particular $\mathbb{C}_a$ is a trivial $\mathfrak{sl}(\infty)$-module. These are the only simple finite dimensional representations of $\mathfrak{gl}(\infty)$, and any finite dimensional weight representation is a finite direct sum of these. In order to find more interesting representations, we need to look for infinite dimensional modules. Finitely generated modules of $\mathfrak{gl}(\infty)$ do not form an abelian category, since $U(\mathfrak{gl}(\infty))$ is not left-noetherian, so we need to look for an alternative notion of a "small" $\mathfrak{gl}(\infty)$-module.

4.9. **The large annihilator condition.** Let $\mathfrak{k} \subset \mathfrak{gl}(\infty)$ be a subalgebra. Let M be a $\mathfrak{gl}(\infty)$-module M and let $m \in M$. We say that m satisfies the *large annihilator condition* (LAC from now on) with respect to $\mathfrak{k}$ if there exists a finite dimensional subalgebra $\mathfrak{t} \subset \mathfrak{k}$ such that $[\mathfrak{k}^\mathfrak{t}, \mathfrak{k}^\mathfrak{t}]$, the derived subalgebra of the centraliser of $\mathfrak{t}$ in $\mathfrak{k}$, acts trivially on $\mathbb{C}m$. In other words, the annihilator of m contains the "large" subalgebra $[\mathfrak{k}^\mathfrak{t}, \mathfrak{k}^\mathfrak{t}]$. We say that M satisfies the LAC if every vector in M satisfies the LAC.

The adjoint representation of $\mathfrak{gl}(\infty)$ satisfies the LAC with respect to itself, and hence with respect to any other subalgebra $\mathfrak{k}$. It follows that if $m \in M$ satisfies the LAC with respect to $\mathfrak{k}$ then the $\mathfrak{gl}(\infty)$-module generated by m also satisfies the LAC. The tensor product of two representations satisfying the LAC again satisfies the LAC.

Denote by $(\mathfrak{gl}(\infty),\mathfrak{h})\text{-Mod}^{\mathfrak{k}}_{\mathsf{LA}}$ the full subcategory of modules satisfying the LAC with respect to $\mathfrak{k}$. The natural inclusion of this category in $(\mathfrak{gl}(\infty),\mathfrak{h})\text{-Mod}$ has a right adjoint, which we denote by $\Phi_{\mathfrak{k}}$, or just Φ for simplicity. Being right adjoint to an exact functor, this functor is left exact and sends direct limits to direct limits and injective objects to injective objects.

Let us consider the case $\mathfrak{k} = \mathfrak{gl}(\infty)$ in more detail. For simplicity we fix the exhaustion $\mathfrak{g}(\mathbb{V})_r$ associated to the order $I = \mathbb{Z}^{\times}$. If M satisfies the large annihilator condition and $m \in M$, the finite dimensional Lie algebra $\mathfrak{t}$ is contained in $\mathfrak{g}(\mathbb{V})_r$ for some $r \gg 0$, so we might as well take $\mathfrak{t} = \mathfrak{g}(\mathbb{V})_r$. Now the algebra $\mathfrak{gl}(\infty)^{\mathfrak{t}}$ is the subalgebra we denoted by $\mathfrak{g}(\mathbb{V})[r]$ and is isomorphic to $\mathfrak{gl}(\infty)$. Thus $\mathbb{C}m$ is a 1-dimensional representation of $\mathfrak{g}(\mathbb{V})_r$ and hence isomorphic as such to $\mathbb{C}_a$ for some $a \in \mathbb{C}$.

4.10. **The large annihilator condition over $\mathfrak{l}$.** Recall we have set $\mathfrak{l} = \mathfrak{g}_0$ for the $\mathbb{Z}^n$-grading introduced in Section 3.5. In the sequel we will mostly consider modules satisfying the large annihilator condition with respect to $\mathfrak{l}$, so we now focus on this class. Since $\mathfrak{l} = \mathfrak{g}(\mathbb{V}^{(1)}) \oplus \cdots \oplus \mathfrak{g}(\mathbb{V}^{(n)})$, its one-dimensional representations are external tensor products of the form $\mathbb{C}_{a_1} \boxtimes \cdots \boxtimes \mathbb{C}_{a_n}$, with $\chi = (a_1, \ldots, a_n) \in \mathbb{C}^n$.

Let M be an $\mathfrak{l}$-module. A vector $m \in M$ satisfying the LAC with respect to $\mathfrak{l}$ must span a one-dimensional module over $\mathfrak{l}[r] \cong \mathfrak{l}$ for some $r \gg 0$. This forces the weight of m to be r-eligible, i.e. of the form

$$\sum_{k=1}^{n} a_k \omega^{(k)} + \sum_{k=1}^{n} \sum_{i \in \pm[\![r]\!]} b_i^{(k)} \varepsilon_i^{(k)} \in \mathfrak{h}(\mathbb{V}^n)^{\circ}.$$

We call the n-tuple $\chi = (a_1, \ldots, a_n)$ the *level* of this weight, and by a slight abuse of notation we also refer to it as the level of m. Any vector in the module generated by m has the same level as m, and denoting by M^{χ} the space of all vectors of level χ we see that $\bigoplus_{\chi \in \mathbb{C}^n} M^{\chi} \subset M$; this inclusion is an equality if M satisfies the LAC with respect to $\mathfrak{l}$. Thus every vector in an indecomposable module M satisfying the LAC with respect to $\mathfrak{l}$ must have the same level, and we define this to be the level of M.

For each $r \geq 1$ we set $\Phi_r : (\mathfrak{g},\mathfrak{h})\text{-Mod} \to (\mathfrak{g}_r,\mathfrak{h}_r)\text{-Mod}$ to be the functor that assigns to a module M the space of invariants $M^{\mathfrak{l}[r]'}$. If $m \in M_\lambda$ spans a trivial $\mathfrak{l}[r]'$-module then $\mathbb{C}m \cong \mathbb{C}_\lambda$ as $\mathfrak{l}[r]^+$-modules. We can use this observation to get natural isomorphisms

$$\Phi_r \cong \bigoplus_{\lambda} \operatorname{Hom}_{\mathfrak{g}}\Big(U(\mathfrak{g}) \otimes_{\mathfrak{l}[r]^+} \mathbb{C}_\lambda, -\Big);$$

$$\Phi \cong \bigoplus_{\lambda \in \mathfrak{h}(\mathbb{V}^n)^{\circ}} \varinjlim \operatorname{Hom}_{\mathfrak{g}}\Big(U(\mathfrak{g}) \otimes_{\mathfrak{l}[r]^+} \mathbb{C}_\lambda, -\Big),$$

where the first sum is taken over the space of all r-eligible weights, while the second is taken over all eligible weights.

5. Some categories of representations of $\mathfrak{gl}(\infty)$

Fix $n \in \mathbb{N}$ and set $\mathfrak{g} = \mathfrak{g}(\mathbb{V}^n)$. We omit the symbol $\mathbb{V}^n$ from this point on and simply write $\mathfrak{b}, \mathfrak{s}, \mathfrak{g}_r$, etc. for the subalgebras introduced in Section 3.6.

5.1. **Category $\overline{\mathcal{O}}$ for Dynkin Borel subalgebras.** In this subsection we review some results from Nampaisarn's thesis [15], where he introduced and studied category $\overline{\mathcal{O}}$. We will state his results for the Dynkin subalgebra $\mathfrak{s} \subset \mathfrak{g}$ since this will be the only example we will need. We set $\mathfrak{b}_\mathfrak{s} = \mathfrak{b} \cap \mathfrak{s}$ and $\mathfrak{n}_\mathfrak{s} = [\mathfrak{b}_\mathfrak{s}, \mathfrak{b}_\mathfrak{s}]$. For each $r \in \mathbb{Z}_{>0}$ we set $\mathfrak{s}_r = \mathfrak{s} \cap \mathfrak{g}_r$, which is isomorphic to the reductive Lie algebra $\mathfrak{gl}(r, \mathbb{C}) \oplus \mathfrak{gl}(2r, \mathbb{C})^{n-1} \oplus \mathfrak{gl}(r, \mathbb{C})$.

Definition 5.1. An $\mathfrak{s}$-module M lies in $\overline{\mathcal{O}} = \overline{\mathcal{O}}^{\mathfrak{s}}_{\mathfrak{b}_\mathfrak{s}}$ if it is $\mathfrak{h}$-semisimple, $\mathfrak{n}_\mathfrak{s}$-torsion, and $\dim M_\lambda < \infty$ for each $\lambda \in \mathfrak{h}^*$.

This definition makes sense for an arbitrary locally reductive algebra with a fixed Borel subalgebra, and many results from category $\mathcal{O}_{\mathfrak{s}_r}$ extend to $\overline{\mathcal{O}}$ in a straightforward manner. On the other hand the category lacks several obvious modules, for example the adjoint representation; in the non-Dynkin case Verma modules are also excluded. Still, it contains many interesting objects, and turns out to be an important stepping stone in the study of further categories of representations.

Since $\mathfrak{b}_\mathfrak{s}$ is a Dynkin Borel subalgebra of $\mathfrak{s}$, given $\lambda \in \mathfrak{h}^*$ the Verma module $M_\mathfrak{s}(\lambda)$ and its simple quotient $L_\mathfrak{s}(\lambda)$ belong to $\overline{\mathcal{O}}$. The category is also closed under semisimple duals, so the dual Verma module $M_\mathfrak{s}(\lambda)^\vee$ also belongs to $\overline{\mathcal{O}}$. A weight $\lambda \in \mathfrak{h}^*$ is *integral* if $(\lambda, \alpha) \in \mathbb{Z}$ for any simple root α of $\mathfrak{s}$, *dominant* if $(\lambda + \rho, \alpha) \notin \mathbb{Z}_{<0}$, and *almost dominant* if $(\lambda + \rho, \alpha) \in \mathbb{Z}_{<0}$ for only finitely many simple roots α.

Theorem 5.2 ([15, Theorem 6.7, Proposition 8.11])**.** *Let $\lambda, \mu \in \mathfrak{h}^*$.*

(a) If $M_\mathfrak{s}(\mu) \subset M_\mathfrak{s}(\lambda)$ then $\lambda \succeq \mu$ and there exists $\sigma \in \mathcal{W}(\mathfrak{s})$ such that $\mu = \sigma \cdot \lambda$.

(b) If M is a highest weight module with highest weight vector $m \in M_\lambda$ then for $r \gg 0$

$$[M : L_\mathfrak{s}(\mu)] = [U(\mathfrak{s}_r)m : L_{\mathfrak{s}_r}(\mu|_r)].$$

It follows that $m(\lambda, \mu) = [M_\mathfrak{s}(\lambda) : L_\mathfrak{s}(\mu)]$ coincides with the Kazhdan–Lusztig multiplicities of the corresponding Verma modules for the finite-dimensional reductive Lie algebra $\mathfrak{s}_r$.

Let λ be a dominant weight and denote by $\overline{\mathcal{O}}[\lambda]$ the full subcategory of $\overline{\mathcal{O}}$ whose objects are the modules whose support is contained in $\{\lambda - \mu \mid \mu \succeq 0\}$. It follows from the previous theorem that $\overline{\mathcal{O}}[\lambda]$ is a block of the category $\overline{\mathcal{O}}$. By [15, Theorem 9.9] this block has enough injectives, and by [15, Proposition 9.21] these injective objects have finite filtrations by dual Verma modules (notice Nampaisarn refers to dual Verma modules as costandard modules), and their multiplicities are given by BGG-reciprocity. We put this down as a theorem for future reference.

Theorem 5.3 ([15, Theorem 9.9, Proposition 9.21])**.** *Let λ be an almost dominant weight. Then $L_\mathfrak{s}(\lambda)$ has an injective envelope $I_\mathfrak{s}(\lambda)$ in $\overline{\mathcal{O}}$. The injective envelope has a finite filtration whose layers are modules of the form $M_\mathfrak{s}(\lambda^{(k)})^\vee$ with $k = 0, \ldots, r$, and such that $\lambda^{(0)} = \lambda$ and $\lambda^{(k)} >_{\mathsf{fin}} \lambda$. The multiplicity of $M_\mathfrak{s}(\mu)^\vee$ in this filtration equals $m(\lambda, \mu)$.*

5.2. **The categories $\mathbb{T}_\mathfrak{g}$ and $\mathbb{T}_\mathfrak{l}$.** We now return to our study of representations of $\mathfrak{gl}(\infty)$. Throughout this section we identify $\mathfrak{gl}(\infty)$ with $\mathfrak{g}(\mathbb{V})$, and so $\mathbb{V}$ and $\mathbb{V}_*$ are $\mathfrak{gl}(\infty)$-modules. A *tensor* module of $\mathfrak{gl}(\infty)$ is a subquotient of a finite direct sum of modules of the form $\mathbb{V}^p \otimes \mathbb{V}_*^q$ for $p, q \in \mathbb{N}_0$. The category of tensor modules is denoted by $\mathbb{T}^0_{\mathfrak{gl}(\infty)}$, and was first studied by Penkov and Styrkas in [20], and later by Dan-Cohen, Penkov and Serganova in [7]; in [21] Sam and Snowden study the equivalent category $\mathrm{Rep}^{st}\, \mathbf{GL}(\infty)$. It follows from [7, Section 4] that a module is a tensor module if and only if it is an integrable $\mathfrak{gl}(\infty)$-module of finite length satisfying the LAC, and it decomposes as a sum of indecomposables of level 0.

The simple tensor modules are parametrised by pairs of partitions (λ, μ). The corresponding simple module, which we denote by $L(\lambda, \mu)$, is the simple highest weight module with respect to the Borel subalgebra $\mathfrak{b}(\mathbb{V})$ with highest weight $\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n - \mu_n\varepsilon_{-n} - \cdots - \mu_1\varepsilon_{-1}$. The module $I(\lambda, \mu) = \mathbb{S}_\lambda(\mathbb{V}) \otimes \mathbb{S}_\mu(\mathbb{V}_*)$ belongs to $\mathbb{T}^0_{\mathfrak{gl}(\infty)}$ and is injective in this category [7, Corollary 4.6]. The layers of its socle filtration are given by [20, Theorem 2.3]

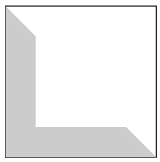
$$\overline{\mathrm{soc}}^{(r+1)} I(\lambda, \mu) = \bigoplus_{\gamma \vdash r} \bigoplus_{\lambda', \mu'} c^{\lambda}_{\lambda', \gamma} c^{\mu}_{\mu', \gamma} L(\lambda', \mu').$$

In particular $I(\lambda, \mu)$ is the injective envelope of $L(\lambda, \mu)$ in $\mathbb{T}^0_{\mathfrak{gl}(\infty)}$.

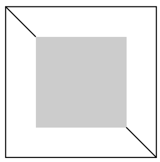
Lemma 5.4. *Let M be an integrable weight $\mathfrak{gl}(\infty)$-module satisfying the large annihilator condition with respect to $\mathfrak{gl}(\infty)$. Then M has finite length if and only if M is finitely generated.*

Proof. We will use the isomorphism $\mathfrak{gl}(\infty) \cong \mathfrak{g}(\mathbb{V})$ to see M as a $\mathfrak{g}(\mathbb{V})$-module. If M has finite length then it is finitely generated. To prove the reverse implication it is enough to prove it when M is cyclic and indecomposable so we assume that M is generated by a weight vector v of weight λ and level a.

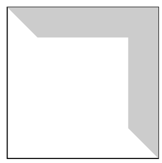
Take $r \gg 0$ so v spans a 1-dimensional $\mathfrak{g}(\mathbb{V})[r]$-module isomorphic to $\mathbb{C}_a$. Take $\mathfrak{r}$ to be the locally nilpotent ideal of $\mathfrak{g}(\mathbb{V})[r] + \mathfrak{b}$, and $\bar{\mathfrak{r}}$ to be the opposite ideal, so $\mathfrak{gl}(\infty) = \mathfrak{r} \oplus \mathfrak{g}(\mathbb{V})[r]^+ \oplus \bar{\mathfrak{r}}$, where $\mathfrak{g}(\mathbb{V})[r]^+ = \mathfrak{g}(\mathbb{V})[r] + \mathfrak{h}$.



$\bar{\mathfrak{r}}$



$\mathfrak{g}(\mathbb{V})[r]^+$



$\mathfrak{r}$

As $\mathfrak{g}(\mathbb{V})[r]$-modules, M is isomorphic to a quotient of $\mathbf{S}^\bullet(\bar{\mathfrak{r}}) \otimes \mathbf{S}^\bullet(\mathfrak{r}) \otimes \mathbb{C}_a$, while $\mathfrak{r}$ and $\bar{\mathfrak{r}}$ decompose as $\mathbb{V}[r]^r \oplus \mathbb{V}_*[r]^r \oplus \mathbb{C}^{r(2r-1)}$. By [7, Lemma 4.1] there exists a finite dimensional subspace $X \subset \mathfrak{r}$ such that $\mathbf{S}^p(\mathfrak{r})$ is generated over $\mathfrak{g}(\mathbb{V})[r]$ by $\mathbf{S}^p(X)$ for all $p \geq 0$. On the other hand, since M is integrable every element of X acts nilpotently on M, and so for $p \gg 0$ we have $\mathbf{S}^p(X)v = 0$, which implies $\mathbf{S}^p(\mathfrak{r})v = 0$. By an analogous reasoning, $\mathbf{S}^p(\bar{\mathfrak{r}})v = 0$, for $p \gg 0$.

It follows that M is in fact isomorphic as $\mathfrak{g}(\mathbb{V})[r]$-module to a quotient of $\mathbf{S}^{\leq p}(\bar{\mathfrak{r}}) \otimes \mathbf{S}^{\leq p}(\mathfrak{r}) \otimes \mathbb{C}_a$ for some $p \gg 0$. Since $\mathbf{S}^{\leq p}(\bar{\mathfrak{r}}) \otimes \mathbf{S}^{\leq p}(\mathfrak{r})$ is a tensor $\mathfrak{g}(\mathbb{V})[r]$-module it has finite length as $\mathfrak{g}(\mathbb{V})[r]$-module, so M has finite length over $\mathfrak{g}(\mathbb{V})[r]$ and hence over $\mathfrak{g}(\mathbb{V})$. □

We now turn back to the case $\mathfrak{g} = \mathfrak{g}(\mathbb{V}^n)$. We say that an $\mathfrak{l}$-module is a *tensor module* if it is a subquotient of the tensor algebra $T(\mathbb{V}^{(1)} \oplus \mathbb{V}^{(1)}_* \oplus \cdots \oplus \mathbb{V}^{(n)} \oplus \mathbb{V}^{(n)}_*)$. An indecomposable tensor module is isomorphic to an external tensor product $M_1 \boxtimes M_2 \boxtimes \cdots \boxtimes M_n$ with each M_i a tensor $\mathfrak{g}(\mathbb{V}^{(k)})$-module. Simple tensor modules are parametrised by pairs $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ with $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ n-tuples of partitions, and

$$L(\boldsymbol{\lambda}, \boldsymbol{\mu}) \cong L(\lambda_1, \mu_1) \boxtimes L(\lambda_2, \mu_2) \boxtimes \cdots \boxtimes L(\lambda_n, \mu_n).$$

Tensor $\mathfrak{l}$-modules have finite filtrations whose layers are simple tensor modules, and hence they have finite length. It is also clear they are integrable and satisfy the LAC by the characterisation of $\mathfrak{gl}(\infty)$ tensor modules given above.

Definition 5.5. The category $\mathbb{T}_{\mathfrak{l}}$ is the full subcategory of integrable $\mathfrak{g}$-modules of finite length satisfying the LAC. For each $\chi \in \mathbb{C}^n$ we define $\mathbb{T}_{\mathfrak{l}}^{\chi}$ to be the subcategory of $\mathbb{T}_{\mathfrak{l}}$ formed by modules of level χ.

Tensor $\mathfrak{l}$-modules belong to $\mathbb{T}_{\mathfrak{l}}^0$. Each $\mathbb{T}_{\mathfrak{l}}^{\chi}$ is a block of $\mathbb{T}_{\mathfrak{l}}$. Also $\mathbb{C}_\chi \in \mathbb{T}_{\mathfrak{l}}^{\chi}$, and tensoring with $\mathbb{C}_\chi$ gives an equivalence between $\mathbb{T}_{\mathfrak{l}}^0$ and $\mathbb{T}_{\mathfrak{l}}^{\chi}$.

Proposition 5.6. *Let M be an object of $\mathbb{T}_{\mathfrak{l}}$.*

(a) If M is simple then it is isomorphic to $\mathbb{C}_\chi \otimes L(\boldsymbol{\lambda}, \boldsymbol{\mu})$ for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathsf{Part}^n$.
(b) If M is a highest weight module with highest weight λ then $(\lambda, \alpha) \in \mathbb{Z}_{\geq 0}$ for α a positive root of $\mathfrak{l}$.

Proof. Since tensoring with a fixed one-dimensional module gives an equivalence between blocks of $\mathbb{T}_{\mathfrak{l}}$, to prove (a) it is enough to prove that a simple object L in $\mathbb{T}_{\mathfrak{l}}^0$ is a simple tensor module. Take $v \in L$, and set L_k to be the $\mathfrak{g}(\mathbb{V}^{(k)})$-submodule of L spanned by v. By definition L_k is integrable and satisfies the LAC with respect to $\mathfrak{g}(\mathbb{V}^{(k)})$, and so by Lemma 5.4 it is a tensor $\mathfrak{g}(\mathbb{V}^{(k)})$-module. Since the $\mathfrak{g}(\mathbb{V}^{(k)})$ commute with each other inside $\mathfrak{l}$, there is a surjective map $L_1 \boxtimes L_2 \boxtimes \cdots \boxtimes L_n \to L$, so L is a quotient of a tensor module, and hence a simple tensor module. To prove (b), notice that if M is a highest weight module in $\mathbb{T}_{\mathfrak{l}}$ then its unique simple quotient also lies in $\mathbb{T}_{\mathfrak{l}}$ and so it must be of the form $\mathbb{C}_\chi \otimes L$ with L a simple tensor module. The statement is easily proved for the highest weight of this simple module. $\square$

5.3. **The category $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$.** Since $\mathfrak{l}[r] \cong \mathfrak{l}$ for any $r \in \mathbb{Z}_{>0}$, we can consider tensor modules over $\mathfrak{l}[r]$. We will say that a module over $\mathfrak{l}[r]^+ = \mathfrak{l}[r] + \mathfrak{h}$ is a tensor module if its restriction to $\mathfrak{l}[r]$ is a tensor module.

Recall that weights in $\mathfrak{h}^\circ$ are called eligible weights, and a weight is r-eligible if it is in the span of $\{\omega^{(k)}, \varepsilon_i^{(k)} \mid k \in [\![n]\!], i \in \pm[\![r]\!]\}$(Section 3.8). Let $\lambda \in \mathfrak{h}^*$. We denote by $\mathcal{D}(\lambda)$ the set of all weights $\lambda - \mu$ with $\mu \succeq 0$. Let M be a weight $\mathfrak{g}$-module. We say λ is *extremal* if $M_\lambda \neq 0$ and $M_\mu = 0$ for all $\mu \succ \lambda$. If a module M has finitely many extremal weights $\lambda_1, \ldots, \lambda_k$ then its support is contained in the union of the $\mathcal{D}(\lambda_i)$.

Definition 5.7. The category $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$ is the full subcategory of $(\mathfrak{l}[r]^+, \mathfrak{h})$-Mod whose objects are modules M satisfying the following conditions:

(i) M has finitely many extremal weights.
(ii) For each $\mu \in \mathfrak{h}^*$ the $\mathfrak{l}[r]$-module generated by $M_{\succeq \mu} = \bigoplus_{\nu \succeq \mu} M_\nu$ lies in $\mathbb{T}_{\mathfrak{l}[r]}$.

The following lemma is an easy consequence of the definition.

Lemma 5.8. *The category $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$ is closed under direct summands, direct sums and tensor products.*

Example. We have decompositions

$$\mathbb{V}^{(k)} = V_{-,r}^{(k)} \oplus \mathbb{V}^{(k)}[r] \oplus V_{+,r}^{(k)} \qquad \mathbb{V}_*^{(k)} = \left(V_{-,r}^{(k)}\right)^* \oplus \mathbb{V}_*^{(k)}[r] \oplus \left(V_{+,r}^{(k)}\right)^*$$

where $V_{\pm,r}^{(k)} = \langle v_i \otimes e_k \mid i \in \pm[\![r]\!]\rangle$. Each of these modules is a tensor $\mathfrak{l}[r]$-module: $\mathbb{V}^{(k)}[r]$ is the natural representation of $\mathfrak{g}(\mathbb{V}^{(k)}[r])$ while $\mathbb{V}_*^{(l)}[r]$ is its conatural representation, and the other four spaces are finite direct sums of 1-dimensional $\mathfrak{h}$-modules on which $\mathfrak{l}[r]$ acts trivially. Since tensor modules are closed by finite direct sums, it follows that both are tensor modules over $\mathfrak{l}[r]$.

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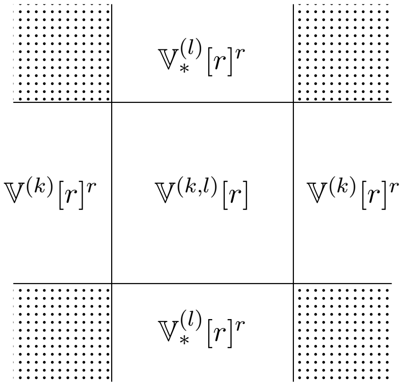
Now consider $\mathbb{V}^{(k,l)} = \mathbb{V}^{(k)} \otimes \mathbb{V}_*^{(l)}$. From the previous example we get a decomposition of this space into nine summands, illustrated in the picture below.



$\mathbb{V}^{(k)} \otimes \mathbb{V}_*^{(l)}$ decomposes as $\mathfrak{l}[r]$-module as a direct sum of:

- $\mathbb{V}^{(k,l)}[r] = \mathbb{V}^{(k)}[r] \otimes \mathbb{V}_*^{(l)}[r]$;
- $2r$ copies of $\mathbb{V}^{(k)}[r]$;
- $2r$ copies of $\mathbb{V}_*^{(l)}[r]$;
- a trivial module of dimension $4r^2$.

The fact that $\mathbb{T}_{\mathfrak{l}[r]}$ is closed by direct sums, direct summands and tensor products implies that $\mathbf{S}^p(\mathbb{V}^{(k,l)})$ is again in $\mathbb{T}_{\mathfrak{l}[r]}$. We claim that if $k > l$ then $\mathbf{S}^\bullet(\mathbb{V}^{(k,l)})$ is in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$. Indeed, its support is contained in $\mathcal{D}(0)$, and furthermore for each $\lambda \in \mathfrak{h}^\circ$ we have $\mathbf{S}^\bullet(\mathbb{V}^{(k,l)})_{\succeq\lambda} \subset \bigoplus_{p=1}^{-\varphi(\lambda)} \mathbf{S}^p(\mathbb{V}^{(k,l)})$, which lies in $\mathbb{T}_{\mathfrak{l}[r]}$; here φ is the map introduced in Example 4.6. On the other hand if $k \leq l$ then the symmetric algebra is not in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$, as it has no extremal weights.

We will mostly be interested in studying $\mathfrak{g}$-modules whose restriction to $\mathfrak{l}[r]^+$ lies in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$. We now show that such a module has LCS as defined in Section 4.7. The statement and proof are similar to [9, Proposition 3.2], but we replace the dimension of $M_{\succeq\lambda}$, which could be infinite in our case, with the length of the $\mathfrak{l}[r]$-module it spans.

Proposition 5.9. *Let M be a $\mathfrak{g}$-module lying in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$. Then M has LCS.*

Proof. Fix μ in the support of M and denote by $N(\mu)$ the $\mathfrak{l}[r]$-submodule spanned by $M_{\succeq\mu}$. The proof proceeds by induction on the length of $N(\mu)$, which we denote by ℓ. If $\ell = 0$ then $0 \subset M$ is the desired composition series. In any other case we have $\mu \prec \lambda$ for some λ which is maximal in the support of M. Taking $v \in M_\lambda$ we see that $V = U(\mathfrak{g})v$ is a highest weight module with a unique maximal submodule V'.

Since $v \in N(\mu)$ by hypothesis, $N(\mu) \cap V' \subsetneq N(\mu)$ has length strictly less than ℓ. Using this module and the induction hypothesis we know there exists an LCS at μ for V'. The same applies to $N(\mu)/(N(\mu) \cap V) \subseteq M/V$, so M/V also has a LCS at μ. These LCS can be used to refine the filtration $0 \subset V' \subset V \subset M$ into an LCS of M at μ. $\square$

5.4. **The symmetric algebras of parabolic radicals.** Recall that we have introduced for each $r \geq 0$ the subalgebra $\mathfrak{p}(r) = \mathfrak{l}[r] + \mathfrak{b}$. This subalgebra has a Levi-type decomposition $\mathfrak{p}(r) = \mathfrak{l}[r]^+ \ni \mathfrak{u}(r)$, which in turn induces a decomposition $\mathfrak{g} = \bar{\mathfrak{u}}(r) \oplus \mathfrak{l}[r]^+ \oplus \mathfrak{u}(r)$. The pictures of these subalgebras are given in Section 3.4. We will now study the structure of $\mathbf{S}^\bullet(\bar{\mathfrak{u}}(r))$ as $\mathfrak{l}[r]^+$-module, as this will be essential in the sequel.

Using the decomposition of $\mathbb{V}^{(k,l)}$ given in Example 5.3 it is easy to see that $\bar{\mathfrak{u}}(r)$ is isomorphic as $\mathfrak{l}[r]$-module to

$$Z \oplus \left(\bigoplus_{k=1} \mathbb{V}^{(k)}[r] \otimes Y_k\right) \oplus \left(\bigoplus_{l=1} X_l \otimes \mathbb{V}_*^{(l)}[r]\right) \oplus \left(\bigoplus_{n\geq k>l\geq 1} \mathbb{V}^{(k)}[r] \otimes \mathbb{V}_*^{(l)}[r]\right)$$

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for some trivial $\mathfrak{l}[r]$-modules X_k, Y_l, Z. This is in fact a decomposition into indecomposable $\bar{\mathfrak{b}}_r \oplus \mathfrak{l}[r]$-modules, where $\bar{\mathfrak{b}}_r = \bar{\mathfrak{b}} \cap \mathfrak{g}_r$. Indeed, Z coincides with the subalgebra $\bar{\mathfrak{n}}_r = \bar{\mathfrak{n}} \cap \mathfrak{g}_r$. The modules X_k and Y_l can be described recursively as

$$X_n = V_{+,r}^{(n)} \qquad X_{k-1} = X_k \oplus V_{-,r}^{(k)} \oplus V_{+,r}^{(k-1)};$$
$$Y_1 = \left(V_{+,r}^{(1)}\right)^* \qquad Y_{k+1} = Y_k \oplus \left(V_{-,r}^{(k-1)}\right)^* \oplus \left(V_{+,r}^{(k)}\right)^*.$$

It follows from this that Y_k is isomorphic to the $\bar{\mathfrak{b}}_r$-submodule of the conatural representation of $\mathfrak{g}_r$ spanned by the unique vector of weight $-\varepsilon_r^{(k)}$, and X_l is isomorphic to the $\bar{\mathfrak{b}}_r$-submodule of the natural representation of $\mathfrak{g}_r$ spanned by the unique vector of weight $\varepsilon_{-r}^{(l)}$.

Proposition 5.10. *The symmetric algebra $\mathbf{S}^\bullet(\bar{\mathfrak{u}}(r))$ is an object of $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$.*

Proof. To study the symmetric algebra $\mathbf{S}^\bullet(\bar{\mathfrak{u}}(r))$ we look at the symmetric algebra of each summand in the decomposition of $\bar{\mathfrak{u}}(r)$ as $\bar{\mathfrak{b}}_r \oplus \mathfrak{l}[r]$-module separately.

$$\begin{aligned}
\mathbf{S}^\bullet(\bar{\mathfrak{n}}_r) &= \bigoplus_{k\geq 0} \mathbf{S}^k(\bar{\mathfrak{n}}_r) \\
\mathbf{S}^\bullet\left(\mathbb{V}^{(k)}[r] \otimes Y_k\right) &= \bigoplus_{\nu_k} \mathbb{S}_{\nu_k}\left(\mathbb{V}^{(k)}[r]\right) \otimes \mathbb{S}_{\nu_k}(Y_k) && (k = 1, \dots, n) \\
\mathbf{S}^\bullet\left(X_l \otimes \mathbb{V}_*^{(l)}[r]\right) &= \bigoplus_{\gamma_l} \mathbb{S}_{\gamma_l}(X_l) \otimes \mathbb{S}_{\gamma_l}\left(\mathbb{V}_*^{(l)}[r]\right) && (l = 1, \dots, n) \\
\mathbf{S}^\bullet\left(\mathbb{V}^{(k)}[r] \otimes \mathbb{V}_*^{(l)}[r]\right) &= \bigoplus_{\rho_{k,l}} \mathbb{S}_{\rho_{k,l}}\left(\mathbb{V}^{(k)}[r]\right) \otimes \mathbb{S}_{\rho_{k,l}}\left(\mathbb{V}_*^{(l)}[r]\right) && (1 \leq l < k \leq n)
\end{aligned}$$

By Lemma 5.8 it is enough to see that each symmetric algebra lies in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$, and in each case we will show that if $\mu \in \mathfrak{h}^*$ is in the support then it is in the support of finitely many of the summands in the decompositions displayed above.

Case 1: $\mathbf{S}^\bullet(\bar{\mathfrak{n}}_r)$. If μ is any weight in the support of this module then $\mathbf{S}^\bullet(\bar{\mathfrak{n}}_r)_{\succeq\mu}$ is finite dimensional, so it is contained in $\mathbf{S}^{\leq k}(\bar{\mathfrak{n}}_r)$ for some $k \in \mathbb{Z}_{>0}$.

Case 2: $\mathbf{S}^\bullet(\mathbb{V}^{(k)}[r] \otimes Y_k)$. Fix a weight μ in the support. Recall that $\mu|_r$ and $\mu[r]$ denote the restriction of μ to $\mathfrak{h}_r$ and $\mathfrak{h}[r]$, respectively. Then

$$\left(\mathbb{S}_\nu\left(\mathbb{V}^{(k)}[r]\right) \otimes \mathbb{S}_\nu(Y_k)\right)_\mu = \mathbb{S}_\nu\left(\mathbb{V}^{(k)}[r]\right)_{\mu[r]} \otimes \mathbb{S}_\nu(Y_k)_{\mu|_r}.$$

Since the support of Y_k consists entirely of negative weights, $\mathbb{S}_\nu(Y_k)_{\succeq\mu|_r} = 0$ whenever $|\nu|$ is larger than the height of $\mu|_r$ as a weight of $\bar{\mathfrak{b}}_r$.

Case 3: $\mathbf{S}^\bullet(X_k \otimes \mathbb{V}_*^{(k)}[r])$. As in case 2 we have a decomposition

$$\left(\mathbb{S}_\nu\left(\mathbb{V}_*^{(l)}[r]\right)\right) \otimes \mathbb{S}_\nu(X_l)\Big)_\mu = \mathbb{S}_\nu\left(\mathbb{V}^{(l)}[r]_*\right)_{\mu[r]} \otimes \mathbb{S}_\nu(X_l)_{\mu|_r}.$$

Now the support of X_l consists entirely of positive weights, and the argument is analogous to the previous case.

Case 4: $\mathbf{S}^\bullet(\mathbb{V}^{(k)}[r] \otimes \mathbb{V}_*^{(l)}[r])$. This was done in Example 5.3. □

Notice that while $\mathbf{S}^p(\mathfrak{u}(r))$ is a tensor $\mathfrak{l}[r]$-module for any p, $\mathbf{S}^\bullet(\mathfrak{u}(r))$ is *not* in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$.

5.5. **Large annihilator duality.** The categories $\mathbb{T}_{\mathfrak{g}}$ and $\mathbb{T}_{\mathfrak{l}}$ are not closed under semisimple duals. For example $\mathfrak{h}^* = \mathfrak{g}_0^\vee$ contains uncountably many vectors that do not satisfy the LAC, so $\mathfrak{g}^\vee$ is not in $\mathbb{T}_{\mathfrak{g}}$. For this reason we introduce the *large annihilator dual* of an $\mathfrak{l}$-module, which we denote by $M^\#$ and define as $\Phi_{\mathfrak{l}}(M^\vee)$ (the functor $\Phi_{\mathfrak{l}}$ was introduced in Section 4.9). We use the same notation when M is a $\mathfrak{g}$-module, seen as $\mathfrak{l}$-module by restriction. It follows from the properties of semisimple duality and $\Phi_{\mathfrak{l}}$ that large annihilator duality is left exact and sends inverse limits to direct limits.

Given two n-tuples of partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ we define the $\mathfrak{l}$-module

$$I(\boldsymbol{\lambda}, \boldsymbol{\mu}) = I(\lambda_1, \mu_1) \boxtimes \cdots \boxtimes I(\lambda_n, \mu_n).$$

The socle filtration of these modules can be deduced from the socle filtration of the $I(\lambda_i, \mu_i)$, and as a direct consequence we see that there is a nonzero morphism $I(\boldsymbol{\lambda}, \boldsymbol{\mu}) \to \mathbb{C}$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\mu}$, and in that case this map is unique.

Lemma 5.11. *The large annihilator dual of $I(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is isomorphic to*

$$\bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta}} \bigoplus_{\boldsymbol{\gamma}} \left(\prod_k c^{\lambda_k}_{\alpha_k,\gamma_k} c^{\mu_k}_{\beta_k,\gamma_k} \right) I(\boldsymbol{\alpha}, \boldsymbol{\beta}),$$

where the sum runs over all n-tuples of partitions $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$. Thus $I(\boldsymbol{\lambda}, \boldsymbol{\mu})^\#$ lies in $\mathbb{T}_{\mathfrak{l}}^0$.

Proof. Set $\mathfrak{l}_r = \mathfrak{l} \cap \mathfrak{g}_r$. There exist decompositions of $\mathfrak{l}_r \oplus \mathfrak{l}[r]$-modules

$$\mathbb{V}^{(k)} = V_r^{(k)} \oplus \mathbb{V}^{(k)}[r]; \qquad \mathbb{V}_*^{(k)} = \left(V_r^{(k)}\right)^* \oplus \mathbb{V}_*^{(k)}[r].$$

Using the formula for the image of a direct sum by a Schur functor we get

$$I(\lambda, \mu) = \bigoplus_{\alpha,\beta,\gamma,\delta} c^\lambda_{\alpha,\gamma} c^\mu_{\beta,\delta} \mathbb{S}_\alpha\left(V_r^{(k)}\right) \otimes \mathbb{S}_\beta\left(\left(V_r^{(k)}\right)^*\right) \boxtimes I(\gamma, \delta)[r],$$

where $I(\gamma, \delta)[r] = \mathbb{S}_\gamma(\mathbb{V}^{(k)}[r]) \otimes \mathbb{S}_\delta(\mathbb{V}_*^{(k)}[r])$. This is an isomorphism of $\mathfrak{l}_r \oplus \mathfrak{l}[r]$-modules.

Set

$$T^{(k)}_{\alpha,\beta,\gamma,\delta} = \bigoplus_{\gamma,\delta} c^\lambda_{\alpha,\delta} c^\mu_{\beta,\gamma} \left(\mathbb{S}_\alpha\left(V_r^{(k)}\right) \otimes \mathbb{S}_\beta\left(\left(V_r^{(k)}\right)^*\right)\right).$$

Since this is a finite dimensional $\mathfrak{l}_r$-module, it is isomorphic to its own semisimple dual. Using the fact that duals distribute over tensor products if one of the factors is finite dimensional, we obtain an isomorphism

$$I(\lambda, \mu)^\vee \cong \bigoplus_{\alpha,\beta,\gamma,\delta} T^{(k)}_{\alpha,\beta,\gamma,\delta} \boxtimes \underline{\mathrm{Hom}}_{\mathbb{C}}(I(\gamma, \delta)[r], \mathbb{C}),$$

and so

$$I(\boldsymbol{\lambda}, \boldsymbol{\mu})^\vee \cong \bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}} \left(\bigotimes_{k=1}^n T^{(k)}_{\alpha_k,\beta_k,\gamma_k,\delta_k} \right) \boxtimes \underline{\mathrm{Hom}}_{\mathbb{C}}(I(\boldsymbol{\gamma}, \boldsymbol{\delta})[r], \mathbb{C})$$

as $\mathfrak{l}_r \oplus \mathfrak{l}[r]$-modules.

We now apply Φ_r to this module, obtaining

$$\Phi_r(\underline{\mathrm{Hom}}_{\mathbb{C}}(I(\boldsymbol{\gamma}, \boldsymbol{\delta})[r], \mathbb{C})) \cong \underline{\mathrm{Hom}}_{\mathfrak{l}[r]}(I(\boldsymbol{\gamma}, \boldsymbol{\delta})[r], \mathbb{C}) \cong \underline{\mathrm{Hom}}_{\mathfrak{l}}(I(\boldsymbol{\gamma}, \boldsymbol{\delta}), \mathbb{C}).$$

As mentioned in the preamble this is the zero vector space unless $\boldsymbol{\gamma} = \boldsymbol{\delta}$, in which case it is one-dimensional. Thus $\Phi_r(I(\boldsymbol{\lambda}, \boldsymbol{\mu})^\vee)$ is isomorphic to

$$\bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \left(\bigotimes_{k=1}^{n} T^{(k)}_{\alpha_k,\beta_k,\gamma_k,\gamma_k} \right) = \bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \bigotimes_{k=1}^{n} c^{\lambda_k}_{\alpha_k,\gamma_k} c^{\mu_k}_{\beta_k,\gamma_k} \left(\mathbb{S}_{\alpha_k}\left(V_r^{(k)}\right) \otimes \mathbb{S}_{\beta_k}\left(\left(V_r^{(k)}\right)^*\right)\right).$$

Taking the limit as r goes to infinity, we see that $I(\boldsymbol{\lambda}, \boldsymbol{\mu})^\#$ is isomorphic to

$$\bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \bigotimes_{k=1}^{n} c^{\lambda_k}_{\alpha_k,\gamma_k} c^{\mu_k}_{\beta_k,\gamma_k} \left(\mathbb{S}_{\alpha_k}(\mathbb{V}^{(k)}) \otimes \mathbb{S}_{\beta_k}\left(\mathbb{V}^{(k)}_*\right)\right),$$

and this is precisely the module in the statement. □

The following is an immediate consequence of the last two results.

Proposition 5.12. *The large annihilator dual of* $\mathbf{S}^\bullet(\bar{\mathfrak{u}}(r))$ *lies in* $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$.

6. Category $\mathcal{O}_{\mathsf{LA}}$: first definitions

As in the previous section we fix $n \in \mathbb{Z}_{>0}$ and denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{g}(\mathbb{V}^n)$. We will also continue to omit $\mathbb{V}^n$ from the notation for subalgebras of $\mathfrak{g}$.

6.1. Introduction to $\mathcal{O}_{\mathsf{LA}}$.

We now introduce a category of representations of $\mathfrak{g}$ whose finite length objects form an analogue of category $\mathcal{O}$.

Definition 6.1. The category $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{g}$ is the full subcategory of $\mathfrak{g}$-Mod whose objects are the $\mathfrak{g}$-modules M satisfying the following conditions:

(i) M is $\mathfrak{h}$-semisimple.
(ii) M is $\mathfrak{n}$-torsion.
(iii) M satisfies the LAC with respect to $\mathfrak{l}$.

We often write $\mathcal{O}_{\mathsf{LA}}$ for $\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{g}$. If M is an object of $\mathcal{O}_{\mathsf{LA}}$ then so is any subquotient of M. It follows from the definition that the support of an object M in $\mathcal{O}_{\mathsf{LA}}$ must be contained in $\mathfrak{h}^\circ$.

Recall from Example 4.6 that there is a map $\psi : \mathfrak{h}^\circ \to \mathbb{C}$ that sends the finite roots to 0, induces a $\mathbb{Z}$-grading on $\mathfrak{g}$, and turns a weight module M into a graded module M^ψ. In particular objects in $\mathcal{O}_{\mathsf{LA}}$ can be seen as $\mathbb{Z}$-graded modules. The following lemma shows that finitely generated objects in $\mathcal{O}_{\mathsf{LA}}$ have LCS and hence well-defined Jordan–Hölder multiplicities.

Lemma 6.2. *Let M be any $\mathfrak{g}$-module and let $v \in M$ be an $\mathfrak{h}$-semisimple and $\mathfrak{n}$-torsion vector satisfying the LAC with respect to $\mathfrak{l}$. Then the submodule $N = U(\mathfrak{g})v$ lies in $\mathcal{O}_{\mathsf{LA}}$, has LCS, and $N^\psi_z = 0$ for $z \gg 0$.*

Proof. Take $r \gg 0$ such that $\mathbb{C}v$ is a trivial $\mathfrak{l}[r]$-module. Using the PBW theorem we have a surjective map of $\mathfrak{l}[r]$-modules

$$\begin{aligned} p : \mathbf{S}^\bullet(\bar{\mathfrak{u}}(r)) \otimes \mathbf{S}^\bullet(\mathfrak{u}(r)) \otimes \mathbf{S}^\bullet(\mathfrak{l}[r]^+) \otimes \mathbb{C}_\lambda &\longrightarrow N \\ u \otimes u' \otimes l \otimes 1_\lambda &\longmapsto uu'lv. \end{aligned}$$

Since v is $\mathfrak{n}$-nilpotent and satisfies the LAC we have $\mathbf{S}^\bullet(\mathfrak{l}[r]^+)v = \mathbb{C}v$ and $\mathbf{S}^k(\mathfrak{u}(r))v = 0$ for $k \gg 0$. Thus the restriction

$$p' : \mathbf{S}^\bullet(\bar{\mathfrak{u}}(r)) \otimes \mathbf{S}^{\leq k}(\mathfrak{u}(r)) \otimes \mathbb{C}_\lambda \longrightarrow N$$

is surjective. By Propositions 5.9 and 5.10 the domain of this last map lies in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$ and has LCS. Since $\bar{\mathfrak{u}}(r)^{\psi}$ has a right-bounded grading, the same holds for the domain of p' and hence for N. By the PBW theorem

$$\mathfrak{n} \cdot \mathbf{S}^i(\bar{\mathfrak{u}}(r)) \otimes \mathbf{S}^j(\mathfrak{u}(r)) \subset \bigoplus_{i',j',t'} \mathbf{S}^{i'}(\bar{\mathfrak{u}}(r)) \otimes \mathbf{S}^{j'}(\mathfrak{u}(r)) \otimes \mathbf{S}^{t'}(\mathfrak{l}[r]^+)$$

where $i'+j'+t' = i+j+1$ and $i' \leq i$. Induction on i shows that $p'(\mathbf{S}^i(\bar{\mathfrak{u}}(r))\otimes\mathbf{S}^j(\mathfrak{u}(r))\otimes\mathbb{C}_\lambda)$ is annihilated by a large enough power of $\mathfrak{n}$, and so N is $\mathfrak{n}$-torsion. □

This lemma has a very useful consequence: every module in $(\mathfrak{g},\mathfrak{h})$-Mod has a largest submodule contained in $\mathcal{O}_{\mathsf{LA}}$, namely the submodule spanned by its $\mathfrak{n}$-torsion elements satisfying the LAC with respect to $\mathfrak{l}$. Categorically, this means that the inclusion functor of $\mathcal{O}_{\mathsf{LA}}$ in $(\mathfrak{g},\mathfrak{h})$-Mod has a right adjoint π_{LA}. We will come back to this observation later when we study categorical properties of $\mathcal{O}_{\mathsf{LA}}$.

6.2. **Simple objects in $\mathcal{O}_{\mathsf{LA}}$.** Let $\lambda \in \mathfrak{h}^*$. The Verma module $M(\lambda)$ does not belong to $\mathcal{O}_{\mathsf{LA}}$ since the highest weight vector does not satisfy the LAC. However, if λ is r-eligible there exists a 1-dimensional $\mathfrak{l}[r]^+$-module $\mathbb{C}_\lambda$, which we can inflate it to a $\mathfrak{p}(r)$-module setting the action of $\mathfrak{u}(r)$ to be zero. Set $M_r(\lambda) = \operatorname{Ind}_{\mathfrak{p}(r)}^{\mathfrak{g}} \mathbb{C}_\lambda$, which is clearly a highest weight module of weight λ.

Since the highest weight vector of $M_r(\lambda)$ satisfies the LAC, Lemma 6.2 implies that $M_r(\lambda)$ lies in $\mathcal{O}_{\mathsf{LA}}$. Notice also that we have surjective maps $M_{r+1}(\lambda) \to M_r(\lambda)$, that given a weight μ in the support the restriction $M_{r+1}(\lambda)_\mu \to M_r(\lambda)_\mu$ is an isomorphism for $r \gg 0$, and that $M(\lambda)$ is the inverse limit of this system. This shows in particular that $\mathcal{O}_{\mathsf{LA}}$ is not closed under inverse limits. With these parabolic Verma modules in hand, we are ready to prove our first result.

Theorem 6.3. *The simple objects in $\mathcal{O}_{\mathsf{LA}}$ are precisely the simple highest weight modules whose highest weight is in $\mathfrak{h}^\circ$.*

Proof. Suppose L is a simple object in $\mathcal{O}_{\mathsf{LA}}$. Since L is $\mathfrak{n}$-torsion it has a highest weight vector v of weight $\lambda \in \mathfrak{h}^\circ$, so $L \cong L(\lambda)$. On the other hand, if λ is an r-eligible weight then $M_r(\lambda)$ lies in $\mathcal{O}_{\mathsf{LA}}$ and so does its unique simple quotient $L(\lambda)$. □

Remark 6.4. A more subtle difference between the definition of $\mathcal{O}_{\mathsf{LA}}$ and that of category $\mathcal{O}$ for finite dimensional reductive Lie algebras is that we ask for M to be $\mathfrak{n}$-torsion and not just locally $\mathfrak{n}$-nilpotent. In the finite-dimensional case, and even in the case $n = 1$, these two conditions are equivalent (see [19, Proposition 4.2] for $\mathfrak{sl}(\infty)$, the proof is the same for $\mathfrak{gl}(\infty)$).

This is no longer true as soon as $n \geq 2$. Indeed, when $n = 2$ the simple *lowest* weight module with lowest weight $\lambda = -\omega^{(1)} + \omega^{(2)}$ is the limit of the simple finite dimensional lowest weight modules $\widetilde{L}(\lambda|_k)$, where each embeds in the next by sending the lowest weight vector to the lowest weight vector. Thus $\widetilde{L}(\lambda) = \varinjlim \widetilde{L}(\lambda|_k)$ is generated by a weight vector satisfying the LAC, and the construction shows that $\widetilde{L}(\lambda)$ is locally $\mathfrak{n}$-nilpotent, but not $\mathfrak{n}$-torsion. Notice that this module can not be a highest weight module: indeed, if it had a highest weight vector v then v would belong to $\widetilde{L}(\lambda|_k)$ for all $k \gg 0$ and be a highest weight vector, but in this construction the image of a highest weight vector of $\widetilde{L}(\lambda|_k)$ in $\widetilde{L}(\lambda|_{k+1})$ is never a highest weight vector.

6.3. **Highest weight modules in $\mathcal{O}_{\mathsf{LA}}$.** As a consequence of Lemma 6.2 a highest weight module M in $\mathcal{O}_{\mathsf{LA}}$ has finite Jordan–Hölder multiplicities, and furthermore if the highest weight vector generates a 1-dimensional $\mathfrak{l}[r]$-module then M lies in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$. We will now show that highest weight modules have finite length.

Recall from Section 3.6 that we denote by $\mathfrak{s}$ and $\mathfrak{m}$ the subalgebras of $\mathfrak{g}$ spanned by root spaces corresponding to finite and positive infinite roots, respectively, and that $\mathfrak{s} = \mathfrak{g}_0^\psi$ and $\mathfrak{m} = \mathfrak{g}_{>0}^\psi$. If M is an object of $\mathcal{O}_{\mathsf{LA}}$ we can see it as a $\mathbb{Z}$-graded module through the map ψ, and each homogeneous component is an $\mathfrak{s}$-module. If the grading on M is right-bounded, for example if M is finitely generated, we will denote by M^+ the top nonzero homogeneous component.

We have also set $\mathfrak{q} = \mathfrak{s} \oplus \mathfrak{m}$. An $\mathfrak{s}$-module can be inflated into a $\mathfrak{q}$-module by imposing a trivial $\mathfrak{m}$-action. The following result shows that the problem of computing Jordan–Hölder multiplicities of a highest weight module in $\mathcal{O}_{\mathsf{LA}}$ reduces to computing Jordan–Hölder multiplicities of highest weight $\mathfrak{s}$-modules in $\overline{\mathcal{O}}_{\mathfrak{s}}$, which were discussed in Section 5.1. This is very useful since weight components of finite length $\mathfrak{s}$-modules are finite dimensional, and hence characters can be given in terms of these dimensions. To formulate our results we recall the notation $m(\lambda, \mu) = [M_{\mathfrak{s}}(\lambda) : L_{\mathfrak{s}}(\mu)]$, which is the asymptotic value of the corresponding Kazhdan–Lusztig coefficient.

Proposition 6.5. *Let $M \in \mathcal{O}_{\mathsf{LA}}$ be a highest weight module of highest weight λ.*

(i) If $N \subset M$ is a nontrivial module then $N^+ = N \cap M^+$.
(ii) M is simple if and only if M^+ is a simple $\mathfrak{s}$-module.
(iii) If μ is an eligible weight and $[M : L(\mu)] \neq 0$ then $\mu <_{\mathsf{fin}} \lambda$ and $[M : L(\mu)] = [M^+ : L_{\mathfrak{s}}(\mu)] \leq m(\lambda, \mu)$. In particular M has finite length.

Proof. (i). Denote by v the highest weight vector of M. If N is a nontrivial submodule of M then it contains a highest weight vector, say w of weight μ. For each $r \geq 0$ denote by M_r the $\mathfrak{g}_r$-module generated by v. Then M_r is a highest weight $\mathfrak{g}_r$-module and for r large enough $w \in M_r$, and it is a highest weight vector. Thus $\mu|_r$ and $\lambda|_r$ are linked for all large r. Lemma 3.1 tells us that $\mu <_{\mathsf{fin}} \lambda$, and so $w \in N \cap M^+ \neq \emptyset$ and $N^+ = N \cap M^+$. This proves (i).

(ii). If M is simple then M^+ is a $U(\mathfrak{g})_0^\psi$ simple module. Using the PBW theorem we have a decomposition $U(\mathfrak{g})_0^\psi = \bigoplus_{k \in \mathbb{N}} U(\overline{\mathfrak{m}})_{-k}^\psi U(\mathfrak{s}) U(\mathfrak{m})_k^\psi$, and since $\mathfrak{m}$ acts trivially on M^+ it follows that M^+ is a simple $U(\mathfrak{s})$-module. Conversely, if M^+ is simple then any nonzero submodule $N \subset M$ must have $N^+ = M^+$ by (i), so it must contain the highest weight vector. Thus $N = M$ and M is simple.

(iii). We have already seen that $\mu <_{\mathsf{fin}} \lambda$. Since M^+ is a highest weight module in $\overline{\mathcal{O}}_{\mathfrak{s}}$, it has a finite Jordan–Hölder filtration $\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = M^+$. Set $M_i = U(\mathfrak{g}) F_i$ and $\overline{M}_i = M_i / M_{i-1}$. Then $\overline{M}_i^+ \cong F_i / F_{i-1}$ which is simple, so by the previous item $\overline{M}_i$ is also simple. It follows that the M_i are a Jordan–Hölder filtration of M, and so M has finite length. It also follows from this that $[M : L(\mu)] = [M^+ : L_{\mathfrak{s}}(\mu)]$. ☐

This in particular implies that parabolic Verma modules have finite length. We now compute their Jordan–Hölder multiplicities in terms of Kazhdan–Lusztig multiplicities of $\mathfrak{g}_k$.

Corollary 6.6. *Let $\lambda, \mu \in \mathfrak{h}^\circ$ with λ r-eligible. If $L(\mu)$ is a simple constituent of $M_r(\lambda)$ then the following hold.*

(i) μ lies in the dot orbit of λ by $\mathcal{W}(\mathfrak{s}_{r+1})$.
(ii) If μ is r-eligible then $[M_r(\lambda) : L(\mu)] = m(\lambda, \mu)$.

Proof. We have already seen in Proposition 6.5 that μ and λ must be linked. Since $M_r(\lambda)$ is an object of $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$ so is $L(\mu)$, and in particular $\mu|_r$ must be the highest weight of a highest weight module in $\mathbb{T}_{\mathfrak{l}[r]}$. Thus (λ, α) and (μ, α) are positive for any finite root α of $\mathfrak{l}[r]$, and Lemma 3.1 implies that $\mu = \sigma \cdot \lambda$ for $\sigma \in W(\mathfrak{s}_{r+1})$.

We also know from Proposition 6.5 that $[M_r(\lambda) : L(\mu)] = [M_r(\lambda)^+ : L_{\mathfrak{s}}(\mu)]$. The surjective map of $\mathfrak{s}$-modules $M_{\mathfrak{s}}(\lambda) \to M_r(\lambda)^+$ restricts to a bijection of the weight components $\succeq \mu$, and so

$$\begin{aligned}\dim M_r(\lambda)^+_\mu &= \sum_{\nu \succeq \mu} [M_r(\lambda)^+ : L_{\mathfrak{s}}(\nu)] \dim L_{\mathfrak{s}}(\nu)_\mu \\ &= \dim M_{\mathfrak{s}}(\lambda)_\mu = \sum_{\nu \succeq \mu} m(\lambda, \nu) \dim L_{\mathfrak{s}}(\nu)_\mu\end{aligned}$$

Thus $[M_r(\lambda)^+ : L_{\mathfrak{s}}(\mu)] = m(\lambda, \mu)$. ☐

6.4. **The structure of a general object in $\mathcal{O}_{\mathsf{LA}}$.** We now turn to more general objects of $\mathcal{O}_{\mathsf{LA}}$. First we show that finitely generated objects in $\mathcal{O}_{\mathsf{LA}}$ must have finite length.

Proposition 6.7. *Let M be an object of $\mathcal{O}_{\mathsf{LA}}$. The following are equivalent.*

(a) M is finitely generated.
(b) M has a finite filtration whose layers are highest weight modules.
(c) M has finite length.

Proof. We will first show that (a) $\Rightarrow$ (b) for M cyclic. Suppose M is generated by a weight vector v of weight λ such that $\mathfrak{l}[r]'v = 0$. As seen in Lemma 6.2 M is in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$, so we can proceed with the proof by induction on the length of the $\mathfrak{l}[r]$-module spanned by $M_{\succeq\lambda}$. This submodule contains a highest weight vector, say w, and we set $N = U(\mathfrak{g})w$. Clearly N is a highest weight module and M/N has a filtration by highest weight modules by hypothesis, so the same holds for M. A standard induction on the number of generators now shows that (a) $\Rightarrow$ (b) in general. The case (b) $\Rightarrow$ (c) follows from Corollary 6.6, and (c) $\Rightarrow$ (a) is obvious. ☐

We now give a general approach to compute the Jordan–Hölder multiplicities of an arbitrary object of $\mathcal{O}_{\mathsf{LA}}$. We will use the following tool.

Definition 6.8. Let $M \in \mathcal{O}_{\mathsf{LA}}$. For each $i \in \mathbb{Z}$ we set $\mathcal{F}_i M$ to be the submodule of M generated by $\bigoplus_{j \geq i} M_j^\psi$. The family $\{\mathcal{F}_i M \mid i \in \mathbb{Z}\}$ is the *ψ-filtration* of M. We also define $\mathcal{S}_i M$ to be the $\mathfrak{s}$-module $(\mathcal{F}_i M / \mathcal{F}_{i-1} M)^+$, which is the top component of the corresponding layer of the filtration.

By construction the layers of the ψ-filtration of any module are generated by their top degree component, and hence their multiplicities can be computed using Proposition 6.5.

Proposition 6.9. *Let M be an object in $\mathcal{O}_{\mathsf{LA}}$ with LCS and let λ be an eligible weight with $\psi(\lambda) = p$. Then $[M : L(\lambda)] = [\mathcal{S}_p M : L_{\mathfrak{s}}(\lambda)]$.*

Proof of Proposition 6.9. It is enough to prove the result for finitely generated objects, and by Proposition 6.7 it is enough to prove it for finite length objects. Let M be a finite length object, and take $q \in \mathbb{Z}$ such that $M^+ = M^\psi_q$. A simple induction on $p - q$ shows that $[\mathcal{S}_p M : L_{\mathfrak{s}}(\lambda)]$ is an additive function on M, with the base case a consequence of Proposition 6.5. Since both $[M : L(\lambda)]$ and $[\mathcal{S}_p M : L_{\mathfrak{s}}(\lambda)]$ are additive functions on M it is enough to show they are equal when M is simple, which again follows from Proposition 6.5. $\square$

6.5. **Categorical properties of $\mathcal{O}_{\mathsf{LA}}$.** We now focus on the general categorical properties of $\mathcal{O}_{\mathsf{LA}}$. Let M be a $\mathfrak{h}$-semisimple $\mathfrak{g}$-module. By Lemma 6.2 the submodule spanned by all its $\mathfrak{n}$-torsion vectors satisfying the large annihilator condition is an object of $\mathcal{O}_{\mathsf{LA}}$, and is in fact the largest submodule of M lying in $\mathcal{O}_{\mathsf{LA}}$. We thus have a diagram of functors, where each arrow going from left to right is an embedding of categories and each arrow going from right to left is the corresponding left adjoint

$$\mathcal{O}^{\mathfrak{l}}_{\mathsf{LA}}\mathfrak{g} \underset{\longrightarrow}{\xleftarrow{\Gamma_{\mathfrak{n}}}} (\mathfrak{g}, \mathfrak{h})\text{-Mod}^{\mathfrak{l}}_{\mathsf{LA}} \underset{\longrightarrow}{\xleftarrow{\Phi}} (\mathfrak{g}, \mathfrak{h})\text{-Mod} \underset{\longrightarrow}{\xleftarrow{\Gamma_{\mathfrak{h}}}} \mathfrak{g}\text{-Mod}$$

It follows that we have a functor $\pi_{\mathsf{LA}} = \Gamma_{\mathfrak{n}} \circ \Phi = \Phi \circ \Gamma_{\mathfrak{n}} : (\mathfrak{g}, \mathfrak{h})\text{-Mod} \to \mathcal{O}_{\mathsf{LA}}$, which is right adjoint to the exact inclusion functor $\mathcal{O}_{\mathsf{LA}} \to (\mathfrak{g}, \mathfrak{h})\text{-Mod}$. In particular it preserves direct limits and sends injectives to injectives.

Recall that an abelian category $\mathcal{A}$ is locally artinian if every object is the limit of its finite length objects. Also, $\mathcal{A}$ has the Grothendieck property if it has direct limits, and for every object M, every subobject $N \subset M$ and every directed family of subobjects $(A_\alpha)_{\alpha \in I}$ of M it holds that $N \cap \varinjlim A_\alpha = \varinjlim N \cap A_\alpha$. The following result is an easy consequence of the properties of π_{LA}.

Theorem 6.10. *Category $\mathcal{O}_{\mathsf{LA}}$ is a locally artinian category with direct limits, enough injective objects, and the Grothendieck property.*

7. Category $\mathcal{O}_{\mathsf{LA}}$: Standard objects

In this section we will introduce the standard objects of $\mathcal{O}_{\mathsf{LA}}$. We then compute their simple multiplicities through their ψ-filtrations.

7.1. **Large-annihilator dual Verma modules.** We now introduce the modules that will play the role of standard objects on $\mathcal{O}_{\mathsf{LA}}$. In the finite dimensional case this role is played by the semisimple duals of Verma modules, so it is natural to consider the “best approximation” to these modules in $\mathcal{O}_{\mathsf{LA}}$.

Definition 7.1. For every $\lambda \in \mathfrak{h}^\circ$ we set $A(\lambda) = M(\lambda)^{\#}$.

The module $A(\lambda)$ can be hard to grasp. For example, it is not clear at first sight that it lies in $\mathcal{O}_{\mathsf{LA}}$. As a first approximation, set $A_r(\lambda)$ to be the large annihilator dual of the parabolic Verma $M_r(\lambda)$. The Verma module $M(\lambda)$ is the inverse limit of the $M_r(\lambda)$, and since large annihilator duality sends inverse limits to direct limits, $A(\lambda) = \varinjlim A_r(\lambda)$. Also, the natural maps $A_r(\lambda) \to A(\lambda)$ are injective, and for any μ in the support of $A(\lambda)$ the map $A_r(\lambda)_{\succeq \mu} \to A(\lambda)_{\succeq \mu}$ is an isomorphism for and $r \gg 0$; this follows from the dual facts for $M_r(\lambda)$ and $M(\lambda)$. This approach allows us to prove that the $A(\lambda)$ are indeed in $\mathcal{O}_{\mathsf{LA}}$.

Proposition 7.2. *For every eligible weight λ and every $r \geq 0$ the modules $A(\lambda), A_r(\lambda)$ lie in $\mathcal{O}_{\mathsf{LA}}$ and have LCS. Furthermore, we have $[A(\lambda) : L(\mu)] = [A_r(\lambda) : L(\mu)]$ for $r \gg 0$.*

Proof of Proposition 7.2. By definition the space $M_r(\lambda)^\#$ is an $\mathfrak{h}$-semisimple module satisfying the LAC. Also it is isomorphic to $\mathbf{S}^\bullet(\bar{\mathfrak{u}}(r))^\# \otimes \mathbb{C}_\lambda$ as $\mathfrak{l}[r]^+$-module, and hence lies in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$ by Proposition 5.12, and by Lemma 5.9 it has LCS. To see that it is $\mathfrak{n}$-torsion, first observe that it is $\mathfrak{n}(r) = \mathfrak{l}[r] \cap \mathfrak{n}$-torsion by virtue of being in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$. Now recall the map θ from Example 4.6. It induces a right-bounded grading on $A_r(\lambda)$, and since $\mathfrak{g}^\theta_{>0} = \mathfrak{u}(r)$, it follows that this subalgebra acts nilpotently on $A_r(\lambda)$ and hence $A_r(\lambda)$ is $\mathfrak{n} = \mathfrak{n}(r) \ni \mathfrak{u}(r)$-torsion. This completes the proof for $A_r(\lambda)$.

Since $\mathcal{O}_{\mathsf{LA}}$ is closed by direct limits it follows that $A(\lambda)$ belongs to $\mathcal{O}_{\mathsf{LA}}$. Given a weight μ we build an LCS for $A(\lambda)$ at μ as follows: start with an LCS at μ for $A_r(\lambda)$ such that $A_r(\lambda)_\mu \to A(\lambda)_\mu$ is an isomorphism. Applying the natural inclusion of $A_r(\lambda)$ into $A(\lambda)$ to this filtration and adding $A(\lambda)$ at the top of it we obtain the desired LCS. This also shows that the multiplicities coincide as in the statement. ☐

This result shows that $A(\lambda)$ is the projection of $M(\lambda)^\vee$ to $\mathcal{O}_{\mathsf{LA}}$. By Proposition 4.1 we get that $A(\lambda) \cong \Phi(\underline{\mathrm{Coind}}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda) \cong \pi_{\mathsf{LA}}(\underline{\mathrm{Coind}}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda)$. The following theorem shows that the $A(\lambda)$ have the usual properties associated to standard objects in highest weight categories.

Theorem 7.3. *For each $\lambda \in \mathfrak{h}^\circ$ the following hold:*

(i) $A(\lambda)$ is indecomposable and $\operatorname{soc} A(\lambda) \cong L(\lambda)$.

(ii) The composition factors of $A(\lambda)/L(\lambda)$ are of the form $L(\mu)$ with $\mu \prec \lambda$.

(iii) For each $\mu \in \mathfrak{h}^\circ$ we have $\dim \operatorname{Hom}_{\mathcal{O}_{\mathsf{LA}}}(A(\mu), A(\lambda)) < \infty$.

Proof. Let $\mu \in \mathfrak{h}^\circ$. Since π_{LA} is right adjoint to the inclusion of $\mathcal{O}_{\mathsf{LA}}$ in $(\mathfrak{g}, \mathfrak{h})$-Mod there are isomorphisms

$$\operatorname{Hom}_{\mathcal{O}_{\mathsf{LA}}}(L(\mu), A(\lambda)) \cong \operatorname{Hom}_{\mathfrak{g},\mathfrak{h}}\Big(L(\mu),\ \underline{\mathrm{Coind}}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda\Big) \cong \operatorname{Hom}_{\bar{\mathfrak{b}},\mathfrak{h}}(L(\mu), \mathbb{C}_\lambda),$$

and since $L(\mu)$ has $\mathbb{C}_\mu$ as its unique simple quotient as $\bar{\mathfrak{b}}$-module, it follows that the last Hom space has dimension 1 when $\lambda = \mu$ and zero otherwise. Hence $L(\lambda)$ is the socle of $A(\lambda)$, which implies (i). Since the support of $A(\lambda)/L(\lambda)$ is contained in the set of weights $\mu \prec \lambda$, (ii) follows.

Since $A(\mu)$ has a local composition series at λ, there exists a finite length $\mathfrak{g}$-module $N \subset A(\mu)$ such that $(A(\mu)/N)_\lambda = 0$. Now consider the following exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathsf{LA}}}(A(\mu)/N, A(\lambda)) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathsf{LA}}}(A(\mu), A(\lambda)) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathsf{LA}}}(N, A(\lambda)).$$

Since any nonzero map to $A(\lambda)$ must contain $L(\lambda)$ in its image, the first Hom-space in the long exact sequence is zero and the map $\operatorname{Hom}_{\mathcal{O}_{\mathsf{LA}}}(A(\mu), A(\lambda)) \to \operatorname{Hom}_{\mathcal{O}_{\mathsf{LA}}}(N, A(\lambda))$ is injective. Now a simple induction shows that $\dim \operatorname{Hom}_{\mathcal{O}_{\mathsf{LA}}}(X, A(\lambda))$ is finite for any X of finite length. This proves (iii). ☐

From now on we will refer to the $A(\lambda)$ as *standard* modules of $\mathcal{O}_{\mathsf{LA}}$. We point out that the order $\prec$ on $\mathfrak{h}^\circ$ is not interval-finite, and so it is not yet clear that the $A(\lambda)$ are standard in the sense of highest weight categories. In the coming subsections we will compute the Jordan–Hölder multiplicities of the standard modules explicitly, and in the process find an interval-finite order for its simple constituents.

7.2. **The submodules $T_r(\lambda)$.** Our computation of the simple multiplicities of the module $A(\lambda)$ is rather long and technical. The idea is to find an exhaustion of $A(\lambda)$, compute the ψ-filtrations of the modules in this exhaustion, and from this obtain the layers of the ψ-filtration of $A(\lambda)$.

Let us start by introducing the exhaustion. Set $T_r(\lambda) = \Phi_r(M_r(\lambda)^\vee)$; this is clearly a $\mathfrak{g}_r$-module, and since it is spanned by $\mathfrak{h}$-semisimple vectors it is in fact a $\mathfrak{g}_r^+ = \mathfrak{g}_r + \mathfrak{h}$-module. Through ψ we can see it as a $\mathbb{Z}$-graded module.

Lemma 7.4. *For each $r \gg 0$ there are injective maps of $\mathbb{Z}$-graded $\mathfrak{g}_r^+$-modules $T_r(\lambda) \hookrightarrow T_{r+1}(\lambda)$ and $A(\lambda) \cong \varinjlim T_r(\lambda)$ as $\mathfrak{g}$-modules.*

Proof. Identifying $M_r(\lambda)^\vee$ with its image inside $M(\lambda)^\vee$ it is clear that $M_r(\lambda)^\vee \subset M_{r+1}(\lambda)^\vee$ and that

$$T_r(\lambda) = \Phi_r\big(M_r(\lambda)^\vee\big) \subset \Phi_{r+1}\big(M_r(\lambda)^\vee\big) \subset \Phi_{r+1}\big(M_{r+1}(\lambda)^\vee\big) = T_{r+1}(\lambda).$$

The desired maps are given by the inclusions. Now if $m \in \Phi_r(M(\lambda)^\vee)$ then there exists $s \geq r$ such that $m \in \Phi_r(M_s(\lambda)^\vee) \subset T_s(\lambda)$, so $A(\lambda) = \bigcup_{r\geq 0} T_r(\lambda)$. $\square$

The definition of ψ-filtrations given in Definition 6.8 is easily adapted to ψ-graded $\mathfrak{g}_r^+$-modules. The fact that $A(\lambda)$ is the direct limit of the $T_r(\lambda)$ in the category of ψ-graded $\mathfrak{g}_r^+$-modules implies that the p^{th} module in the ψ-filtration of $A(\lambda)$ is the limit of the p^{th} module in the ψ filtrations of the $T_r(\lambda)$. Furthermore, since direct limits are exact we can recover the $\mathfrak{s}$-module $\mathcal{S}_p(A(\lambda))$ as the direct limit of the $\mathfrak{s}_r$-modules $\mathcal{S}_p(T_r(\lambda))$. This will allow us to compute their characters and, using Proposition 6.9, the simple multiplicities of $A(\lambda)$.

Remark 7.5. Notice that it is not true that the Jordan–Hölder multiplicities of the $T_r(\lambda)$ can be recovered from the multiplicities of the $\mathfrak{s}_r$-modules $\mathcal{S}_p(T_r(\lambda))$. Indeed, Proposition 6.9 follows from Proposition 6.5, and the analogous statement is clearly false for $\mathfrak{g}_r$.

Set $R_i = \mathfrak{g}^\psi_{\leq -i}$, i.e. the subspace of $\mathfrak{g}$ spanned by all root-spaces corresponding to roots α with $\psi(\alpha) \leq -i$, and set $R_{i,r} = R_i \cap \mathfrak{g}_r$. We see R_i as $\mathfrak{b}$-module through the isomorphism $R_i \cong \mathfrak{g}/\mathfrak{g}^\psi_{>-i}$. We set $\mathcal{R} = \bigoplus_{k=1}^n 2^{k-1} R_k$ and $\mathcal{R}_r = \bigoplus_{k=1}^n 2^{k-1} R_{k,r}$.

Lemma 7.6. *For each $r \geq 0$ and each $p \leq \psi(\lambda)$ there exist isomorphisms of $\mathfrak{s}_r$-modules*

$$\mathcal{S}_p T_r(\lambda) \cong \left(\operatorname{Ind}^{\mathfrak{s}_r}_{\mathfrak{s}_r\cap\mathfrak{b}} \mathbf{S}^\bullet(\mathcal{R}_r)_{p-\psi(\lambda)} \otimes \mathbb{C}_\lambda\right)^\vee \cong \underline{\operatorname{Coind}}^{\mathfrak{s}_r}_{\mathfrak{s}_r\cap\bar{\mathfrak{b}}_r} \mathbf{S}^\bullet(\mathcal{R}_r^\vee)_{p-\psi(\lambda)} \otimes \mathbb{C}_\lambda$$

The proof of this statement is quite technical and not particularly illuminating, so we postpone it until the end of this section. It is, however the main step to compute the Jordan–Hölder multiplicities of standard modules.

Theorem 7.7. *Let λ and μ be eligible weights. Then $[A(\lambda) : L(\mu)]$ is nonzero if and only if there exist $\nu \in \mathfrak{h}^\circ$ with $\psi(\nu) \in \mathbb{Z}_{<0}$ and $\sigma \in \mathcal{W}^\circ$ such that $\mu = \sigma \cdot (\lambda + \nu)$. In that case the multiplicity is given by*

$$[A(\lambda) : L(\mu)] = m(\lambda + \nu, \mu) \dim \mathbf{S}^\bullet(\mathcal{R}^\vee)_\nu.$$

Proof. Since $A(\lambda)^\psi_p = \bigcup_{r\geq 0} T_r(\lambda)^\psi_p$, it follows that $\mathcal{F}_p A(\lambda) = \varinjlim \mathcal{F}_p T_r(\lambda)$. Exactness of direct limits of vector spaces implies that

$$\frac{\mathcal{F}_p A(\lambda)}{\mathcal{F}_{p-1} A(\lambda)} \cong \varinjlim \frac{\mathcal{F}_p T_r(\lambda)}{\mathcal{F}_{p-1} T_r(\lambda)}$$

and so taking top degree components we get $\mathcal{S}_p A(\lambda) = \varinjlim \mathcal{S}_p T_r(\lambda)$. Thus

$$[A(\lambda) : L(\mu)] = [\mathcal{S}_p A(\lambda) : L_{\mathfrak{s}}(\mu)] = \lim_{r\to\infty} [\mathcal{S}_p T_r(\lambda) : L_{\mathfrak{s}_r}(\lambda)]$$

 https://doi.org/10.5802/art.42

for large r.

Using Lemma 7.6 and standard results on category $\mathcal{O}$ for the reductive algebra $\mathfrak{s}_r$, see for example [14, Section 3.6, Theorem], $\mathcal{S}_p T_r(\lambda)$ has a filtration by dual Verma modules of the form $M_{\mathfrak{s}_r}(\lambda+\nu)^\vee$, and each of these modules appears with multiplicity $\dim \mathbf{S}^\bullet(\mathcal{R}^\vee)_\nu$. Thus

$$[\mathcal{S}_p T_r(\lambda) : L_{\mathfrak{s}_r}(\mu|_r)] = [M_{\mathfrak{s}_r}(\lambda+\nu|_r) : L_{\mathfrak{s}_r}(\mu|_r)] \dim \mathbf{S}^\bullet(\mathcal{R}_r^\vee)_\nu,$$

and for large r this equals $m(\lambda+\nu,\mu)\dim \mathbf{S}^\bullet(\mathcal{R}^\vee)_\nu$. The support of $\mathbf{S}^\bullet(\mathcal{R}^\vee)$ is precisely the set of $\nu \in \mathfrak{h}^\circ$ with $\psi(\nu) \in \mathbb{Z}_{<0}$, and $m(\lambda+\nu,\mu)$ is nonzero if and only if $\mu = \sigma\cdot(\lambda+\nu)$ for some $\sigma \in \mathcal{W}^\circ$, which proves the result. □

7.3. **An interval finite order on $\mathfrak{h}^\circ$.** We are now ready to show that there is an interval finite order for the simple constituents in standard objects of $\mathcal{O}_{\mathsf{LA}}$. In view of Theorem 7.7 there is only one reasonable choice.

Definition 7.8. Let $\lambda,\mu \in \mathfrak{h}^\circ$. We write $\mu <_{\mathsf{inf}} \lambda$ if there exist a weight ν with $\psi(\nu) \in \mathbb{Z}_{<0}$ and $\sigma \in \mathcal{W}(\mathfrak{s})$ such that $\mu = \sigma\cdot(\lambda+\nu) \preceq \lambda$.

Lemma 7.9. *The order $<_{\mathsf{inf}}$ is interval finite.*

Proof. Let us write $\mu \lhd \lambda$ if either:

(i) there is a simple root α such that $\mu = s_\alpha\cdot\lambda \prec \lambda$ or
(ii) $\mu = \lambda+\nu$ with ν a root such that $\psi(\nu) < 0$.

Notice that $\mu <_{\mathsf{inf}} \lambda$ implies $\mu \prec \lambda$. Since the support of $\mathcal{R}$ is the $\mathbb{Z}_{>0}$-span of the space of roots of $\mathfrak{g}^\psi_{<0}$, it follows that $\lhd$ is a sub-relation of $<_{\mathsf{inf}}$, and in fact this order is the reflexive transitive closure of $\lhd$. To prove the statement, it is enough to show that there are only finitely many weights γ such that $\mu \lhd \gamma <_{\mathsf{inf}} \lambda$, and that there is a global bound on the length of chains of the form $\mu \lhd \lambda_1 \lhd \cdots \lhd \lambda_l \lhd \lambda$.

If we are in case (i) then $\gamma = s_\alpha\cdot\mu$, and since $\gamma \prec \lambda$ it follows that $\alpha \in \mathcal{W}(\mathfrak{s}_r)$, so there are only finitely many options in this case. Notice also that $(\mu,\rho_{r+1}) < (\gamma,\rho_{r+1}) \leq (\lambda,\rho_{r+1})$, where ρ_{r+1} is the sum of all positive roots of $\mathfrak{s}_{r+1}$. Thus in a chain as above there can be at most $(\lambda-\mu,\rho_{r+1})$ inequalities of type (i).

If we are in case (ii), then $\lambda-\mu \succ 0$ and $\gamma = \mu-\nu$ for some ψ-negative root ν. Since $\lambda-\gamma = \lambda-\mu-\nu \succeq 0$, it is enough to check that there are only finitely many ν such that the last inequality holds. Now since λ and μ are r-eligible, $(\lambda-\mu,\alpha) = 0$ for all roots outside of the root space of $\mathfrak{g}_{r+1}$. This means that ν must be a negative root of $\mathfrak{g}_{r+1}$, of which there are finitely many. Notice also that in this case $\psi(\mu) < \psi(\gamma) \leq \psi(\lambda)$, and hence in any chain as above there are at most $\psi(\lambda-\mu)$ inequalities of type (ii). Thus any chain is of length at most $\psi(\lambda-\mu)+(\lambda-\mu,\rho_{r+1})$ and we are done. □

8. Category $\mathcal{O}_{\mathsf{LA}}$: injective objects and further categorical properties

8.1. **Large annihilator coinduction.** In previous sections we have used inflation functors to turn objects in $\overline{\mathcal{O}}_{\mathfrak{s}}$ into $\mathfrak{q}$-modules and then used induction to construct objects in $\mathcal{O}_{\mathsf{LA}}$. In this section we will use a dual construction. Recall that $I_{\mathfrak{s}}(\lambda)$ denotes the injective envelope of $L_{\mathfrak{s}}(\lambda)$ in $\overline{\mathcal{O}}_{\mathfrak{s}}$.

Definition 8.1. We denote by $\mathcal{C} : \overline{\mathcal{O}}_{\mathfrak{s}} \to (\mathfrak{g},\mathfrak{h})$-Mod the functor $\mathcal{C} = \Phi \circ \underline{\mathrm{Coind}}_{\overline{\mathfrak{q}}}^{\mathfrak{g}} \circ \mathcal{I}_{\mathfrak{s}}^{\overline{\mathfrak{q}}}$. For each $\lambda \in \mathfrak{h}^\circ$ we set $I(\lambda) = \mathcal{C}(I_{\mathfrak{s}}(\lambda))$.

Notice that $\mathcal{C}$ does not automatically fall in $\mathcal{O}_{\mathsf{LA}}$. However from the definition and Proposition 4.1 it follows that $\mathcal{C}(M(\lambda)^\vee) = A(\lambda)$, so standard objects from the category $\overline{\mathcal{O}}_{\mathfrak{s}}$ are sent to standard objects of $\mathcal{O}_{\mathsf{LA}}$. We will show that the same holds for injective modules, and that in fact the standard filtrations of injective modules in $\overline{\mathcal{O}}_{\mathfrak{s}}$ also lift to standard filtrations in $\mathcal{O}_{\mathsf{LA}}$.

8.2. **Injective objects.** We now begin our study of injective envelopes in $\mathcal{O}_{\mathsf{LA}}$. For each eligible weight λ set $J(\lambda) = \underline{\mathrm{Coind}}_{\mathfrak{q}}^{\mathfrak{g}} \mathcal{I}_{\mathfrak{s}}^{\bar{\mathfrak{q}}} I_{\mathfrak{s}}(\lambda)$. Just as we did for standard modules, we will show that $\Phi(J(\lambda)) = \pi_{\mathsf{LA}}(J(\lambda))$, and that this is the injective envelope of the simple module $L(\lambda)$.

Lemma 8.2. *Let $\lambda \in \mathfrak{h}^\circ$ and let $I_{\mathfrak{s}}(\lambda)$ be the injective envelope of $L_{\mathfrak{s}}(\lambda)$ in $\overline{\mathcal{O}}_{\mathfrak{s}}$. Then for any object M in $\mathcal{O}_{\mathsf{LA}}$ we have $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, J(\lambda)) = 0$.*

Proof. It is enough to prove the result for $M = L(\mu)$ with μ an eligible weight. First we point out that the simplicity of $L(\mu)$ implies that $L(\mu)/\overline{\mathfrak{m}}L(\mu) \cong L(\mu)^+$, which by Proposition 6.5 is isomorphic to $L_{\mathfrak{s}}(\mu)$. Now since $I_{\mathfrak{s}}(\lambda)$ is an injective object it is acyclic for the functor $\underline{\mathrm{Coind}}_{\mathfrak{q}}^{\mathfrak{g}} \circ \mathcal{I}_{\mathfrak{s}}^{\bar{\mathfrak{q}}}$ so by basic homological algebra

$$\begin{aligned}\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(L(\mu), J(\lambda)) &\cong \mathrm{Ext}^1_{\mathfrak{s},\mathfrak{h}}(U(\mathfrak{s}) \otimes_{\bar{\mathfrak{q}}} L(\mu), I_{\mathfrak{s}}(\lambda))\\ &\cong \mathrm{Ext}^1_{\mathfrak{s},\mathfrak{h}}(L(\mu)/\overline{\mathfrak{m}}L(\mu), I_{\mathfrak{s}}(\lambda)) \cong \mathrm{Ext}^1_{\mathfrak{s},\mathfrak{h}}(L_{\mathfrak{s}}(\mu), I_{\mathfrak{s}}(\lambda))\end{aligned}$$

Since $\overline{\mathcal{O}}_{\mathfrak{s}}$ is closed under semisimple extensions, this last Ext-space must be 0. ☐

Set $\omega = \omega_1 + \cdots + \omega_n$. Then for any $a \in \mathbb{C}$ the weight $a\omega$ is dominant. By Theorem 5.3 $I_{\mathfrak{s}}(\omega) = M_{\mathfrak{s}}(\omega)^\vee$ and hence $J(\omega) = M(\omega)^\vee$.

Lemma 8.3. *Let M be a weight $\mathfrak{g}$-module. Suppose M has a finite filtration $F_1M \subset F_2M \subset \cdots \subset F_nM$ such that its i^{th} layer is isomorphic to $M(\lambda_i)^\vee$ for $\lambda_i \in \mathfrak{h}^\circ$. Then $\Phi(M) = \pi_{\mathsf{LA}}(M)$, and $\pi_{\mathsf{LA}}(F_iM)/\pi_{\mathsf{LA}}(F_{i-1}M) \cong A(\lambda_i)$.*

Proof. Recall that $\pi_{\mathsf{LA}} = \Gamma_{\mathfrak{n}} \circ \Phi$. If we show that the dual Verma modules $M(\lambda)^\vee$ are acyclic for both Φ and π_{LA} then the result follows from induction on the length of the filtration. Indeed, the base case is the fact that $A(\lambda) = \Phi(M(\lambda)^\vee) = \pi_{\mathsf{LA}}(M(\lambda)^\vee)$, and the inductive step follows from a standard application of the five lemma.

Let us start by showing that $M(\lambda)^\vee$ is Φ-acyclic. Since λ is admissible, it can be written as $\widetilde{\lambda} + a_1\omega_1 + \cdots + a_n\omega_n$ with $(a_1, \ldots, a_n)$ the level of λ. Fix $r \gg 0$ so λ is r-admissible and set $M_k = M_{\mathfrak{l}^{(k)}[r]}(a_k\omega_k[k])$. Using Proposition 4.1 we have isomorphisms

$$M(\lambda) \cong \mathrm{Ind}_{\mathfrak{p}(r)}^{\mathfrak{g}} \mathcal{I}_{\mathfrak{l}[r]}^{\mathfrak{p}(r)} M_1 \boxtimes \cdots \boxtimes M_n;$$

$$M(\lambda)^\vee \cong \underline{\mathrm{Coind}}_{\overline{\mathfrak{p}(\mathfrak{r})}}^{\mathfrak{g}} \mathcal{I}_{\mathfrak{l}[r]}^{\overline{\mathfrak{p}(\mathfrak{r})}} M_1^\vee \boxtimes \cdots \boxtimes M_n^\vee.$$

Recall that Φ can be written as the direct limit of functors Φ_r taking $\mathfrak{l}[r]'$-invariant submodules, and that there are natural isomorphisms

$$\Phi_r \cong \bigoplus_{\mu} \mathrm{Hom}_{\mathfrak{g},\mathfrak{h}}\Big(U(\mathfrak{g}) \otimes_{\mathfrak{l}[r]^+} \mathbb{C}_\mu, -\Big)$$

where the sum is taken over all r-eligible weights $\mu \in \mathfrak{h}^\circ$. Since derived functors commute with direct limits, and in particular with direct sums, it is enough to show that $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(U(\mathfrak{g}) \otimes_{\mathfrak{l}[r]^+} \mathbb{C}_\mu, M(\lambda)^\vee) = 0$.

Using that inflation functors are exact and semisimple coinduction is exact on semisimple duals, we get

$$\begin{aligned}\operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}&\Big(U(\mathfrak{g})\otimes_{\mathfrak{l}[r]^+}\mathbb{C}_\mu, M(\lambda)^\vee\Big)\\ &\cong \operatorname{Ext}^1_{\overline{\mathfrak{p}(\mathfrak{r})},\mathfrak{h}}\Big(U(\mathfrak{g})\otimes_{\mathfrak{l}[r]^+}\mathbb{C}_\mu, \mathcal{I}^{\overline{\mathfrak{p}(\mathfrak{r})}}_{\mathfrak{l}[r]} M_1^\vee\boxtimes\cdots\boxtimes M_n^\vee\Big)\\ &\cong \operatorname{Ext}^1_{\mathfrak{l}[r]^+,\mathfrak{h}}\Big(U(\mathfrak{l}[r])^+\otimes_{\overline{\mathfrak{p}}(r)} U(\mathfrak{g})\otimes_{\mathfrak{l}[r]^+}\mathbb{C}_\mu, M_1^\vee\boxtimes\cdots\boxtimes M_n^\vee\otimes\mathbb{C}_{\lambda_r}\Big)\\ &\cong \operatorname{Ext}^1_{\mathfrak{l}[r],\mathfrak{h}}(\mathbf{S}^\bullet(\mathfrak{u}(r))\otimes\mathbb{C}_{\mu-\lambda}, M_1^\vee\boxtimes\cdots\boxtimes M_n^\vee).\end{aligned}$$

Now $\mathbf{S}^\bullet(\mathfrak{u}(r))\otimes\mathbb{C}_{\mu-\lambda}$ decomposes as a direct sum of modules of the form $T_1\boxtimes\cdots\boxtimes T_n$ with T_k a tensor $\mathfrak{l}[r]$-module. Thus the Ext-space decomposes as a direct sum of vector spaces of the form

$$\operatorname{Ext}^1_{\mathfrak{l}^{(1)}[r],\mathfrak{h}^{(1)}}(T_1, M_1^\vee)\otimes\cdots\otimes\operatorname{Ext}^n_{\mathfrak{l}^{(n)}[r],\mathfrak{h}^{(n)}}(T_n, M_n^\vee).$$

Finally, M_k is the Verma $\mathfrak{g}(\mathbb{V}^{(k)})[r]$-module with highest weight $a_k\omega_k|_r$. Since $\mathfrak{l}^{(k)}[r]\cong\mathfrak{g}(\mathbb{V})$, we can apply both the special case $n=1$ of both Lemma 8.2 and the observation in the preamble to Lemma 8.3, and conclude that $M_k(a_k\omega_k|_r)^\vee = J(a_k\omega_k|_r)^\vee$ satisfies that $\operatorname{Ext}_{\mathfrak{l}^{(k)}[r],\mathfrak{h}^{(k)}}(T_k, M_k^\vee)=0$. Thus $M(\lambda)^\vee$ is Φ-acyclic.

Recall that $\Phi:(\mathfrak{g},\mathfrak{h})\text{-Mod}\to(\mathfrak{g},\mathfrak{h})\text{-Mod}^{\mathfrak{l}}_{\mathsf{LA}}$ is right adjoint to the natural inclusion functor, and hence sends injective objects to injective objects. It follows that to show that $M(\lambda)^\vee$ is π_{LA}-acyclic it is enough to show that $\Phi(M(\lambda)^\vee)=A(\lambda)$ is a $\Gamma_{\mathfrak{n}}$-acyclic object in $(\mathfrak{g},\mathfrak{h})\text{-Mod}^{\mathfrak{l}}_{\mathsf{LA}}$. To show this we use that the functor $\Gamma_{\mathfrak{n}}:(\mathfrak{g},\mathfrak{h})\text{-Mod}^{\mathfrak{l}}_{\mathsf{LA}}\to\mathcal{O}_{\mathsf{LA}}$ can be represented as

$$\Gamma_{\mathfrak{n}}\cong\varinjlim\bigoplus_{\lambda\in\mathfrak{h}^\circ}\operatorname{Hom}_{\mathfrak{g},\mathfrak{h}}(U(\mathfrak{g})/J_{r,n}\otimes\mathbb{C}_\lambda,-),$$

where $J_{r,n}$ is the ideal of $U(\mathfrak{g})$ generated by $\mathfrak{l}[r]$ and $\mathfrak{u}(r)^n$, and the limit is taken over both r and n. The details of this second argument are similar to the previous one and left to the interested reader. □

We had already observed by a general argument that $\mathcal{O}_{\mathsf{LA}}$ has enough injectives. We can now give a more explicit construction of the injective envelopes of the simple modules, and show that they have standard filtrations.

Theorem 8.4. *Let $\lambda\in\mathfrak{h}^\circ$. The module $I(\lambda)=\Phi(J(\lambda))$ belongs to $\mathcal{O}_{\mathsf{LA}}$ and is the injective envelope for $L(\lambda)$. Furthermore it has a finite filtration whose first layer is $A(\lambda)$ and whose higher layers are isomorphic to $A(\mu)$ with $\mu>_{\mathsf{fin}}\lambda$.*

Proof. Since inflation is exact and semisimple coinduction is exact over semisimple duals, the costandard filtration of $I_{\mathfrak{s}}(\lambda)$ induces a finite filtration over $J(\lambda)$. Furthermore the first layer of this filtration is isomorphic to $M(\lambda)^\vee$, and the higher layers are isomorphic to $M(\mu)^\vee$ for $\mu>_{\mathsf{fin}}\lambda$. Applying Lemma 8.3 it follows that $I(\lambda)=\Phi(J(\lambda))=\pi_{\mathsf{LA}}(J(\lambda))$ and hence belongs to $\mathcal{O}_{\mathsf{LA}}$, and that this module has the required filtration. Finally, reasoning as in the proof of Lemma 8.2 we have isomorphisms

$$\operatorname{Ext}^i_{\mathcal{O}_{\mathsf{LA}}}(L(\mu),I(\lambda))\cong\operatorname{Ext}^i_{\mathfrak{g},\mathfrak{h}}(L(\mu),J(\lambda))\cong\operatorname{Ext}^i_{\mathfrak{s},\mathfrak{h}}(L_{\mathfrak{s}}(\mu),I_{\mathfrak{s}}(\lambda)).$$

Taking $i=0$ and varying μ we see that $L(\lambda)$ is the socle of $I(\lambda)$, and taking $i=1$ it follows that $I(\lambda)$ is injective in $\mathcal{O}_{\mathsf{LA}}$. □

8.3. **Highest weight structure and blocks.** We are now ready to prove the main result of this paper, namely that $\mathcal{O}_{\mathsf{LA}}$ is a highest weight category in the sense of Cline, Parshall and Scott [4].

Theorem 8.5. *Category $\mathcal{O}_{\mathsf{LA}}$ is a highest weight category with indexing set $(\mathfrak{h}^\circ, <_{\mathsf{inf}})$. Its simple modules are the simple highest weight modules $L(\lambda)$ with $\lambda \in \mathfrak{h}^\circ$, standard objects $A(\lambda)$ have infinite length, and the corresponding injective envelopes $I(\lambda)$ have finite standard filtrations.*

Proof. We only need to refer to previously proved results. We have proved that $<_{\mathsf{inf}}$ is an interval-finite order in Lemma 7.9. The categorical properties of $\mathcal{O}_{\mathsf{LA}}$ were summarized in Theorem 6.10. That eligible weights parametrise simple modules was proved in Theorem 6.3, and standard modules exist and have the desired properties by Theorems 7.3 and 7.7. Finally injective modules exist and have finite standard filtrations with the required properties by Theorem 8.4. □

We also have the following analogue of BGG-reciprocity in $\mathcal{O}_{\mathsf{LA}}$.

Corollary 8.6. *Let $\lambda, \mu \in \mathfrak{h}^\circ$. The multiplicity of $A(\mu)$ in any standard filtration of $I(\lambda)$ equals $m(\lambda, \mu)$.*

Proof. It is enough to check the result for one standard filtration, and as shown in the proof of Theorem 8.4 we obtain such a filtration by applying the functor $\mathcal{C}$ to the standard filtration of $I_{\mathfrak{s}}(\lambda)$. By Lemmas 8.2 and 8.3 the multiplicity of $A(\mu)$ in this filtration $I(\lambda)$ coincides with the multiplicity of $M_{\mathfrak{s}}(\mu)^\vee$ in the standard filtration of $I_{\mathfrak{s}}(\lambda)$. By Theorem 5.3 this multiplicity equals $[M_{\mathfrak{s}}(\mu) : L_{\mathfrak{s}}(\lambda)] = m(\lambda, \mu)$. □

We finish with a description of the irreducible blocks of $\mathcal{O}_{\mathsf{LA}}$. Recall that we denote by Λ the root lattice of $\mathfrak{g}$. For each $\kappa \in \mathfrak{h}^\circ/\Lambda$ set $\mathcal{O}_{\mathsf{LA}}(\kappa)$ to be the subcategory formed by those modules in $\mathcal{O}_{\mathsf{LA}}$ whose support is contained in κ. Notice in particular that each class has a fixed level, and so the decomposition $\mathcal{O}_{\mathsf{LA}} = \prod_\kappa \mathcal{O}_{\mathsf{LA}}(\kappa)$ refines the decomposition by levels.

Proposition 8.7. *The blocks $\mathcal{O}_{\mathsf{LA}}(\kappa)$ are indecomposable.*

Proof. The support of $\mathfrak{g}^\psi_{-1} \subset \mathcal{R}$ spans the root lattice. It follows from Theorem 7.7 that $[A(\lambda) : L(\lambda)] = [A(\lambda) : L(\lambda + \nu)]$ for any root ν in the support of $\mathfrak{g}^\psi_{-1}$. Thus whenever two weights λ, μ lie in the same class modulo the root lattice, there is a finite chain of weights $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_m = \mu$ such that $L(\lambda_{i+1})$ is a simple constituent of the indecomposable module $A(\lambda_i)$. This shows that all simple modules in the block $\mathcal{O}_{\mathsf{LA}}(\kappa)$ are linked and hence the block is indecomposable. □

Appendix A. A proof of Lemma 7.6

As promised, we now compute the layers of the ψ-filtration of $T_r(\lambda)$.

A.1. **A result on Schur functors.** We begin by setting some notation for Schur functors and Littlewood–Richardson coefficients. Recall that given two vector spaces V, W and partitions λ, μ we have isomorphisms, natural in both variables,

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) = \bigoplus_\nu c^\nu_{\lambda,\mu} \mathbb{S}_\nu(V), \qquad \mathbb{S}_\lambda(V \oplus W) = \bigoplus_{\alpha,\beta} c^\gamma_{\alpha,\beta} \mathbb{S}_\alpha(V) \otimes \mathbb{S}_\beta(W).$$

Given an m-tuple of partitions $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and an m-tuple of vector spaces $U_1, \ldots, U_m$ we set $\mathbb{S}_{\boldsymbol{\alpha}}(U_1, \ldots, U_m) = \mathbb{S}_{\alpha_1}(U_1) \otimes \cdots \otimes \mathbb{S}_{\alpha_m}(U_m)$. It follows that for each $\gamma \in \mathsf{Part}$ there exists $c_{\boldsymbol{\alpha}}^{\gamma} \in \mathbb{Z}_{\geq 0}$ such that the following isomorphisms hold

$$\mathbb{S}_\gamma(U_1 \oplus \cdots \oplus U_m) \cong \bigoplus_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}}^{\gamma} \mathbb{S}_{\boldsymbol{\alpha}}(U_1, \ldots, U_m);$$
$$\mathbb{S}_{\boldsymbol{\alpha}}(U, U, \ldots, U) \cong \bigoplus_{\gamma} c_{\boldsymbol{\alpha}}^{\gamma} \mathbb{S}_\gamma(U).$$

These easily imply the following lemma.

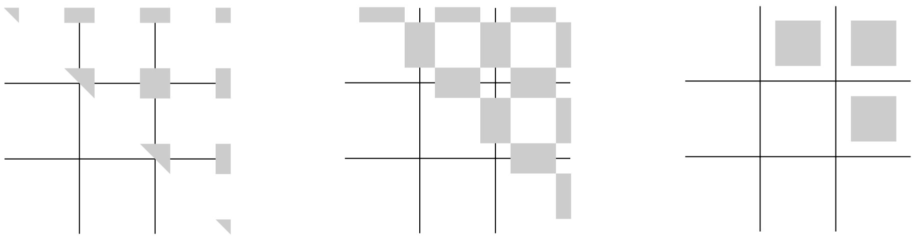
Lemma A.1. *Fix $\boldsymbol{\alpha} \in \mathsf{Part}^m$. Given vector spaces $U_1, \ldots, U_m$ there is an isomorphism, natural in all variables,*

$$\bigoplus_{\gamma \in \mathsf{Part}} \bigoplus_{\boldsymbol{\beta} \in \mathsf{Part}^m} c_{\boldsymbol{\alpha}}^{\gamma} c_{\boldsymbol{\beta}}^{\gamma} \mathbb{S}_{\boldsymbol{\beta}}(U_1, U_2, \ldots, U_m) \cong \mathbb{S}_{\boldsymbol{\alpha}}(U, U, \ldots, U)$$

where $U = U_1 \oplus U_2 \oplus \cdots \oplus U_m$.

A.2. **The module $\mathcal{H}$.** In order to understand the module $T_r(\lambda)$ we will need to look under the hood of the dual modules $M_r(\lambda)^\vee$. For this subsection only we write $\mathbb{W} = \mathbb{V}[r]$, so in particular $\mathbb{W}^{(k)} = \mathbb{V}^{(k)}[r]$ and $\mathbb{W}_*^{(l)} = \mathbb{V}_*^{(l)}[r]$, etc.

We begin by decomposing the parabolic algebra $\mathfrak{p}(r)$ as $\mathfrak{b}_r \oplus \mathfrak{f}_r \oplus \mathfrak{l}[r] \oplus \mathfrak{a}[r]$, where $\mathfrak{b}_r = \mathfrak{b} \cap \mathfrak{g}_r$, $\mathfrak{a}[r] = \bigoplus_{k<l} \mathbb{W}^{(k)} \otimes \mathbb{W}^{(l)}$, and $\mathfrak{f}_r$ is the unique root subspace completing the decomposition.



$\mathfrak{b}_r$ $\mathfrak{f}_r$ $\mathfrak{a}[r]$

By transposition, we obtain a decomposition of $\bar{\mathfrak{b}}_r \oplus \mathfrak{l}[r] \oplus \bar{\mathfrak{a}}[r]$-modules

$$\bar{\mathfrak{p}}(r) = \bar{\mathfrak{b}}_r \oplus \overline{\mathfrak{f}_{\mathfrak{r}}} \oplus \mathfrak{l}[r] \oplus \bar{\mathfrak{a}}[r].$$

We can also decompose $\mathfrak{g} = \mathfrak{g}[r] \oplus (\mathfrak{f}_r \oplus \bar{\mathfrak{f}}_r) \oplus \mathfrak{g}_r$. This is a decomposition of $\mathfrak{g}$ as $\mathfrak{g}[r] \oplus \mathfrak{g}_r$-module, since over this subalgebra we have

$$\mathfrak{f}_r \oplus \bar{\mathfrak{f}}_r \cong (\mathbb{W}^n \otimes (V_r)^*) \oplus (V_r \otimes \mathbb{W}_*^n).$$

Set $\mathfrak{q}_r = \mathfrak{s}_r + \mathfrak{b}_r \subset \mathfrak{g}_r$. Since any element of $\mathfrak{g}_r$ commutes with any element of $\mathfrak{g}[r]$, the subspaces $\mathfrak{a}[r]$ and $\bar{\mathfrak{a}}[r]$ are trivial $\mathfrak{q}_r$-submodules of $\mathfrak{g}$. The subspace $\mathfrak{f}_r$ is also a $\mathfrak{q}_r$-submodule of $\mathfrak{g}$, and we endow $\bar{\mathfrak{f}}_r$ with the structure of a $\mathfrak{q}_r$-module through the vector space isomorphism $\bar{\mathfrak{f}}_r \cong \mathfrak{f}_r \oplus \bar{\mathfrak{f}}_r/\mathfrak{f}_r$. We set $\mathcal{H} = \mathbf{S}^\bullet(\bar{\mathfrak{f}}_r \oplus \bar{\mathfrak{a}}[r])$.

By the PBW theorem we have an isomorphism of $\mathfrak{g}_r \oplus \mathfrak{l}[r]$-modules

$$M_r(\lambda) \cong U(\mathfrak{g}) \otimes_{\mathfrak{p}(r)} \mathbb{C}_\lambda \cong U(\mathfrak{g}_r) \otimes_{\mathfrak{b}_r} (\mathcal{H} \otimes \mathbb{C}_\lambda) = \operatorname{Ind}_{\mathfrak{b}_r}^{\mathfrak{g}_r} \mathcal{H} \otimes \mathbb{C}_\lambda.$$

where the $\mathfrak{l}[r]$-module structure is determined by that of $\mathcal{H}$. Thus by Proposition 4.1 we see that $M_r(\lambda)^\vee \cong \underline{\operatorname{Coind}}_{\mathfrak{b}_r}^{\mathfrak{g}_r} \mathcal{H}^\vee \otimes \mathbb{C}_\lambda$; applying Φ_r and using the fact that the $\mathfrak{l}[r]$-module

 https://doi.org/10.5802/art.42

structure is completely determined by $\mathcal{H}$ we get

$$T_r(\lambda) = \Phi_r(M_r(\lambda)^\vee) \cong \underline{\mathrm{Coind}}_{\mathfrak{b}_r}^{\mathfrak{g}_r} \Phi_r(\mathcal{H}^\vee) \otimes \mathbb{C}_\lambda.$$

Thus we have to compute $\Phi_r(\mathcal{H}^\vee)$, and for this we will need a finer description of $\bar{\mathfrak{f}}_r$.

A.3. **The invariants of $\mathcal{H}^\vee$.** Since $\mathcal{H} = \mathbf{S}^\bullet(\bar{\mathfrak{f}}_r \oplus \bar{\mathfrak{a}}[r]) = \mathbf{S}^\bullet(\bar{\mathfrak{f}}_r) \otimes \mathbf{S}^\bullet(\bar{\mathfrak{a}}[r])$ we will now decompose each of these symmetric algebras into irreducible $\mathfrak{s}_r \oplus \mathfrak{l}[r]$-modules.

We introduce a family of $\mathfrak{s}_r$-modules B_k for $k \in [\![0, n]\!]$

$$\begin{aligned} B_0 &= \langle v_i \otimes e_1 \mid i \in [\![r]\!]\rangle = V_{r,+}^{(1)} \\ B_k &= \langle v_{-i} \otimes e_k, v_i \otimes e_{k+1} \mid i \in [\![r]\!]\rangle = V_{r,-}^{(k)} \oplus V_{r,+}^{(k+1)} \\ B_n &= \langle v_{-i} \otimes e_n \mid i \in [\![r]\!]\rangle = V_{r,-}^{(n)}. \end{aligned}$$

Set also $A_k = B_k^*$. Each of these modules is simple. Indeed, $\mathfrak{s}_k$ is the direct sum of $n+1$ reductive Lie algebras of type A sitting in the diagonal of $\mathfrak{g}_k$, and B_k is the natural representation of the k^{th} summand.

The modules A_k, B_k appear when we decompose $\bar{\mathfrak{f}}_r$ as $\mathfrak{l}[r] \oplus \mathfrak{s}_r$-module

$$\bar{\mathfrak{f}}_r \cong \bigoplus_{k \geq l} \Big(\mathbb{W}^{(k)} \otimes A_l\Big) \oplus \Big(B_k \otimes \mathbb{W}_*^{(l)}\Big)$$

and hence when we take the symmetric algebra of this space we get

$$\mathbf{S}^\bullet(\bar{\mathfrak{f}}_r) \cong \bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta}} \bigotimes_{k \geq l} \mathbb{S}_{\alpha_{k,l}}(A_l) \otimes \mathbb{S}_{\beta_{k,l}}(B_k) \otimes \mathbb{S}_{\alpha_{k,l}}\Big(\mathbb{W}^{(k)}\Big) \otimes \mathbb{S}_{\beta_{k,l}}\Big(\mathbb{W}_*^{(k)}\Big),$$

where the sum runs over all families $\boldsymbol{\alpha} = (\alpha_{k,l})_{1 \leq k \leq l \leq n}$ and $\boldsymbol{\beta} = (\beta_{k,l})_{1 \leq k \leq l \leq n}$. On the other hand we have decompositions

$$\bar{\mathfrak{a}}[r] = \bigoplus_{k > l} \mathbb{W}^{(k)} \otimes \mathbb{W}_*^{(l)}; \qquad \mathbf{S}^\bullet(\bar{\mathfrak{a}}[r]) = \bigoplus_{\boldsymbol{\gamma}} \bigotimes_{k > l} \mathbb{S}_{\gamma_{k,l}}\Big(\mathbb{W}^{(k)}\Big) \otimes \mathbb{S}_{\gamma_{k,l}}\Big(\mathbb{W}_*^{(l)}\Big);$$

where again the sum runs over all families of partitions $\boldsymbol{\gamma} = (\gamma_{k,l})_{1 \leq l < k \leq n}$. We extend these families by setting $\gamma_{k,k} = \emptyset$, so $\gamma_{k,l}$ is defined for all $k \geq l$.

We now introduce the following notation: given a family of partitions $\boldsymbol{\alpha} = (\alpha_{k,l})_{1 \leq l \leq k \leq n}$ we set $\boldsymbol{\alpha}_{k,\bullet} = (\alpha_{k,l})_{l \leq k \leq n}$ and $\boldsymbol{\alpha}_{\bullet,l} = (\alpha_{k,l})_{1 \leq l \leq k}$. By collecting all terms of the form $\mathbb{S}_{\eta_k}(\mathbb{W}^{(k)}) \otimes \mathbb{S}_{\nu_k}(\mathbb{W}_*^{(k)})$ we obtain the following isomorphism of $\mathfrak{s}_r \oplus \mathfrak{l}[r]$-modules

$$\mathcal{H} \cong \bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\eta},\boldsymbol{\nu}} \bigotimes_{k=1}^{n} c^{\eta_k}_{\boldsymbol{\alpha}_{k,\bullet},\boldsymbol{\gamma}_{k,\bullet}} c^{\nu_k}_{\boldsymbol{\beta}_{\bullet,k},\boldsymbol{\gamma}_{\bullet,k}} Q(\boldsymbol{\alpha}_{k,\bullet}, \boldsymbol{\beta}_{\bullet,k}) \boxtimes \mathbb{S}_{\eta_k}\Big(\mathbb{W}^{(k)}\Big) \otimes \mathbb{S}_{\nu_k}\Big(\mathbb{W}_*^{(k)}\Big)$$

where the sum is taken over all collections of partitions and

$$Q(\boldsymbol{\alpha}_{k,\bullet}, \boldsymbol{\beta}_{\bullet,k}) = \mathbb{S}_{\boldsymbol{\alpha}_{k,\bullet}}(A_1, A_2, \ldots, A_k) \otimes \mathbb{S}_{\boldsymbol{\beta}_{\bullet,k}}(B_k, B_{k+1}, \ldots, B_n).$$

Since $\mathcal{H}$ is contained in $\mathbf{S}^\bullet(\bar{\mathfrak{u}}(r))$ it must lie in $\widetilde{\mathbb{T}}_{\mathfrak{l}[r]}$. It follows that each weight component intersects at most finitely many of the direct summands in the decomposition of $\mathcal{H}$, and so semisimple duality will commute with both the direct sum and the big tensor product. Furthermore, since each $Q(\boldsymbol{\alpha}_{k,\bullet}, \boldsymbol{\beta}_{\bullet,k})$ has finite dimensional components we have

$$\mathcal{H}^\vee \cong \bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\eta},\boldsymbol{\nu}} \bigotimes_{k=1}^{n} c^{\eta_k}_{\boldsymbol{\alpha}_{k,\bullet},\boldsymbol{\gamma}_{k,\bullet}} c^{\nu_k}_{\boldsymbol{\beta}_{\bullet,k},\boldsymbol{\gamma}_{\bullet,k}} Q(\boldsymbol{\alpha}_{k,\bullet}, \boldsymbol{\beta}_{\bullet,k})^\vee \boxtimes I(\boldsymbol{\eta}, \boldsymbol{\nu})[r]^\vee.$$

Now recall that $\Phi_r(I(\boldsymbol{\eta},\boldsymbol{\nu})[r]^\vee) \cong \underline{\mathrm{Hom}}_{\mathfrak{l}[r]'}(I(\boldsymbol{\eta},\boldsymbol{\nu})[r],\mathbb{C}) \cong \delta_{\boldsymbol{\eta},\boldsymbol{\nu}}$. Thus

$$\begin{aligned}\Phi_r(\mathcal{H}^\vee) &\cong \bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\eta}} \bigotimes_{k=1}^n c^{\eta_k}_{\boldsymbol{\alpha}_{k,\bullet},\gamma_{k,\bullet}} c^{\eta_k}_{\boldsymbol{\beta}_{\bullet,k},\boldsymbol{\gamma}_{\bullet,k}} Q(\boldsymbol{\alpha}_{k,\bullet},\boldsymbol{\beta}_{\bullet,k})^\vee \\ &\cong \left(\bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\eta}} \bigotimes_{k=1}^n c^{\eta_k}_{\boldsymbol{\alpha}_{k,\bullet},\gamma_{k,\bullet}} c^{\eta_k}_{\boldsymbol{\beta}_{\bullet,k},\boldsymbol{\gamma}_{\bullet,k}} Q(\boldsymbol{\alpha}_{k,\bullet},\boldsymbol{\beta}_{\bullet,k})\right)^\vee.\end{aligned}$$

Thus we have shown that $\Phi_r(\mathcal{H}^\vee) \cong Q^\vee$ as $\mathfrak{s}_r$-modules, where Q is the module inside the semisimple dual.

A.4. **The structure of Q.** Let us analyse the factors of the tensor product defining Q one by one. When $k = 1$ the Littlewood–Richardson coefficient $c^{\eta_k}_{\boldsymbol{\alpha}_{k,\bullet},\gamma_{k,\bullet}}$ simplifies to $c^{\eta_1}_{\alpha_{1,1},\emptyset}$, and the only way for this to be nonzero is that $\eta_1 = \alpha_{1,1}$. Thus the first factor in the tensor product has the form

$$c^{\eta_1}_{\boldsymbol{\beta}_{\bullet,1},\boldsymbol{\gamma}_{\bullet,1}} \mathbb{S}_{\eta_1}(A_1) \otimes \mathbb{S}_{\boldsymbol{\beta}_{\bullet,1}}(B_1, B_2, \dots, B_n).$$

We can sum all these factors over η_1 and $\beta_{l,1}$ for every $l \geq 1$ since these do not appear in any other factors. Using Lemma A.1 we obtain

$$\begin{aligned}&\bigoplus_{\boldsymbol{\beta}_{\bullet,1},\eta_1} c^{\eta_1}_{\boldsymbol{\beta}_{\bullet,1},\boldsymbol{\gamma}_{\bullet,1}} \mathbb{S}_{\eta_1}(A_1) \otimes \mathbb{S}_{\boldsymbol{\beta}_{\bullet,1}}(B_1, B_2, \dots, B_n) \\ &\qquad \cong \bigoplus_{\boldsymbol{\beta}_{\bullet,1}} \mathbb{S}_{\boldsymbol{\beta}_{\bullet,1}}(A_1, A_1, \dots, A_1) \otimes \mathbb{S}_{\boldsymbol{\gamma}_{\bullet,1}}(A_1, A_1, \dots, A_1) \otimes \mathbb{S}_{\boldsymbol{\beta}_{\bullet,1}}(B_1, B_2, \dots, B_n) \\ &\qquad \cong \mathbf{S}^\bullet(A_1 \otimes (B_1 \oplus B_2 \oplus \cdots \oplus B_n)) \otimes \mathbb{S}_{\boldsymbol{\gamma}_{\bullet,1}}(A_1, A_1, \dots, A_1).\end{aligned}$$

We see thus that Q is isomorphic to

$$\mathbf{S}^\bullet(A_1 \otimes (B_1 \oplus B_2 \oplus \cdots \oplus B_n)) \otimes \bigoplus_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\eta}} \bigotimes_{k=2}^n c^{\eta_k}_{\boldsymbol{\alpha}_{k,\bullet},\gamma_{k,\bullet}} c^{\eta_k}_{\boldsymbol{\beta}_{\bullet,k},\boldsymbol{\gamma}_{\bullet,k}} \mathbb{S}_{\gamma_{k,1}}(A_1) \otimes Q(\boldsymbol{\alpha}_{k,\bullet},\boldsymbol{\beta}_{\bullet,k})$$

where the sum is now restricted to the subsequences where $k \geq 2$.

Let us now focus on the new factor corresponding to $k = 2$. We can again sum over η_2 and $\boldsymbol{\beta}_{\bullet,2}$ to get

$$\begin{aligned}&\bigoplus_{\eta_2,\boldsymbol{\beta}_{\bullet,2}} c^{\eta_2}_{\boldsymbol{\beta}_{\bullet,2},\boldsymbol{\gamma}_{\bullet,2}} c^{\eta_2}_{\alpha_{2,1},\alpha_{2,2},\gamma_{2,1}} \mathbb{S}_{\gamma_{k,1}}(A_1) \otimes \mathbb{S}_{\alpha_{2,1},\alpha_{2,2}}(A_1, A_2) \otimes \mathbb{S}_{\boldsymbol{\beta}_{\bullet,2}}(B_2, \dots, B_n) \\ &\qquad \cong \bigoplus_{\boldsymbol{\beta}_{\bullet,2}} \mathbb{S}_{\boldsymbol{\beta}_{\bullet,2}}(\overline{A}_2) \otimes \mathbb{S}_{\boldsymbol{\gamma}_{\bullet,2}}(\overline{A}_2) \otimes \mathbb{S}_{\boldsymbol{\beta}_{\bullet,2}}(B_2, \dots, B_n) \\ &\qquad \cong \mathbf{S}^\bullet(\overline{A}_2 \otimes (B_2 \oplus \cdots \oplus B_n)) \otimes \mathbb{S}_{\boldsymbol{\gamma}_{\bullet,2}}(\overline{A}_2).\end{aligned}$$

where $\overline{A}_2 = A_1 \oplus A_1 \oplus A_2$. After n steps of this recursive process we obtain

$$Q \cong \bigotimes_{k=1}^n \mathbf{S}^\bullet(\overline{A}_k \otimes \underline{B}_k) \cong \mathbf{S}^\bullet\left(\bigoplus_{k=1}^n \overline{A}_k \otimes \underline{B}_k\right)$$

where

$$\underline{B}_k = B_k \oplus B_{k+1} \oplus \cdots \oplus B_n$$

and

$$\begin{aligned}\bar{A}_k &= A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus \bar{A}_1 \oplus \bar{A}_2 \oplus \cdots \oplus \bar{A}_{k-1} \\ &= 2^{k-1}A_1 \oplus 2^{k-2}A_2 \oplus \cdots \oplus A_k.\end{aligned}$$

Thus in the direct sum the term $A_k \otimes B_l$ appears $2^{l-k} + 2^{l-k-1} + \cdots + 2 + 1 = 2^{l-k+1} - 1$ times. Now the sum of all the $A_k \otimes B_l$ with $l-k+1 = i$ fixed is the space $R_{i,r}$ introduced in the preamble of Lemma 7.6 and so $Q \cong \mathbf{S}^\bullet(\mathcal{R}_r)$. The naturality of Schur functors implies that this is an isomorphism of $\mathfrak{s}_r$-modules.

A.5. **Finishing the proof.** Recall that $T_r(\lambda) \cong \underline{\mathrm{Coind}}_{\mathfrak{b}_r}^{\mathfrak{g}_r} Q^\vee \otimes \mathbb{C}_\lambda$ as $\mathfrak{g}_r$-modules. The p^{th} module of the ψ-filtration of $T_r(\lambda)$ is thus

$$\mathcal{F}_p T_r(\lambda) \cong \underline{\mathrm{Coind}}_{\mathfrak{b}_r}^{\mathfrak{g}_r} Q^\vee_{\geq p-\psi(\lambda)} \otimes \mathbb{C}_\lambda \cong \left(\mathrm{Ind}_{\mathfrak{b}_r}^{\mathfrak{g}_r} Q_{\geq p-\psi(\lambda)} \otimes \mathbb{C}_\lambda\right)^\vee.$$

Since induction and semisimple duality are exact functors, we can use this to compute the p^{th} layer of this filtration

$$\frac{\mathcal{F}_p T_r(\lambda)}{\mathcal{F}_{p-1} T_r(\lambda)} \cong \left(\frac{\mathrm{Ind}_{\mathfrak{b}_r}^{\mathfrak{g}_r} Q_{\geq p-\psi(\lambda)} \otimes \mathbb{C}_\lambda}{\mathrm{Ind}_{\mathfrak{b}_r}^{\mathfrak{g}_r} Q_{\geq p-\psi(\lambda)-1} \otimes \mathbb{C}_\lambda}\right)^\vee \cong \left(\mathrm{Ind}_{\mathfrak{b}_r}^{\mathfrak{g}_r} Q_{p-\psi(\lambda)} \otimes \mathbb{C}_\lambda\right)^\vee.$$

We can use this to compute the top layer

$$\mathcal{S}_p T_r(\lambda) \cong \left(\mathrm{Ind}_{\mathfrak{s}_r \cap \mathfrak{b}}^{\mathfrak{s}_r} Q_{p-\psi(\lambda)} \otimes \mathbb{C}_\lambda\right)^\vee.$$

The isomorphism from the previous subsection tells us that $Q_{p-\psi(\lambda)} \cong \mathbf{S}^\bullet(\mathcal{R}_r)_{p-\psi(\lambda)}$ as $\mathfrak{s}_r \cap \mathfrak{b}$-modules, which gives us the statement of Lemma 7.6.

Acknowledgments

I thank Ivan Penkov for introducing me to the study of locally-finite Lie algebras and posing the questions that led to this work. Vera Serganova answered several questions and provided valuable references. I also thank both anonymous referees for their thorough reading and suggestions.

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