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Continuous Stability Conditions of Type $\mathbb{A}$ and Measured Laminations of the Hyperbolic Plane

Kiyoshi Igusa * and Job Daisie Rock

Dedicated to Idun Reiten for her kind support and encouragement.

Abstract. We introduce stability conditions (in the sense of King) for representable modules of continuous quivers of type $\mathbb{A}$ along with a special criteria called the four point condition. The stability conditions are defined using a generalization of δ functions, called half-δ functions. We show that for a continuous quiver of type $\mathbb{A}$ with finitely many sinks and sources, the stability conditions satisfying the four point condition are in bijection with measured laminations of the hyperbolic plane. Along the way, we extend an earlier result by the first author and Todorov regarding continuous cluster categories for linear continuous quivers of type $\mathbb{A}$ and laminations of the hyperbolic plane to all continuous quivers of type $\mathbb{A}$ with finitely many sinks and sources. We also give a formula for the continuous cluster character.

Introduction

History. We introduce a new version of stability conditions inspired by King stability conditions [18], which are used to study the moduli space of finitely generated representations of a finite-dimensional algebra. The stability conditions introduced in the present paper are used to study representations of the real line.

There is recent work connecting stability conditions to wall and chamber structures for finite-dimensional algebras [6] and real Grothendieck groups [2]. There is also work studying the linearity of stability conditions for finite-dimensional algebras [9].

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Keywords. stability condition, quiver representation, continuous quiver representation, cluster category, lamination, measured lamination.

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In 2015, the first author and Todorov introduced the continuous cluster category for type $\mathbb{A}$ [15]. More recently, both authors and Todorov introduced continuous quivers of type $\mathbb{A}$ and a corresponding weak cluster category [13, 14]. The second author also generalized the Auslander–Reiten quiver of type $\mathbb{A}_n$ to the Auslander–Reiten space for continuous type $\mathbb{A}$ and a geometric model to study these weak cluster categories [21, 22].

Contributions and Organization. In the present paper, we generalize stability conditions, in the sense of King, to continuous quivers of type $\mathbb{A}$. In Section 1 we recall facts about stability conditions and reformulate them for our purposes. In Section 2 we recall continuous quivers of type $\mathbb{A}$, representable modules, and then introduce our continuous stability conditions.

At the beginning of Section 2.2 we define a half-δ function, which can be thought of as a Dirac δ function that only exists on the "minus side" or "plus side" of a point. We use the half-δ functions to define useful functions (Definition 2.8), which are equivalent to functions with bounded variation but better suited to our purposes. Then we define a stability condition as an equivalence class of pairs of useful functions with particular properties, modulo shifting the pair of functions up and down by a constant (Definitions 2.14 and 2.17).

We use some auxiliary constructions to define a semistable module (Definition 2.22). Then we recall $\mathbf{N}_\pi$-compatibility (Definition 2.26), which can be thought of as the continuous version of rigidity in the present paper. We link stability conditions to maximally $\mathbf{N}_\pi$-compatible sets using a criteria called the four point condition (Definition 2.24). By $\mathcal{S}_{\mathrm{fpc}}(Q)$ we denote the set of stability conditions of a continuous quiver Q of type $\mathbb{A}$ that satisfy the four point condition.

Theorem A (Theorem 2.28)**.** *Let Q be a continuous quiver of type $\mathbb{A}$ with finitely many sinks and sources and let $\sigma \in \mathcal{S}(Q)$. Then the following are equivalent.*

- $\sigma \in \mathcal{S}_{\mathrm{fpc}}(Q)$.
- *The set of σ-semistable indecomposables is maximally $\mathbf{N}_\pi$-compatible.*

In Section 3 we define a continuous version of tilting. That is, for a continuous quiver Q of type $\mathbb{A}$ we define a new continuous quiver Q' of type $\mathbb{A}$ together with an induced map on the set of indecomposable representable modules. This is not to be confused with reflection functors for continuous quivers of type $\mathbb{A}$, introduced by Liu and Zhao [19]. For each stability condition σ of Q that satisfies the four point condition, we define a new stability condition σ' of Q' (Definition 3.12). We show that continuous tilting induces a bijection on indecomposable representable modules, preserves $\mathbf{N}_\pi$-compatibility, and includes a bijection on stability conditions for Q and Q' that satisfy the four point condition. Denote by $\mathrm{mod}^{\mathrm{r}}(Q)$ the category of representable modules over Q.

Theorem B (Theorems 3.2 and 3.17)**.** *Let Q and Q' be continuous quivers of type $\mathbb{A}$ with finitely many sinks and sources. Continuous tilting yields a triple of bijections: ϕ, Φ, and Ψ.*

- *A bijection $\phi : \mathrm{Ind}(\mathrm{mod}^{\mathrm{r}}(Q)) \to \mathrm{Ind}(\mathrm{mod}^{\mathrm{r}}(Q'))$.*
- *A bijection Φ from maximal $\mathbf{N}_\pi$-compatible sets of $\mathrm{mod}^{\mathrm{r}}(Q)$ to maximal $\mathbf{N}_\pi$-compatible sets of $\mathrm{mod}^{\mathrm{r}}(Q')$. Furthermore if $\mu : T \to T'$ is a mutation then so is $\Phi(\mu) : \Phi T \to \Phi T'$ given by $\phi(M_I) \mapsto \phi(\mu(M_I))$.*
- *A bijection $\Psi : \mathcal{S}_{\mathrm{fpc}}(Q) \to \mathcal{S}_{\mathrm{fpc}}(Q')$ such that if T is the set of σ-semistable modules then $\Phi(T)$ is the set of $\Psi(\sigma)$-semistable modules.*

In Section 4 we define a measured lamination to be a lamination of the (Poincaré disk model of the) hyperbolic plane together with a particular type of measure on the set of geodesics (Definition 4.2). We denote the Poincaré disk model of the hyperbolic plane by $\mathfrak{h}^2$. Then we recall the correspondence between laminations of $\mathfrak{h}^2$ and maximally $\mathbf{N}_\pi$-compatible sets of indecomposable representable modules over the straight descending continuous quiver of type $\mathbb{A}$, from the first author and Todorov (Theorem 4.5) [15]. We extend this correspondence to stability conditions that satisfy the four point condition and measured laminations (Theorem 4.13). Combining this with Theorems A and B, we have the last theorem.

Theorem C (Corollary 4.14)**.** *Let Q be a continuous quiver of type $\mathbb{A}$ and $\mathcal{L}$ the set of measured laminations of $\mathfrak{h}^2$. There are three bijections: ϕ, Φ, and Ψ.*

- *The bijection ϕ from $\mathrm{Ind}(\mathrm{mod}^{\mathrm{r}}(Q))$ to geodesics in $\mathfrak{h}^2$.*
- *The bijection Φ from maximally $\mathbf{N}_\pi$-compatible sets to (unmeasured) laminations of $\mathfrak{h}^2$ such that, for each maximally $\mathbf{N}_\pi$-compatible set T, $\phi|_T$ is a bijection from the indecomposable modules in T to the geodesics in $\Phi(T)$.*
- *The bijection $\Psi : \mathcal{S}_{\mathrm{fpc}}(Q) \to \mathcal{L}$ such that if T is the set of σ-semistable modules then $\Phi(T)$ is the set of geodesics in $\Psi(\sigma)$.*

In Section 4.3, we give a formula for a continuous cluster character $\chi(M_{ab})$. This is a formal expression in formal variables x_t, one for every real number t. We verify some cluster mutation formulas, but leave further work for a future paper.

In Section 4.5, we relate continuous tilting to cluster categories of type $\mathbb{A}_n$. In particular, we discuss how a particular equivalence between type $\mathbb{A}_n$ cluster categories is compatible with continuous tilting. We conclude our contributions with an example for type $\mathbb{A}_4$ (Section 4.5.1). Then we briefly describe some directions for further related research.

1. The Finite Case

There is a relation between stability conditions and generic decompositions which will become more apparent in the continuous case. Here we examine the finite case and impose continuous structures onto the discrete functions in order to give a preview of what will happen in the continuous quiver case.

We start with the generic decomposition of a dimension vector for any finite quiver of type $\mathbb{A}_n$. This is an elegant combinatorial construction introduced by Abeasis [1] which arranges spots in a planar array and gives the components of the generic decomposition as sequences of horizontal spots. The spots are “top aligned” for left arrow and “bottom aligned” for right arrows. See (1.3) below for an example.

Next, we interpret Abeasis' discrete arrangement as the graph of a piecewise linear function F on the closed interval $[0, n+1]$. When the arrangement of spots is “flipped” from top aligned to bottom aligned (or vise-versa) we introduce a discontinuity in the piecewise linear function. See Figure 1.2. The derivative of this function is the continuous stability function θ so that the components of the generic decomposition are θ-semistable. The standard definition of θ-semistability, given in [18] and later related to generic decompositions in [2, 6, 9] and explained in Section 1.1 below using a discrete stability function θ, is artificially transformed into a continuous stability function as outlined above and in more detail in Section 1.2 below, as motivation for our “continuous stability function” which, more precisely, is a pair of functions of bounded variation on $\mathbb{R}$ (Definition 2.14), the topic of this paper.

We recall that APR-tilting is the process of reversing the direction of all arrows at a vertex which is a sink, making it into a source. This is named after the authors Auslander, Platzeck and Reiten who, in [3], give a formula relating representations of a quiver with those of its APR-tilt. In Section 1.3 we reformulate this well-known formula into stability conditions using our piecewise continuous stability functions. This is a precursor of the continuous tilting process which we discuss in Section 3. In Section 1.4, we review the Caldero–Chapoton cluster character [7] for representations of a quiver of type $\mathbb{A}_n$ to motivate the continuous case in Section 4.3.

1.1. **Semistability condition.** Recall that a stability function is a linear map

$$\theta : K_0\Lambda = \mathbb{Z}^n \longrightarrow \mathbb{R}.$$

A module M is θ-semistable if $\theta(\underline{\dim}M) = 0$ and $\theta(\underline{\dim}M') \leq 0$ for all submodules $M' \subset M$. We say M is θ-stable if, in addition, $\theta(\underline{\dim}M') < 0$ for all $0 \neq M' \subsetneq M$. For Λ of type $\mathbb{A}_n$, we denote by $M_{(a,b]}$ the indecomposable module with support on the vertices $a+1, \ldots, b$. For example $M_{(i-1,i]}$ is the simple module S_i. Let $F : \{0, 1, \ldots, n\} \to \mathbb{R}$ be the function

$$F(k) = \sum_{0<i\leq k} \theta(\underline{\dim}S_i) = \theta(\underline{\dim}M_{(0,k]})$$

Then we have $\theta(M_{(a,b]}) = F(b) - F(a)$.

Thus, for $M_{(a,b]}$ to be θ-semistable we need $F(a) = F(b)$ and another condition to make $\theta(\underline{\dim}M') \leq 0$. For example, take the quiver of type $\mathbb{A}_n$ having a source at vertex c and sinks at $1, n$. Then the indecomposable submodules of $M_{(a,b]}$ are $M_{(a,x]}$ for $a < x < c$, $x \leq b$ and $M_{(y,b]}$ for $c \leq y < b$, $a \leq y$. Therefore, we also need $F(x) \leq F(a) = F(b) \leq F(y)$ for such x, y. (And strict inequalities $F(x) < F(a) = F(b) < F(y)$ to make $M_{(a,b]}$ stable.)

A simple characterization of x, y is given by numbering the arrows. Let α_i be the arrow between vertices $i, i+1$. Then the arrows connecting vertices in $(a, b]$ are α_i for $a < i < b$. $M_{(a,x]} \subset M_{(a,b]}$ if α_x points left (and $a < x < b$). $M_{(y,b]} \subset M_{(a,b]}$ if α_y points to the right (and $a < y < b$). More generally, we have the following.

Proposition 1.1. *$M_{(a,b]}$ is θ-semistable if and only if the following hold.*

(1) $F(a) = F(b)$
(2) $F(x) \leq F(a)$ if α_x points left and $a < x < b$.
(3) $F(y) \geq F(b)$ if α_y points right and $a < y < b$.

Furthermore, if the inequalities in (2), (3) *are strict, $M_{(a,b]}$ is θ-stable.*

For example, take the quiver

$$1 \xleftarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \tag{1.1}$$

with $\theta = (-1, 2, -1, -1)$, $F = (0, -1, 1, 0, -1)$. Then $F(1) < F(0) = F(3) = 0 < F(2)$, with α_1 pointing left and α_2 pointing right. So, $M_{(0,3]}$ is θ-stable. Similarly, $F(1) = F(4) = -1 < F(2), F(3)$ implies $M_{(1,4]}$ is also θ-stable.

One way to visualize the stability condition is indicated in Figure 1.1.

1.2. **Generic decomposition.** Stability conditions for quivers of type $\mathbb{A}_n$ also give the generic decomposition for dimension vectors $\mathbf{d} \in \mathbb{N}^n$. This becomes more apparent for large n and gives a preview of what happens in the continuous case.

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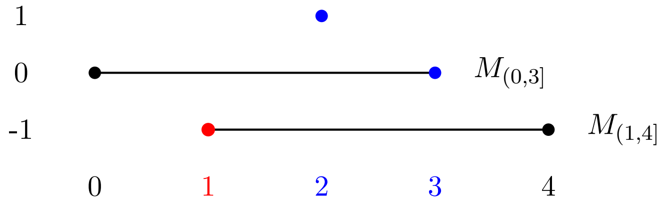


FIGURE 1.1. The graph of $F : \{0, 1, 2, 3, 4\} \to \mathbb{R}$ shows the θ-semistable modules. When $M_{(a,b]}$ is θ-stable, $F(a) = F(b)$ making the line segment connecting $(a, F(a))$ to $(b, F(b))$ horizontal. Also, the intermediate red points are below and the blue points are above the line segment if we draw as red/blue, the spot $(x, F(x))$ for α_x pointing left/right, respectively.

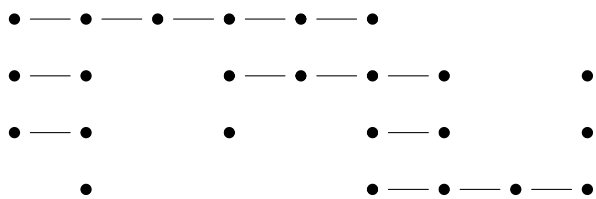
Given a dimension vector $\mathbf{d} \in \mathbb{N}^n$, there is, up to isomorphism, a unique Λ-module M of dimension vector $\mathbf{d}$ which is rigid, i.e., $\mathrm{Ext}^1(M, M) = 0$. The dimension vectors β_i of the indecomposable summands of M add up to $\mathbf{d}$ and the expression $\mathbf{d} = \sum \beta_i$ is called the "generic decomposition" of $\mathbf{d}$. We use the notation $\beta_{ab} = \underline{\dim} M_{(a,b]}$ and $\mathbf{d} = (d_1, \ldots, d_n)$.

There is a well-known formula for the generic decomposition of a dimension vector [1] which we explain with an example. Take the quiver of type $\mathbb{A}_9$:

$$1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xleftarrow{\alpha_3} 4 \xleftarrow{\alpha_4} 5 \xleftarrow{\alpha_5} 6 \xrightarrow{\alpha_6} 7 \xrightarrow{\alpha_7} 8 \xrightarrow{\alpha_8} 9 \tag{1.2}$$

with dimension vector $\mathbf{d} = (3, 4, 1, 3, 2, 4, 3, 1, 3)$. To obtain the generic decomposition for $\mathbf{d}$, we draw d_i spots in vertical columns as shown in (1.3) below.

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5 \longleftarrow 6 \longrightarrow 7 \longrightarrow 8 \longrightarrow 9 \tag{1.3}$$



$\mathbf{d}$: 3 4 1 3 2 4 3 1 3

For arrows going left, such as $3 \leftarrow 4$, $5 \leftarrow 6$ the top spots should line up horizontally. For arrows going right, such as $6 \to 7, 7 \to 8$ the bottom spots should line up horizontally as shown. Consecutive spots in any row are connected by horizontal lines. For example, the spots in the first row are connected giving $M_{(0,6]}$ but the second row of spots is connected in three strings to give $M_{(0,2]}, M_{(3,7]}$ and $S_9 = M_{(8,9]}$. The generic decomposition is given by these horizontal lines. Thus

$$\mathbf{d} = (3, 4, 1, 3, 2, 4, 3, 1, 3) = \beta_{06} + 2\beta_{02} + \beta_{37} + 2\beta_{89} + \beta_{12} + \beta_{34} + \beta_{57} + \beta_{59}$$

is the generic decomposition of $\mathbf{d} = (3, 4, 1, 3, 2, 4, 3, 1, 3)$.

We construct this decomposition using a stability function based on (1.3). We explain this with two examples without proof. The purpose is to motivate continuous stability conditions.

Take real numbers $d_0, d_1, \ldots, d_n, d_{n+1}$ where $d_0 = d_{n+1} = 0$. Draw arrows where the arrow α_i connecting $i, i+1$ where α_0 points in the same direction as α_1 and α_n points in the same direction as α_{n-1}. To each arrow α_i we associate the real number which is d_i of the target minus d_i of the source. We write this difference below the arrow if the arrow points left and above the arrow when the arrow points right. Then we compute the partial sums for the top numbers and the bottom numbers. Let B, R denote these functions. Thus $B(6) = 0, B(7) = -1, B(8) = -3, B(9) = -1, B(10) = -4$ and $R(0) = 0, R(1) = -3$, etc. as shown below.

$$
\begin{array}{lccccccccccccccccccccc}
i: & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & & 10 \\
B: & & & & & & & & & & & & & 0 & & -1 & & -3 & & -1 & & -4 \\
 & & & & & & & & & & & & & & -1 & & -2 & & 2 & & -3 & \\
d_i & 0 & \longleftarrow & 3 & \longleftarrow & 4 & \longleftarrow & 1 & \longleftarrow & 3 & \longleftarrow & 2 & \longleftarrow & 4 & \longrightarrow & 3 & \longrightarrow & 1 & \longrightarrow & 3 & \longrightarrow & 0 \\
 & & -3 & & -1 & & 3 & & -2 & & 1 & & -2 & & & & & & & & & \\
R: & 0 & & -3 & & -4 & & -1 & & -3 & & -2 & & -4 & & & & & & & &
\end{array}
$$

We extend the blue and red functions by $B(x) = B(6) = 0$ for all $x < 6$ and $R(x) = R(6) = -4$ for all $x > 6$.

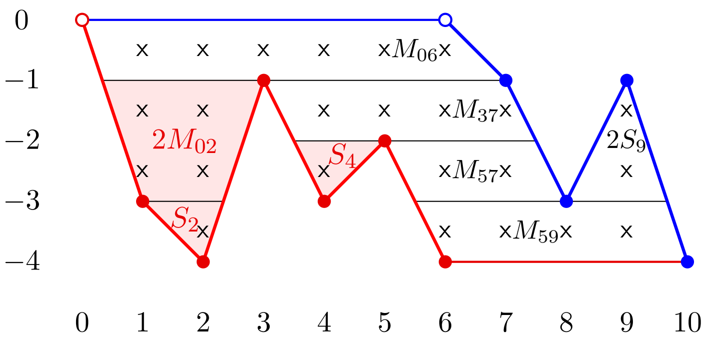


FIGURE 1.2. The function $F : (0, n+1] \to \mathbb{R}$ is given by the red function R on $(0, 6]$ since the first 5 arrows point left and by the blue function B on $(6, 10]$. A module $M_{(a,b]}$ is semistable if there exists a y and a horizontal line from (a, y) to (b, y) so that $R(x) \le y \le B(x)$ for all $a \le x \le b$. "Islands" $M_{(a,b]}$ for $b < 5$ are shaded.

The generic decomposition of $\mathbf{d}$ is given by $\mathbf{d} = \sum a_i \beta_i$ where the coefficient a_i of $\beta_i = \beta_{ab}$ is the linear measure of the set of all $c \in \mathbb{R}$ so that $M_{(x,y]}$ is semistable with $F(x) = c = F(y)$ and so that $\mathbb{Z} \cap (x, y] = \{a+1, \dots, b\}$. For example, in Figure 1.2, the coefficient of β_{02} is the measure of the vertical interval $[-3, -1]$ which is 2. For c in this vertical interval the horizontal line at level c goes from the red line between $\frac{1}{3}$ and 1 to the red line between $\frac{7}{3}$ and 3 with blue lines above and red lines below. (We extend the red and blue functions to the interval $(0, 10]$ as indicated.) We require $R(x) \le B(x)$ for all $x \in \mathbb{R}$.

We interpret the stability function θ to be the derivative of F where we consider R, B separately. So, θ is a step function equal to $-3, -1, 3, -2, 1, -2$ on the six red unit intervals between 0 and 6 and θ is $-1, -2, 2, -3$ on the four blue intervals from 6 to 10. θ is 4 times the dirac delta function at 6. For example,

$$\theta(M_{(a,b]}) = \int_a^b \theta(x)\,\mathrm{d}x = F(b) - F(a) = 0,$$

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for $a = 3+\varepsilon, b = 7+\varepsilon$ with $0 \leq \epsilon \leq 1$ since $F(a) = F(b) = -1-2\varepsilon$ in this range. However, $F(5) = -2$ which is greater than $-1 - 2\varepsilon$ for $\varepsilon > 1/2$. So, $M_{(3+\varepsilon,7+\varepsilon]}$ is semistable only when $0 \leq \varepsilon \leq 1/2$. Taking only the integers in the interval $(3 + \varepsilon, 7 + \varepsilon]$, we get $M_{(3,7]}$ to be semistable.

1.3. **APR-tilting.** For quivers of type $\mathbb{A}_n$, we would like all arrows to be pointing in the same direction. We accomplish this with APR-tilting [3].

We recall that APR-tilting of a quiver Q is given by choosing a sink and reversing all the arrows pointing to that sink, making it a source in a new quiver Q'. Modules M of Q correspond to modules M' of Q' with the property that

$$\mathrm{Hom}(M', N') \oplus \mathrm{Ext}(M', N') \cong \mathrm{Hom}(M, N) \oplus \mathrm{Ext}(M, N)$$

for all pairs of $\Bbbk Q$-modules M, N. This gives a bijection between exceptional sequences for $\Bbbk Q$ and for $\Bbbk Q'$. However, generic modules are given by sets of ext-orthogonal modules. So, we need to modify this procedure.

In our example, we have a quiver Q with 5 arrows pointing left. By a sequence of APR-tilts we can make all of these point to the right. The new quiver Q' will have all arrows pointing to the right. Any $\Bbbk Q$-module $M_{(a,b]}$ with $a \leq 5 < b$ gives a $\Bbbk Q'$-module $M_{(5-a,b]}$. For example $M_{(0,6]}, M_{(3,7]}, M_{(5,7]}, M_{(5,9]}$ become $M_{(5,6]}, M_{(2,7]}, M_{(0,7]}, M_{(0,9]}$. See Figure 1.3. For $a > 5$, such as $M_{(8,9]} = S_9$, the module is "unchanged". For $b \leq 5$, the APR-tilt of $M_{(a,b]}$ is $M_{(5-b,5-a]}$. However, these are not in general ext-orthogonal to the other modules in our collection. For example, the APR-tilt of $S_4 = M_{(3,4]}$ is $M_{(1,2]} = S_2$ which extends $M_{(2,7]}$. So we need to shift it by τ^{-1} to get $\tau^{-1}M_{(5-b,5-a]} = M_{(4-b,4-a]}$. There is a problem when $b = 5$ since, in that case $4-b = -1$. This problem will disappear in the continuous case. We call modules $M_{(a,b]}$ with $b < 5$ *islands*. We ignore the problem case $b = 5$. Islands are shaded in Figure 1.2. Shifts of their APR-tilts are shaded in Figure 1.3.

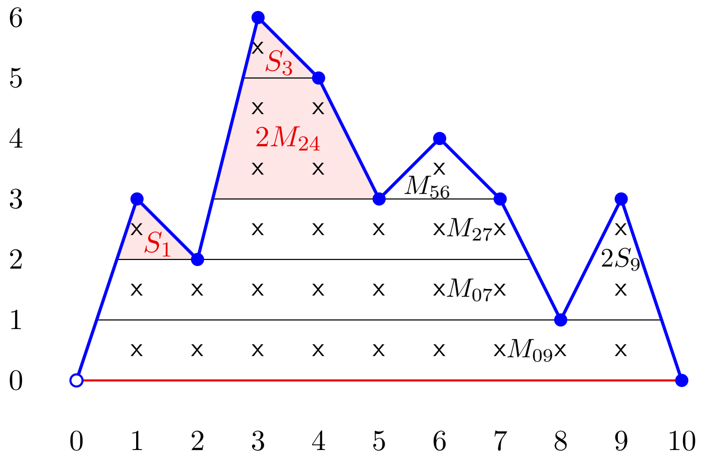

FIGURE 1.3. This is given by APR-tilting of Figure 1.2. The modules $M_{(a,b]}$ from Figure 1.2 with $a \leq 5 < b$ become $M_{(5-a,b]}$ by APR-tilting. The "islands" $M_{(a,b]}$ in Figure 1.2 gave $\tau^{-1}M_{(5-b,5-a]} = M_{(4-b,4-a]}$ above (shaded).

1.4. **Clusters and cluster algebras.** The components of a generic decomposition of any module form a partial tilting object since they do not extend each other. In the example shown in Figure 1.3, we have 8 objects:

$$M_{01}, M_{07}, M_{09}, M_{23}, M_{24}, M_{27}, M_{56}, M_{89}.$$

Since the quiver A_9 has rank 9, we need one more to complete the tilting object. There are two other modules that we could add to complete this tilting object. They are $X = M_{26}$ and $X^* = M_{57}$. There are always at most two objects that will complete a tilting object with $n-1$ components. Tilting objects are examples of clusters and, in the cluster category [5], there are always exactly two objects which complete a cluster with $n-1$ components.

These two objects M_{26}, M_{57} extend each other in the cluster category with extensions:

$$M_{57} \longrightarrow M_{27} \oplus M_{56} \longrightarrow M_{26}$$

and

$$M_{26} \longrightarrow M_{24} \longrightarrow M_{46}[1] = \tau^{-1} M_{57}[1].$$

In the cluster category, a module M over any hereditary algebra is identified with $\tau^{-1}M[1]$. Thus, an exact sequence $\tau^{-1}M \hookrightarrow A \twoheadrightarrow B$ gives an exact triangle $M[-1] \to A \to B \to M$ in the cluster category since $\tau^{-1}M = M[-1]$.

In the cluster algebra [8], which is the subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by "cluster variables", we have a formula due to Caldero and Chapoton [7] which associates a cluster variable $\chi(M)$ to every rigid indecomposable module M and, in this case, satisfies the equation:

$$\chi(X)\chi(X^*) = \chi(M_{27})\chi(M_{56}) + \chi(M_{24}) \tag{1.4}$$

The Caldero–Chapoton formula for the cluster character of M_{ab} for $1 < a < b < n$ with arrows going right is the sum of the inverses of exponential g-vectors of all submodules $x^{g(M_{ib})} = x_i/x_b$ times that of the duals of their quotients $M_{ab}/M_{ib} = M_{ai}$ (see [20]):

$$\chi(M_{ab}) = \sum_{i=a}^{b} x^{-g(M_{ib})} x^{-g(DM_{ai})} = \sum_{i=a}^{b} \frac{x_b x_{a-1}}{x_i x_{i-1}}. \tag{1.5}$$

So, $\chi(M_{aa}) = \chi(0) = 1$. When $b = n+1$, M_{ab} is projective with support $[a, n+1) = [a, n]$. So,

$$\chi(P_a) = \chi(M_{a,n+1}) = \sum_{i=a}^{n+1} \frac{x_{a-1}}{x_i x_{i-1}}$$

where $x_{n+1} = 1$. This yields:

$$\chi(M_{ab}) = x_b \chi(P_a) - x_{a-1}\chi(P_{b+1}).$$

Then, the mutation equation (1.4) becomes the Plücker relation for the 2×4 matrix:

$$\begin{bmatrix} x_1 & x_4 & x_6 & x_7 \\ \chi(P_2) & \chi(P_5) & \chi(P_7) & \chi(P_8) \end{bmatrix}.$$

2. Continuous stability conditions

2.1. **Continuous quivers of type $\mathbb{A}$.** Recall that in a partial order $\preceq$, an element x is a *sink* if $y \preceq x$ implies $y = x$. Dually, x is a *source* if $x \preceq y$ implies $y = x$.

Definition 2.1. Let $\preceq$ be a partial order on $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with finitely many sinks and sources such that, between sinks and sources, $\preceq$ is either the same as the usual order or the opposite. Let $Q = (\mathbb{R}, \preceq)$ where $\preceq$ is the same partial order on $\mathbb{R} \subseteq \overline{\mathbb{R}}$. We call Q a *continuous quiver of type* $\mathbb{A}$. We consider Q as a category where the objects of Q are the points in $\mathbb{R}$ and

$$\mathrm{Hom}_Q(x, y) = \begin{cases} \{*\} & y \preceq x \\ \emptyset & \text{otherwise.} \end{cases}$$

Definition 2.2. Let Q be a continuous quiver of type $\mathbb{A}$. A *pointwise finite-dimensional Q module* over the field $\Bbbk$ is a functor $V : Q \to \mathrm{vec}(\Bbbk)$. Let $I \subset \mathbb{R}$ be an interval. An *interval indecomposable module* M_I is given by

$$M_I(x) := \begin{cases} \Bbbk & x \in I \\ 0 & x \notin I, \end{cases} \qquad M_I(x, y) := \begin{cases} 1_\Bbbk & y \preceq x,\ x, y \in I \\ 0 & \text{otherwise,} \end{cases}$$

where $I \subseteq \mathbb{R}$ is an interval.

By results in [4, 14] we know that every pointwise finite-dimensional Q module is isomorphic to a direct sum of interval indecomposables. In particular, this decomposition is unique up to isomorphism and permutation of summands. In [14] it is shown that the category of pointwise finite-dimensional modules is abelian, interval indecomposable modules are indecomposable, and there are indecomposable projectives P_a for each $a \in \mathbb{R}$ given by

$$P_a(x) = \begin{cases} \Bbbk & x \preceq a \\ 0 & \text{otherwise,} \end{cases} \qquad P_a(x, y) = \begin{cases} 1_\Bbbk & y \preceq x \preceq a \\ 0 & \text{otherwise.} \end{cases}$$

These projectives are representable as functors.

Definition 2.3. Let Q be a continuous quiver of type $\mathbb{A}$. We say V is *representable* if there is a finite direct sum $P = \bigoplus_{i=1}^n P_{a_i}$ and an epimorphism $P \twoheadrightarrow V$ whose kernel is a direct sum $\bigoplus_{j=1}^m P_{a_j}$.

By [14, Theorem 3.0.1], V is isomorphic to a finite direct sum of interval indecomposables. By results in [21], the subcategory of representable modules is abelian (indeed, a wide subcategory) but has no injectives. When $\preceq$ is the standard total order on $\mathbb{R}$, the representable modules are the same as those considered in [15].

Notation 2.4. We denote the abelian subcategory of representable modules over Q by $\mathrm{mod}^{\mathrm{r}}(Q)$. We denote the set of isomorphism classes of indecomposables in $\mathrm{mod}^{\mathrm{r}}(Q)$ by $\mathrm{Ind}^{\mathrm{r}}(Q)$.

Definition 2.5. Let Q be a continuous quiver of type $\mathbb{A}$, $s \in \overline{\mathbb{R}}$ a sink, and $s' \in \overline{\mathbb{R}}$ an adjacent source.

- If $s < s'$ and $x \in (s, s')$ we say x is *red* and (s, s') is *red*.
- If $s' < s$ and $x \in (s', s)$ we say x is *blue* and (s', s) is *blue*.

Let I be an interval in $\mathbb{R}$ such that neither $\inf I$ nor $\sup I$ is a source. We will need to refer to the endpoints of I as being red or blue the following way.

- If $\inf I$ is a sink and $\inf I \in I$ we say $\inf I$ is *blue*.

- If $\inf I$ is a sink and $\inf I \notin I$ we say $\inf I$ is *red*.
- If $\sup I$ is a sink and $\sup I \in I$ we say $\sup I$ is *red*.
- If $\sup I$ is a sink and $\sup I \notin I$ we say $\sup I$ is *blue*.
- If $\inf I$ is not a sink ($\sup I$ is not a sink) then we say $\inf I$ ($\sup I$) is red or blue according to the first part of the definition.

Note that $\inf I$ could be $-\infty$, in which case it is red. Similarly, if $\sup I = +\infty$ then it is blue.

Definition 2.6. We say I is *left red* (respectively, *left blue*) if $\inf I$ is red (respectively, if $\inf I$ is blue).

We say I is *right red* (respectively, *right blue*) if $\sup I$ is red (respectively, if $\sup I$ is blue).

We have the following characterization of support intervals.

Proposition 2.7. *Let $I \subset \mathbb{R}$ be the support of an indecomposable representable module $M_I \in \mathrm{Ind}^{\mathrm{r}}(Q)$. Then an endpoint of I lies in I if and only if it is either left blue or right red (or both, as in the case $I = [s,s]$ where s is a sink).*

2.2. **Half-δ functions and red-blue function pairs.** To define continuous stability conditions we need to introduce half-δ functions. A half-δ function δ_x^- at $x \in \overline{\mathbb{R}}$ has the following property. Let f be some integrable function on $[a,b] \subset \overline{\mathbb{R}}$ where $a < x < b$. Then the following equations hold:

$$\int_a^x (f(t) + \delta_x^-)\,\mathrm{d}t = \left(\int_a^x f(t)\,\mathrm{d}t\right) + 1, \qquad \int_x^b (f(t) + \delta_x^-)\,\mathrm{d}t = \int_x^b f(t)\,\mathrm{d}t.$$

The half-δ function δ_x^+ at $x \in \mathbb{R}$ has a similar property for an f integrable on $[a,b] \subset \overline{\mathbb{R}}$ with $a < x < b$:

$$\int_a^x \left(f(t) + \delta_x^+\right)\mathrm{d}t = \int_a^x f(t)\,\mathrm{d}t, \qquad \int_x^b \left(f(t) + \delta_x^+\right)\mathrm{d}t = \left(\int_x^b f(t)\,\mathrm{d}t\right) + 1.$$

Consider $f + \delta_x^- - \delta_x^+$. Then we have

$$\begin{aligned}
\int_a^x \left(f(t) + \delta_x^- - \delta_x^+\right)\mathrm{d}t &= \left(\int_a^x f(t)\,\mathrm{d}t\right) + 1,\\
\int_x^b \left(f(t) + \delta_x^- - \delta_x^+\right)\mathrm{d}t &= \left(\int_x^b f(t)\,\mathrm{d}t\right) - 1,\\
\int_a^b \left(f(t) + \delta_x^- - \delta_x^+\right)\mathrm{d}t &= \int_a^b f(t)\,\mathrm{d}t.
\end{aligned}$$

For each $x \in \mathbb{R}$, denote the functions

$$\begin{aligned}
\Delta_x^-(z) &= \int_{-\infty}^z \delta_x^-\,\mathrm{d}t = \begin{cases} 0 & z < x\\ 1 & z \geq x,\end{cases}\\
\Delta_x^+(z) &= \int_{-\infty}^z \delta_x^+\,\mathrm{d}t = \begin{cases} 0 & z \leq x\\ 1 & z > x.\end{cases}
\end{aligned}$$

Though not technically correct, we write that a function $f + u_x^-\Delta_x^- + u_x^+\Delta_x^+$ is from $\mathbb{R}$ to $\mathbb{R}$, where u_x^- and u_x^+ are real numbers. See Figure 2.1 for an example.

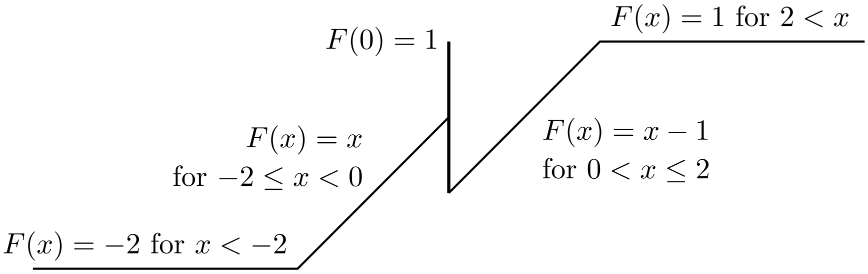

FIGURE 2.1. Graph of $F(x) = f(x) + \Delta_0^-(x) - 2\Delta_0^+(x)$ when f is given by: $f(x) = x$ if $|x| \leq 2$, $f(x) = -2$ for $x < -2$, $f(x) = 2$ if $x > 2$, $u_0^- = 1$, $u_0^+ = -2$, and $u_x^\pm = 0$ for all other $x \in \mathbb{R} \cup \{\pm\infty\}$.

We also allow $\delta_{+\infty}^-$ and $\delta_{-\infty}^+$, which satisfy the relevant parts of the equations above. We don't allow the other half-δ functions at $\pm\infty$ because it does not make sense in terms of integration.

Our stability conditions will be comprised of equivalence classes of pairs of useful functions.

Definition 2.8. We call a function $F : \mathbb{R} \to \mathbb{R}$ *useful* if it satisfies the following.

(1) $F = f + \sum_{x \in \mathbb{R} \cup \{+\infty\}} u_x^- \Delta_x^- + \sum_{x \in \mathbb{R} \cup \{-\infty\}} u_x^+ \Delta_x^+$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function of bounded variation and each u_x^-, u_x^+ are in $\mathbb{R}$.

(2) The sums $\sum_{x \in \mathbb{R} \cup \{+\infty\}} |u_x^-|$ and $\sum_{x \in \mathbb{R} \cup \{-\infty\}} |u_x^+|$ both converge in $\mathbb{R}$.

Remark 2.9. Note Definition 2.8 (2) implies the set $\{u_x^- \mid u_x^- \neq 0\} \cup \{u_x^+ \mid u_x^+ \neq 0\}$ is at most countable. Combining (1) and (2) in Definition 2.8 means we think of F as having the notion of bounded variation.

We think of the value of a useful function F at x as being "the integral from $-\infty$ to x" where the integrand is some function that includes at most countably-many half-δ functions.

Proposition 2.10. *Let F be a useful function and let $a \in \overline{\mathbb{R}}$.*

(1) If $a > -\infty$ then $\lim_{x \to a^-} F(x)$ exists.

(2) If $a < +\infty$ then $\lim_{x \to a^+} F(x)$ exists.

(3) If $a \in \mathbb{R}$ then $F(a) = \lim_{x \to a^-} F(x) + u_a^-$ and $F(a) + u_a^+ = \lim_{x \to a^+} F(x)$.

Proof.

(1) *and* (2). Straightforward computations show that

$$\lim_{x \to a^-} F(x) = \lim_{x \to a^-} f(x) + \sum_{-\infty < x < a} u_x^- + \sum_{-\infty \leq x < a} u_x^+ \quad \text{if } a > -\infty$$

$$\lim_{x \to a^+} F(x) = \lim_{x \to a^+} f(x) + \sum_{-\infty < x \leq a} u_x^- + \sum_{-\infty \leq x \leq a} u_x^+ \quad \text{if } a < +\infty.$$

Thus, (1) and (2) hold.

(3). By definition, we see that

$$F(a) = f(a) + \sum_{-\infty < x \leq a} u_x^- + \sum_{-\infty \leq x < a} u_x^+.$$

Thus, using (1) and (2), we see that (3) holds. □

Notation 2.11. Let F be a useful function. For each $a \in \mathbb{R}$, we define

$$F_{\min}(a) := \min\Big\{F(a), \lim_{x\to a^-} F(x), \lim_{x\to a^+} F(x)\Big\}$$
$$F_{\max}(a) := \max\Big\{F(a), \lim_{x\to a^-} F(x), \lim_{x\to a^+} F(x)\Big\}.$$

We also define

$$F(-\infty) := \lim_{x\to -\infty^+} F(x) - u^+_{-\infty}, \quad F(+\infty) := \lim_{x\to +\infty^-} F(x) + u^-_{+\infty}$$

$$F_{\min}(-\infty) := \min\Big\{F(-\infty), \lim_{x\to -\infty^+} F(x)\Big\}$$
$$F_{\min}(+\infty) := \min\Big\{F(+\infty), \lim_{x\to +\infty^-} F(x)\Big\}$$
$$F_{\max}(-\infty) := \max\Big\{F(-\infty), \lim_{x\to -\infty^+} F(x)\Big\}$$
$$F_{\max}(+\infty) := \max\Big\{F(+\infty), \lim_{x\to +\infty^-} F(x)\Big\}.$$

Definition 2.12. Let F be a useful function. We define the *graph* $\mathcal{G}(F)$ of F to be the following subset of $\mathbb{R}^2$:

$$\{(x, y) \,|\, x \in \mathbb{R},\ F_{\min}(x) \le y \le F_{\max}(x)\}.$$

The *completed graph*, denoted $\overline{\mathcal{G}(\mathcal{F})}$ of F is the following subset of $\overline{\mathbb{R}} \times \mathbb{R}$:

$$\Big\{(x, y) \,\Big|\, x \in \overline{\mathbb{R}},\ F_{\min}(x) \le y \le F_{\max}(x)\Big\}.$$

Remark 2.13. Let $F = f + \sum_{x\in\mathbb{R}\cup\{+\infty\}} u^-_x \Delta^-_x + \sum_{x\in\mathbb{R}\cup\{-\infty\}} u^+_x \Delta^+_x$ be a useful function. For any $a \le b \in \mathbb{R}$ there exists $c \le d \in \mathbb{R}$, such that $\mathcal{G}(F) \cap ([a, b] \times \mathbb{R}) = \mathcal{G}(F) \cap ([a, b] \times [c, d])$.

We now define red-blue function pairs, which are used to define equivalence classes of pairs of useful functions. The red-blue function pairs are analogs of the red and blue functions from Section 1.

Definition 2.14. Let $R = r + \sum_{x\in\mathbb{R}} u^-_x \Delta^-_x + \sum_{x\in\mathbb{R}} u^+_x \Delta^+_x$ and let $B = b + \sum_{x\in\mathbb{R}} v^-_x \Delta^-_x + \sum_{x\in\mathbb{R}} v^+_x \Delta^+_x$ be useful functions. We say the pair (R, B) is a *red-blue function pair* if the following criteria are satisfied.

(1) For all $x \in \mathbb{R}$, we have $R_{\max}(x) = R(x)$ and $B_{\min}(x) = B(x)$.
(2) If s is a source, including possibly at $\pm\infty$, $u^-_s = u^+_s = v^-_s = v^+_s = 0$.
(3) For all $x \in \overline{\mathbb{R}}$,

$$R(x) \le B_{\max}(x) \quad \text{and} \quad R_{\min}(x) \le B(x).$$

(4) We have $R(-\infty) = B(-\infty)$ and $R(+\infty) = B(+\infty)$.
(5) The useful function R is constant on blue intervals. That is: for $s \le x < y < s'$ in $\mathbb{R}$ where (s, s') is blue, we have $r(x) = r(y)$ and $u^-_y = u^+_y = 0$.
(6) The useful function B is constant on red intervals. That is: for $s < x < y \le s'$ in $\mathbb{R}$ where (s, s') is red, we have $b(x) = b(y)$ and $v^-_x = v^+_x = 0$.

Lemma 2.15. *Let (R, B) be a red-blue function pair.*

1. *For any $a \le b$ and $c \le d$ in $\mathbb{R}$, the set $\mathcal{G}(R) \cap ([a, b] \times [c, d])$ is closed in $\mathbb{R}^2$.*
2. *For any $a \le b \in \mathbb{R}$ the useful function R has a local maximum on $[a, b]$ in the sense that there exists $x \in [a, b]$ such that for all $y \in [a, b]$: $R_{\max}(y) \le R_{\max}(x)$.*
3. *For any $a \le b \in \mathbb{R}$ the useful function R has a local minimum on $[a, b]$ in the sense that there exists $x \in [a, b]$ such that for all $y \in [a, b]$: $R_{\min}(x) \le R_{\min}(y)$.*

Statements (1)–(3) *are true when we replace R, r, and u with B, b, and v, respectively.*

Proof. We first prove (1) for R as the proof for B is identical. Let $\{(x_i, y_i)\}$ be a sequence in $\mathcal{G}(R) \cap ([a, b] \times [c, d])$ that converges to (w, z). Since $\{x_i\}$ converges to w we assume, without loss of generality, that $\{x_i\}$ is monotonic. If there exists $i \in \mathbb{N}$ such that $x_i = w$ then, assuming monotonicity, we know $(w, z) \in \mathcal{G}(R)$. Thus, assume $x_i \ne w$ for all $i \in \mathbb{N}$.

Without loss of generality, assume $\{x_i\}$ is increasing. The decreasing case is similar. Since $\sum_{x \in \overline{\mathbb{R}}} |u_x^-| + |u_x^+| < \infty$, we know that

$$\lim_{i \to \infty} |R_{\max}(x_i) - R_{\min}(x_i)| = 0.$$

And so,

$$\lim_{i \to \infty} R_{\max}(x_i) = \lim_{i \to \infty} R_{\min}(x_i) = \lim_{i \to \infty} R(x_i) = \lim_{x \to w^-} R(x).$$

Then we must have

$$\lim_{i \to \infty} y_i = \lim_{x \to w^-} R(x).$$

Therefore, $(w, z) \in \mathcal{G}(R)$.

Next, we only prove (2) for R as the remaining proofs are similar and symmetric. By Remark 2.13 there exists $c \le d \in \mathbb{R}$ such that

$$\mathcal{G}(R) \cap ([a, b] \times \mathbb{R}) = \mathcal{G}(R) \cap ([a, b] \times [c, d]).$$

Then there must be a greatest lower bound $d_0 \ge c$ for all d such that the equation above holds. Since $\mathcal{G}(R) \cap ([a, b] \times [c, d_0])$ must be closed by Lemma 2.15 (1), there must be a point $(x, d_0) \in \mathcal{G}(R)$ for $a \le x \le b$. This is the desired x. □

Remark 2.16. Lemma 2.15 (1) implies that $\mathcal{G}(R)$ is connected. Indeed, if $(a, c), (b, d)$ are in different components of $\mathcal{G}(R)$ then $\mathcal{G}(R) \cap ([a, b] \times [c, d])$ (or instead $\mathcal{G}(R) \cap ([a, b] \times [d, c])$ if $d < c$) is a disjoint union of two or more closed and thus compact subsets. Their images in $[a, b]$ form two or more disjoint compact subsets whose union is $[a, b]$, contradicting that the interval is connected. Similarly $\mathcal{G}(B)$, $\overline{\mathcal{G}(\mathcal{R})}$ and $\overline{\mathcal{G}(\mathcal{B})}$ are connected.

Also, $\overline{\mathcal{G}(\mathcal{R})}$ and $\overline{\mathcal{G}(\mathcal{B})}$ meet at $+\infty$ and $-\infty$. So, we have the intermediate value theorem: given $(x, h) \in \mathbb{R}^2$ so that $R_{\max}(x) < h < B_{\min}(x)$, there are $w < x$ and $z > x$ so that $(w, h), (z, h)$ lie in $\overline{\mathcal{G}(\mathcal{R})} \cup \overline{\mathcal{G}(\mathcal{B})}$. By Lemma 2.15 (1), we can take w maximal and z minimal. Then $R_{\max}(y) < h < B_{\min}(y)$ for all $y \in (w, z)$.

2.3. Stability conditions.

Definition 2.17. Let (R, B) and (R', B') be red-blue function pairs. We say (R, B) and (R', B') are *equivalent* if there exists a constant $\mathfrak{c} \in \mathbb{R}$ such that, for all $x \in \overline{\mathbb{R}}$ and $y \in \mathbb{R}$, we have

$$(x, y) \in \overline{\mathcal{G}(\mathcal{R})} \quad \text{if and only if} \quad (x, y + \mathfrak{c}) \in \overline{\mathcal{G}(\mathcal{R}')}$$

and

$$(x, y) \in \overline{\mathcal{G}(\mathcal{B})} \quad \text{if and only if} \quad (x, y + \mathfrak{c}) \in \overline{\mathcal{G}(\mathcal{B}')}.$$

A *stability condition on* Q, denoted σ, is an equivalence class of red-blue function pairs. We denote by $\mathcal{S}(Q)$ the set of stability conditions on Q.

We now define the *modified* versions of a continuous quiver Q of type $\mathbb{A}$, an interval I of a module M_I in $\mathrm{Ind}^{\mathrm{r}}(Q)$, and graphs of red-blue function pairs. This makes it easier to check whether or not an indecomposable module is semistable with respect to a particular stability condition.

Definition 2.18.

(1) Let Q be a continuous quiver of type $\mathbb{A}$ with finitely many sinks and sources. We define a totally ordered set $\widehat{Q}$, called the *modified quiver* of Q, in the following way.
 First we define the elements:
 - For each $x \in \mathbb{R}$ such that x is not a sink nor a source of Q, $x \in \widehat{Q}$.
 - If $s \in \mathbb{R}$ is a source of Q, including possibly $\pm\infty$, then $s \notin \widehat{Q}$.
 - If $s \in \mathbb{R}$ is a sink of Q, then $s_-, s_+ \in \widehat{Q}$. These are distinct elements, neither of which is in $\mathbb{R}$.
 - If $-\infty$ (respectively, $+\infty$) is a sink then $-\infty_+ \in \widehat{Q}$ (respectively, $+\infty_- \in \widehat{Q}$).

 Now, the partial order on $\widehat{Q}$ is defined in the following way. Let $x, y \in \widehat{Q}$.
 - Suppose $x, y \in \mathbb{R} \cap \widehat{Q}$. Then $x \le y$ in $\widehat{Q}$ if and only if $x \le y$ in $\mathbb{R}$.
 - Suppose $x \in \mathbb{R}$ and $y = s_\pm$, for some sink s of Q in $\mathbb{R}$. If $x < s$ in $\mathbb{R}$ then $x < y$ in $\widehat{Q}$. If $s < x$ in $\mathbb{R}$ then $y < x$ in $\widehat{Q}$.
 - Suppose $x = s_\varepsilon$ and $y = s'_{\varepsilon'}$ for two sinks s, s' of Q in $\mathbb{R}$. We consider $- < +$. Then $x \le y$ if and only if (i) $s < s'$ or (ii) $s = s'$ and $\varepsilon \le \varepsilon'$.
 - If $x = -\infty_+ \in \widehat{Q}$ (respectively, $y = +\infty_- \in \widehat{Q}$), then x is the minimal element (respectively, y is the maximal element) of $\widehat{Q}$.

 If $s \in \mathbb{R}$ is a sink of Q then we say s_- is blue and s_+ is red. If $-\infty_+ \in \widehat{Q}$ we say $-\infty_+$ is red. If $+\infty_- \in \widehat{Q}$ we say $+\infty_-$ is blue. All other $x \in \widehat{Q}$ are red (respectively, blue) if and only if $x \in \mathbb{R}$ is red (respectively, blue).

(2) Let I be an interval of $\mathbb{R}$ such that M_I is in $\mathrm{Ind}^{\mathrm{r}}(Q)$. The *modified interval* $\widehat{I}$ of $\widehat{Q}$ is $\widehat{I} = [\min \widehat{I}, \max \widehat{I}]$ where $\min \widehat{I}$ is given by the following conditions:
 - If $\inf I$ is not $-\infty$ nor a sink of Q then $\min \widehat{I} = \inf I$.
 - If $\inf I$ is a sink s of Q then (i) $\min \widehat{I} = s_-$ if $\inf I \in I$ or (ii) $\min \widehat{I} = s_+$ if $\inf I \notin I$.
 - If $\inf I = -\infty$ then $\min \widehat{I} = -\infty_+$.

 Similarly, $\max \widehat{I}$ is given by these conditions:
 - If $\sup I$ is not $+\infty$ nor a sink of Q then $\max \widehat{I} = \sup I$.
 - If $\sup I$ is a sink s of Q then (i) $\max \widehat{I} = s_+$ if $s \in I$ else (ii) $\max \widehat{I} = s_-$.
 - If $\sup I = +\infty$ then $\max \widehat{I} = +\infty_-$.

 In all cases $\min \widehat{I} \neq \max \widehat{I}$.

(3) Let (R, B) be a red-blue function pair. The *modified graph* of (R, B) is the subset $\widehat{\mathcal{G}}(R, B)$ of $\widehat{Q} \times \mathbb{R}$ defined as follows.
 - For each $x \in \mathbb{R}$ not a sink nor a source of Q and each $y \in \mathbb{R}$,
 $$(x, y) \in \widehat{\mathcal{G}}(R, B)$$
 if and only if
 $$[[(x, y) \in \mathcal{G}(B) \text{ and } x \text{ is blue}] \quad \text{or} \quad [(x, y) \in \mathcal{G}(R) \text{ and } x \text{ is red}]]$$

- For each $s \in \mathbb{R}$ a sink of Q and each $y \in \mathbb{R}$,

$$(s_-, y) \in \widehat{\mathcal{G}}(R, B) \quad \text{if and only if} \quad (s, y) \in \mathcal{G}(B)$$
$$(s_+, y) \in \widehat{\mathcal{G}}(R, B) \quad \text{if and only if} \quad (s, y) \in \mathcal{G}(R).$$

- If $-\infty_+ \in \widehat{Q}$, then for all $y \in \mathbb{R}$,

$$(-\infty_+, y) \in \widehat{\mathcal{G}}(R, B) \quad \text{if and only if} \quad (-\infty, y) \in \overline{\mathcal{G}(\mathcal{R})}.$$

- If $+\infty_- \in \widehat{Q}$, then for all $y \in \mathbb{R}$,

$$(+\infty_-, y) \in \widehat{\mathcal{G}}(R, B) \quad \text{if and only if} \quad (+\infty, y) \in \overline{\mathcal{G}(\mathcal{B})}.$$

Remark 2.19. The doubling of a sink $s \in Q$ into $s_-, s_+ \in \widehat{Q}$ can be interpreted in terms of microlocal sheaves [16]. As in [17], the finitely presented representation of Q with support the half open interval $(a, b]$ contained in the red part of Q can be viewed as a section of a sheaf of representations of Q over $(a, b]$. These representations extend to the sink, e.g., $(s, b]$. The "microlocalization" of this sheaf of red modules is an additional point in the nonpropagation direction of the red sheaf at the boundary point s which we identify with the blue point s_-. Similarly, the red point s_+ can be viewed as the microlocalization of the sheaf of blue modules to the left of s. Since Q is 1-dimensional, the microlocalization is only one point. Analogous constructions in higher dimensions will be explored in another paper [11], a sequel to [12] and [10].

The following proposition follows from straightforward checks.

Proposition 2.20. *There is a bijection between* $\mathrm{Ind}^{\mathrm{r}}(Q)$ *and intervals of* $\widehat{Q}$ *with distinct minimal and maximal element.*

Definition 2.21. Let $\widehat{I}$ be a modified interval in $\widehat{Q}$ for some indecomposable M_I. We say a horizontal line segment $\ell = \widehat{I} \times \{h\}$ in $\widehat{Q} \times \mathbb{R}$ is *perfectly in bounds* if the following two conditions are satisfied.

(1) The endpoints of ℓ touch $\widehat{\mathcal{G}}(R, B)$. That is, $(\min \widehat{I}, h), (\max \widehat{I}, h) \in \widehat{\mathcal{G}}(R, B)$.
(2) The line ℓ may touch but not cross $\widehat{\mathcal{G}}(R, B)$. That is, for each $x \in \widehat{I}$ such that $x \notin \{\max \widehat{I}, \min \widehat{I}\}$, we have

$$R_{\max}(x) \leq h \leq B_{\min}(x),$$

where if $x = s_\pm$ then $R_{\max}(x) = R_{\max}(s)$ and $B_{\min}(x) = B_{\min}(s)$.

In this case, we call the horizontal line segment $\ell = \widehat{I} \times \{h\}$ a *stability indicator* for the indecomposable M_I. There may be more than one stability indicator for the same M_I. (See X in Figure 4.4.)

Using the modified definitions, we now define what it means to be semistable.

Definition 2.22. Let Q be a continuous quiver of type $\mathbb{A}$ with finitely many sinks and sources, $\sigma \in \mathcal{S}(Q)$, and (R, B) be a representative of σ.

We say an indecomposable module M_I in $\mathrm{Ind}^{\mathrm{r}}(Q)$ is $\boldsymbol{\sigma}$*-semistable* if there exists a horizontal line $\ell = \widehat{I} \times \{h\} \subset \widehat{Q} \times \mathbb{R}$ such that ℓ is perfectly in bounds.

Remark 2.23. Notice that M_I is σ-semistable whenever the following are satisfied:

- We have $[F_{\min}(\inf I), F_{\max}(\inf I)] \cap [F'_{\min}(\sup I), F'_{\max}(\sup I)] \neq \emptyset$, where F is R if $\inf I$ is red and is B if $\inf I$ is blue and similarly for F' and $\sup I$.

- For all submodules M_J of M_I, ${F'}_{\min}(\sup J) \leq F_{\min}(\inf J)$, where $F, \inf J$ and $F', \sup J$ are similar to the previous point.

Thus, this is a continuous analogue to the semistable condition in the finite case.

Definition 2.24. Let Q be a continuous quiver of type $\mathbb{A}$ with finitely many sinks and sources, $\sigma \in \mathcal{S}(Q)$, and (R, B) be a representative of σ.

We say σ satisfies the *four point condition* if, for any σ-semistable module M_I, we have $|(\widehat{I} \times \{h\}) \cap \widehat{\mathcal{G}}(R, B)| \leq 3$, where $\widehat{I} \times \{h\}$ is as in Definition 2.22. Equivalently, every line segment $\ell = \widehat{I} \times \{h\}$ that is perfectly in bounds (Definition 2.21) intersects $\widehat{\mathcal{G}}(R, B)$ at most 3 times. We denote the set of stability conditions that satisfy the four point condition as $\mathcal{S}_{\text{fpc}}(Q)$.

The simplest non-example of Definition 2.24 is the constant function at 0.

Recall Definition 2.5.

Lemma 2.25. *Let Q be a continuous quiver of type $\mathbb{A}$ with finitely many sinks and sources and let M_I, M_J be indecomposables in $\text{Ind}^{\text{r}}(Q)$. Let $\widehat{I} = [a, b]$, $\widehat{J} = [c, d]$. Then $\text{Ext}^1(M_J, M_I) \cong \Bbbk \cong \text{Hom}(M_I, M_J)$ if and only if one of the following hold:*

- *$a < c < b < d$, and b, c are red;*
- *$c < a < d < b$, and a, d are blue;*
- *$c < a < b < d$, and a is blue, and b is red; or*
- *$a < c < d < b$, and c is red, and d is blue.*

Proof. It is shown in [14] that Hom and Ext between indecomposables must be 0 or 1 dimensional.

$\Rightarrow$. Since $\text{Hom}(M_I, M_J) \neq 0$ we obtain one of the items in the list where the first or last inequality may not be strict. Since $\text{Ext}(M_I, M_J) \neq 0$ we see all the inequalities must be strict.

$\Leftarrow$. The itemized list implies $\text{Hom}(M_I, M_J) \neq 0$. In the first two cases, there is a short exact sequence $M_I \hookrightarrow M_{I\cup J} \oplus M_{I\cap J} \twoheadrightarrow M_J$ and, in the last two cases we have $M_I \hookrightarrow M_{a,d} \oplus M_{b,c} \twoheadrightarrow M_J$ so, in all cases, $\text{Ext}(M_J, M_I) \neq 0$. ☐

Definition 2.26. Let M_I and M_J be indecomposables in $\text{Ind}^{\text{r}}(Q)$ for some continuous quiver of type $\mathbb{A}$ with finitely many sinks and sources. We say M_I and M_J are $\mathbf{N}_\pi$-*compatible* if both of the following are true:

$$\dim_{\Bbbk}(\text{Ext}(M_J, M_I) \oplus \text{Hom}(M_I, M_J)) \leq 1$$
$$\dim_{\Bbbk}(\text{Ext}(M_I, M_J) \oplus \text{Hom}(M_J, M_I)) \leq 1.$$

One can verify this is equivalent to Igusa and Todorov's compatibility condition in [15] when Q has the straight descending orientation.

In terms of colors and set operations, $\mathbf{N}_\pi$-compatibility can be expressed as follows.

Lemma 2.27. *M_I and M_J are $\mathbf{N}_\pi$-compatible if one of the following is satisfied.*

(1) $I \cap J = \emptyset$,
(2) $I \subset J$ and $J \setminus I$ is connected, or vice versa,
(3) $I \subset J$ and both endpoints of I are the same color, or vice versa,
(4) $I \cap J \neq I$, $I \cap J \neq J$, and $I \cap J$ has endpoints of opposite color.

Theorem 2.28. *Let $\sigma \in \mathcal{S}(Q)$. The following are equivalent.*

- *$\sigma \in \mathcal{S}_{\mathrm{fpc}}(Q)$.*
- *The set of σ-semistable indecomposables is maximally $\mathbf{N}_\pi$-compatible.*

Proof. Let (R, B) be a representative of σ.

$\Leftarrow$. We prove the contrapositive. Suppose σ does not satisfy the four point condition. Then there are $a < b < c < d$ in $\widehat{Q}$ that determine indecomposable modules $M_{a,b}$, $M_{a,c}$, $M_{a,d}$, $M_{b,c}$, $M_{b,d}$, $M_{c,d}$. Here, the notation $M_{x,y}$ means the interval indecomposable with interval I such that $\min \widehat{I} = x$ and $\max \widehat{I} = y$. Using Lemma 2.27 we see that at least two of the modules must be not $\mathbf{N}_\pi$-compatible.

$\Rightarrow$. Now suppose σ satisfies the four point condition. By Lemma 2.27 we see that the set of σ-semistable indecomposables is $\mathbf{N}_\pi$-compatible. We now check maximality.

Let M_J be an indecomposable in $\mathrm{Ind}^{\mathrm{r}}(Q)$ such that M_J is not σ-semistable. We will construct M_I which is σ-semistable but not $\mathbf{N}_\pi$-compatible with M_J. Recall left and right colors from Definition 2.6. There are four cases depending on whether J is left red or left blue and whether J is right red or right blue. However, the case where J is both left red and right red is similar to the case where J is both left and right blue. Furthermore, the cases where J is left red and right blue is similar to the case where J left blue and right red. Thus we reduce to two cases where J is left red: either (1) J is right blue or (2) J is right red. (Notice the case where M_J is a simple projective $M_{[s,s]}$ is similar to the case where J is left red and right blue.)

In both cases we claim that $\widehat{J}$ of M_J satisfies Definition 2.21 (2), i.e., there exists $h \in \mathbb{R}$ so that $R_{\max}(x) \leq h \leq B_{\min}(x)$ for all $x \in J$, $x \notin \{\inf J, \sup J\}$. We claim that if such an h does not exist then $R_{\max}$ goes above $B_{\min}$.

Indeed, suppose that for all $h \in \mathbb{R}$ there is $x \in J \setminus \{\inf J, \sup J\}$ such that either $h > R_{\max}(x)$ or $B_{\min} < h$. Let $K = J \setminus \{\inf J, \sup J\}$ and let $h = \inf_{z \in K}\{B_{\min}(z)\}$. Then there is $y \in K$ such that either $h < R_{\max}(y)$ or $B_{\min}(y) < h$. By definition of h, we know that we must have $h < R_{\max}(y)$. Then, by definition of h, there is some $x \in K$ such that $h \leq B_{\min}(x) < R_{\max}(y)$.

So, there exist $x, y \in J \setminus \{\inf J, \sup J\}$ such that x is blue with $B(x) = B_{\min}(x) < h$ and y is red with $R(y) = R_{\max}(y) > h$. Then, $R_{\min}(x) \leq B(x) < h$ and $B_{\max}(y) \geq R(y) > h$. So, the horizontal line at level h meets the graphs of both R and B in the interior of J. So, there are $(a, h), (b, h) \in \widehat{\mathcal{G}}(R, B)$ with a blue and b red. Choose these so that $|a - b|$ is minimal. Let $I = [a, b]$ or (b, a) depending on whether $a < b$ or $b < a$. Then M_I is σ-semistable and not compatible with M_J since I is contained in the interior of J with endpoints of opposite colors.

Therefore $\widehat{J}$ of M_J satisfies Definition 2.21 (2) and the set of all h satisfying $R_{\max}(x) \leq h \leq B_{\min}(x)$ for all $x \in J$, $x \notin \{\inf J, \sup J\}$ forms a closed interval $[h_{\min}, h_{\max}]$.

Case 1. Since M_J is not σ-semistable, $\widehat{J}$ fails Definition 2.21 (1) but satisfies Definition 2.21 (2). Then, for any $h \in [h_{\min}, h_{\max}]$, the horizontal line at level h either fails to meet $\overline{\mathcal{G}(\mathcal{R})}$ over $\inf J$ or it fails to meet $\overline{\mathcal{G}(\mathcal{B})}$ over $\sup J$. Since $R_{\min}(\inf J) \leq h_{\min} \leq h_{\max} \leq B_{\max}(\sup J)$, there are three possibilities:

(a) $R_{\max}(\inf J) < h < B_{\min}(\sup J)$ for some $h \in [h_{\min}, h_{\max}]$,
(b) $R_{\max}(\inf J) \geq h_{\max} < B_{\min}(\sup J)$ or,
(c) $R_{\max}(\inf J) < h_{\min} \geq B_{\min}(\sup J)$.

Since $R(-\infty) = B(-\infty)$, we cannot have $\inf J = -\infty$ in cases (a), (c). Similarly, $\sup J \neq +\infty$ in cases (a), (b). So, in case (a), by Remark 2.16, there are $w < \inf J$ and $z > \sup J$ so that $(w, h), (z, h)$ lie in $\widehat{\mathcal{G}}(R, B)$. If we take w maximal and z minimal, M_I corresponding to $[w, z] \subset \widehat{Q}$ (Proposition 2.20) will be σ-semistable. Since J lies in the interior of I with endpoints of opposite colors, M_I, M_J are not $\mathbf{N}_\pi$-compatible. This shows that, for any M_J which is not σ-semistable, there exists M_I which is σ-semistable and not N_π-compatible with M_J.

In case (b), since σ satisfies the four-point condition by assumption, any line segment ℓ that is perfectly in bounds touches the modified graph at most 3 times. Thus, J is a subinterval of an interval J' that is perfectly in bounds. So, there is a unique y in the interior of J so that $B(y) = h_{\max}$. Since $\sup J < +\infty$, there is a $z > \sup J$ in $\widehat{Q}$ so that M_I, corresponding to $\widehat{I} = [y, z]$ is σ-semistable. Since $\widehat{I} \cap \widehat{J} = [y, \sup J]$ with endpoints of the same color (blue), M_I, M_J are not $\mathbf{N}_\pi$-compatible.

Case (c) is similar. Thus, in all subcases of Case 1, there is M_I which is σ-semistable so that M_I, M_J are not $\mathbf{N}_\pi$-compatible.

Case 2. Suppose $\widehat{J}$ of M_J satisfies Definition 2.21 (2) but fails Definition 2.21 (1). Then, as in Case 1 either

(a) $R_{\max}(\inf J) < h_{\min}$ and $\inf J \neq -\infty$ or,
(b) $R_{\max}(\sup J) < h_{\min}$ and $\sup J \neq +\infty$.

In case (a), take $x \in J$ minimal so that $R_{\max}(x) = h_{\min}$. Then, there is $w < \inf J$ so that M_I is σ-semistable for $I = [w, x]$. But $\inf J, x$ have the same color. So, M_I, M_J are not $\mathbf{N}_\pi$-compatible. Case (b) is similar.

Therefore, Theorem 2.28 is true. □

Let T and T' be maximally $\mathbf{N}_\pi$-compatible sets. We call a bijection $\mu : T \to T'$ a *mutation* if $T' = (T \setminus \{M_I\}) \cup \{M_J\}$, for some $M_I \in T$ and $M_J \in T'$, and $\mu(M_K) = M_K$ for all $K \neq I$. (Then $\mu(M_I) = M_J$.)

3. Continuous tilting

We construct a continuous version of tilting. Consider a stability condition σ on a continuous quiver of type $\mathbb{A}$ where $-\infty$ is a sink and s is either the smallest source or a real number less than the smallest source. Then continuous tilting at s will replace the red interval $K = [-\infty, s)$ with the blue interval $K^* = (-\infty, s]$ and keep the rest of Q unchanged. Thus, $\widehat{Q} = K \coprod Z$ is replaced with $\widehat{Q}^* = K^* \coprod Z$. We have an order reversing bijection $\mathfrak{t} : K \to K^*$ given by

$$\mathfrak{t}(x) = \tan\left(\tan^{-1} s - \tan^{-1} x - \frac{\pi}{2}\right).$$

This extends, by the identity on Z, to a bijection $\bar{\mathfrak{t}} : \widehat{Q} \to \widehat{Q}^*$.

3.1. **Compatibility conditions.** We start with the continuous compatibility conditions for representable modules over the real line. Given a continuous quiver Q of type $\mathbb{A}$, we consider intervals I in $\mathbb{R}$. Let M_I denote the indecomposable module with support I. We say that I is *admissible* if M_I is representable. It is straightforward to see that I is admissible if and only if the following hold.

(1) $\inf I \in I$ if and only if it is blue, and,
(2) $\sup I \in I$ if and only if it is red.

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By Definition 2.3, neither endpoint of I can be a source. When $I = [s, s]$ is a sink, $\widehat{I} = [s_-, s_+]$. We use notation to state this concisely: for any $a < b \in \widehat{Q}$, let $\widehat{I}(a, b)$ be the unique admissible interval in $\mathbb{R}$ with endpoints a, b. Thus $a \in \widehat{I}(a, b)$if and only if a is blue and $b \in \widehat{I}(a, b)$ if and only if b is red. (Recall that every element of $\widehat{Q}$ is colored red or blue.)

Recall that for each $\sigma \in \mathcal{S}_{\text{fpc}}(Q)$, the set of σ-semistable modules form a maximally $\mathbf{N}_\pi$-compatible set (Theorem 2.28).

3.2. Continuous tilting on modules.

Lemma 3.1.

(a) *Continuous tilting gives a bijection between admissible intervals $I = \widehat{I}(a, b)$ for Q and admissible intervals I' for Q' given by $I' = \widehat{I}'(\bar{\mathfrak{t}}(a), \bar{\mathfrak{t}}(b))$ if $\bar{\mathfrak{t}}(a) < \bar{\mathfrak{t}}(b)$ in $\widehat{Q}'$ and $I' = \widehat{I}'(\bar{\mathfrak{t}}(b), \bar{\mathfrak{t}}(a))$ if $\bar{\mathfrak{t}}(b) < \bar{\mathfrak{t}}(a)$.*

(b) *Furthermore, M_I, M_J are $\mathbf{N}_\pi$-compatible for Q if and only if $M_{I'}, M_{J'}$ are $\mathbf{N}_\pi$-compatible for Q'.*

For each admissible interval I for Q, denote by $\phi(M_I)$ the module $M_{I'}$, where I' is the admissible interval of Q' obtained from I by continuous tilting.

Lemma 3.1 immediately implies the following.

Theorem 3.2. *Continuous tilting gives a bijection Φ between maximal compatible sets of representable indecomposable modules over Q and those of Q'. Furthermore if $\mu : T \to T'$ is a mutation then so is $\Phi(\mu) : \Phi T \to \Phi T'$ given by $\phi(M_I) \mapsto \phi(\mu(M_I))$.*

Proof of Lemma 3.1.

(a). Since $\bar{\mathfrak{t}} : \widehat{Q} \to \widehat{Q}'$ is a bijection and $\widehat{I}(a, b)$ is admissible by notation, we get a bijection with admissible Q' intervals by definition.

(b). Suppose that $I = \widehat{I}(a, b)$ and $J = \widehat{I}(c, d)$ with $a \le c$ by symmetry. We use Lemma 2.27 to check $\mathbf{N}_\pi$-compatibility. For this proof, we say "I and J are compatible" to mean "M_I and M_J are $\mathbf{N}_\pi$-compatible".

(1) If a, b, c, d are not distinct then $\bar{\mathfrak{t}}(a), \bar{\mathfrak{t}}(b), \bar{\mathfrak{t}}(c), \bar{\mathfrak{t}}(d)$ are also not distinct. So, I, J are compatible for Q and I', J' are compatible for Q' in this case. So, suppose $S = \{a, b, c, d\}$ has size $|S| = 4$.

(2) If $S \cap K = \emptyset$ then $I, J \subset Z$. So, $I' = I$ and $J' = J$ are compatible for Q' if and only if I, J are compatible for Q.

(3) If $|S \cap K| = 1$ then $S \cap K = \{a\}$. Then $\bar{\mathfrak{t}}$ does not change the order of a, b, c, d and does not change the colors of b, c, d. So, I, J are compatible for Q if and only if I', J' are compatible for Q'.

(4) If $|S \cap K| = 2$ there are three cases:
   (a) $a < b < c < d$,
   (b) $a < c < b < d$,
   (c) or $a < c < d < b$.

   If I, J are in case (4a) then so are I', J' and both pairs are compatible. If I, J are in case (4b) then I', J' are in case (4c) and vise versa. Since the colors of a, c change in both cases (from red to blue), I, J are compatible for Q if and only if I', J' are compatible for Q'.

(5) If $|S \cap K| = 3$ there are the same three cases as in case (4). If I, J are in case (4a), then I', J' are in case (4c) and vise-versa. Since the middle two vertices are the same color, both pairs are compatible. If I, J are in case (4b) then so are I', J' and both pairs are not compatible.
(6) If $S \subset K$ then a, b, c, d reverse order and all become blue. So, I, J are compatible if and only if they are in cases (4a) or (4c) and I', J' are in the same case and are also compatible.

In all cases, I, J are compatible for Q if and only if I', J' are compatible for Q'. □

We can relate continuous tilting to cluster theories, introduced by the authors and Todorov in [13].

Definition 3.3. Let $\mathcal{C}$ be an additive, $\Bbbk$-linear, Krull–Remak–Schmidt, skeletally small category and let $\mathbf{P}$ be a pairwise compatibility condition on the isomorphism classes of indecomposable objects in $\mathcal{C}$. Suppose that for any maximally $\mathbf{P}$-compatible set T and $X \in T$ there exists at most one $Y \notin T$ such that $(T \setminus \{X\}) \cup \{Y\}$ is $\mathbf{P}$-compatible.

Then we call maximally $\mathbf{P}$-compatible sets $\mathbf{P}$-*clusters*. We call bijections $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ of $\mathbf{P}$-clusters $\mathbf{P}$-*mutations*. We call the groupoid whose objects are $\mathbf{P}$-clusters and whose morphisms are $\mathbf{P}$-mutations (and identity functions) the $\mathbf{P}$-*cluster theory of* $\mathcal{C}$. We denote this groupoid by $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ and denote the inclusion functor into the category of sets and functions by $I_{\mathcal{C},\mathbf{P}} : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathcal{S}et$. We say $\mathbf{P}$ *induces* the $\mathbf{P}$-cluster theory of $\mathcal{C}$.

The isomorphism of cluster theories was introduced by the second author in [22].

Definition 3.4. An *isomorphism of cluster theories* is a pair (F, η) with source $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ and target $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$. Then F is a functor $F : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ such that F induces a bijection on objects and morphisms. The η is a natural transformation $\eta : I_{\mathcal{C},\mathbf{P}} \to I_{\mathcal{D},\mathbf{Q}} \circ F$ such that each component morphism $\eta_T : T \to F(T)$ is a bijection.

We see that, for any continuous quiver Q of type $\mathbb{A}$, the pairwise compatibility condition $\mathbf{N}_\pi$ induces the cluster theory $\mathscr{T}_{\mathbf{N}_\pi}(\mathrm{mod}^{\mathrm{r}}(Q))$. The following corollary follows immediately from Theorem 3.2.

Corollary 3.5 (To Theorem 3.2). *For any pair of continuous quivers Q and Q' of type $\mathbb{A}$ with finitely many sinks and sources, there is an isomorphism of cluster theories*

$$\mathscr{T}_{\mathbf{N}_\pi}(\mathrm{mod}^{\mathrm{r}}(Q)) \longrightarrow \mathscr{T}_{\mathbf{N}_\pi}(\mathrm{mod}^{\mathrm{r}}(Q')).$$

3.3. **Continuous tilting of stability conditions.** Given a stability condition σ for Q, we obtain a stability condition σ' for Q' having the property that the σ'-semistable modules are related to the σ-semistable modules for Q by continuous tilting (the bijection Φ of Theorem 3.2). Later we will see that these stability conditions give the same measured lamination on the Poincaré disk.

We continue with the notation from Sections 3.1 and 3.2 above. If the stability condition σ on Q is given by the red-blue pair (R, B), the tilted stability condition σ' on Q' will be given by (R', B') given as follows:

(1) The pair (R', B') will be the same as (R, B) on $[s, \infty)$.
(2) On $K' = (-\infty, s_-] \subseteq \widehat{Q}'$, the new red function R' will be constantly equal to $R_-(s)$.
(3) On $K' = (-\infty, s_-]$, the new blue function B' can be given by "flipping" R horizontally and flipping each "island" vertically, in either order.

Notation 3.6. Let F be a useful function. By $F_-(a)$ we denote $\lim_{x\to a^-} F(x)$, for any $a \in (-\infty, +\infty]$. By $F_+(a)$ we denote $\lim_{x\to a^+} F(x)$, for any $a \in [-\infty, +\infty)$.

Definition 3.7. A (red) *island* in $K = [-\infty, s) \subseteq \widehat{Q}$ is an open interval (x, y) in K which is either:

(1) (x, s) where $x < s$ so that $R(x) \geq R_-(s)$ and $R(z) < R_-(s)$ for all $x < z < s$.
(2) Or (x, y) where $x < y < s$, $R(x) \geq R(y) \geq R_-(s)$, $R(z) < R(y)$ for all $x < z < y$ and $R(w) \leq R(y)$ for all $y < w < s$.

Lemma 3.8. *$z \in (-\infty, s)$ is in the interior of some island in K if and only if there exists $y \in (z, s)$ so that $R(z) < R(y)$.*

Proof.

$(\Rightarrow)$. If z lies in the interior of an island (x, y) there are two cases:

(1) For $y < s$, $R(z) < R(y)$.
(2) For $y = s$, $R(z) < R_-(s)$.

But $R_-(s)$ is a limit, so there is a $y < s$ arbitrarily close to s so that $R(z) < R(y)$ and $z < y < s$.

$(\Leftarrow)$. Let $y \in (z, s)$ so that $R(z) < R(y)$. Let $r = sup\{R(y) : y \in (z, s)\}$. If $r = R(y)$ for some $y \in (z, s)$, let y be minimal. (By the 4 point condition there are at most 2 such y.) Then z lies in an island (x, y) for some $x < z$.

If the maximum is not attained, there exists a sequence y_i so that $R(y_i)$ converges to r. Then y_i converges to some $w \in [z, s]$. If $w \in (z, s)$ then $R(z) = r$ and we are reduced to the previous case. Since $R(z) < r$, $w \neq z$. So, $w = s$ and $r = R_-(s)$. Then z lies in an island (x, s) for some $x < z$. $(x = \max\{w < z : R(w) \geq r\})$ In both cases, z lies in an island as claimed. □

To define the new blue function B', we need a function H defined as follows.

$$H(z) := \begin{cases} R(y) & \text{if } z \in (x, y] \text{ for some island } (x, y) \text{ where } y < s, \\ R_-(s) & \text{if } z \in (x, s) \text{ and } (x, s) \text{ is an island,} \\ R(z) & \text{for all other } z \in [-\infty, s). \end{cases}$$

Remark 3.9. Note that $H(z) > R(z)$ if z is in the interior of an island and $H(z) = R(z)$ otherwise.

Lemma 3.10. *H is a nonincreasing function, i.e., $H(x) \geq H(y)$ for all $x < y < s$. Also, $H(z) = H_-(z) = \lim_{y\to z-} H(y)$ for all $z < s$ and $H_-(s) = R_-(s)$.*

Remark 3.11. Since H is decreasing and converging to $R_-(s)$ we must have: $H(x) = H_-(x) \geq H_+(x) \geq R_-(s)$ for all $x < s$.

Proof. If $H(u) < H(z)$ for some $u < z < s$ then $R(u) \leq H(u) < H(z)$. But $H(z)$ is equal to either $R(z), R_-(s)$ or $R(y)$ for some $y > z$. So, $R(u) < R(y)$ for some $y \in (u, s)$. By Lemma 3.8, u lies in the interior of some island, say (x, y) and, by definition of H, $H(u) = R(y) \geq R(w)$ for all $w \geq y$ and $H(u) = H(z) = H(y)$ for all $u \leq z \leq y$. Thus, H is nonincreasing.

To see that $H(z) = H_-(z)$ suppose first that $z \in (x, y]$ for some island (x, y). Then $H(z) = R(y)$ is constant on the interval $(x, y]$. So, $H(z) = H_-(z) = R(y)$. Similarly, $H(z) = H_-(z)$ if $z \in (x, s)$ and (x, s) is an island. If z is not in any island, $H(z) = R(z)$ and

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$R(z) = R_-(z)$ since, otherwise, z would be on the right end of an island. And, $H_-(z)$ would be the limit of those $H(x)$ where $x < z$ and $H(x) = R(x)$. So, $H_-(z) = R_-(z) = H(z)$ as claimed.

Since $H(y) \geq R(y)$, we have: $H_-(s) = \lim_{y \to s-} H(y) \geq R_-(s)$. If $H_-(s) > R_-(s)$, say $H_-(s) = R_-(s) + c$ then there is a sequence $z_i \to s-$ so that $H(z_i) > R_-(s) + c/2$. For each z_i there is $y_i \in [z_i, s)$ so that $H(z_i) = R(y_i)$. Then $R(y_i) > R_-(s) + c/2$ for all i which is not possible since $y_i \to s-$. So, $H_-(s) = R_-(s)$. $\square$

The monotonicity of H implies that its variation $\mathsf{var}_H I$ on any interval I is the difference of its limiting values on the endpoints. The formula is:

$$\mathsf{var}_H(a, b) = H_+(a) - H_-(b).$$

Using $H = H_-$ and H_+ we can "flip" the islands up to get $\widetilde{R}$:

$$\widetilde{R}(z) = H(z) + H_+(z) - R(z).$$

Definition 3.12. The new blue function B', shown in Figure 3.1, is given on $K^* = (-\infty, s]$ by

$$B'(z) = \widetilde{R}(\mathfrak{t}(z)).$$

The new red function is constant on K^* with value $R'(x) = R_-(s)$ for all $x \in K^*$. On the complement of K^* in $\widehat{Q}'$, the red-blue pair (R', B') is the same as before.

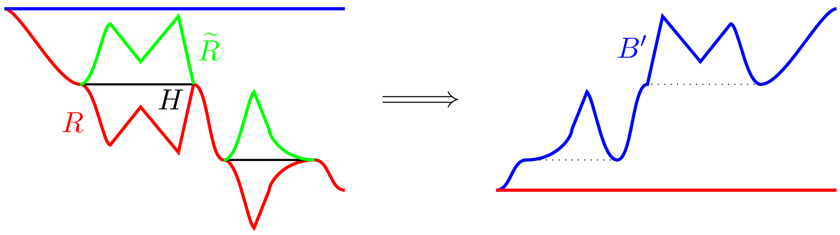

FIGURE 3.1. The function R is in red. H, black, flattens the islands of R. When the islands are flipped up, we get $\widetilde{R}$ in green. The horizontal mirror image of this is the new blue function B' on the right. Figures 4.4, 4.6 give another example.

We will now show B' is a useful function with the same variation on $(-\infty, s]$ as R has on $[-\infty, s)$. More precisely:

Lemma 3.13. *The variation of R on any open interval $(a, b) \subset [-\infty, s)$ is equal to the variation of B' on $(\mathfrak{t}(b), \mathfrak{t}(a))$.*

Proof. Since B' is obtained from $\widetilde{R}$ by reversing the order of the first coordinate, we have $\mathsf{var}_{B'}(\mathfrak{t}(b), \mathfrak{t}(a)) = \mathsf{var}_{\widetilde{R}}(a, b)$. Thus, it suffices to show that $\mathsf{var}_{\widetilde{R}}(a, b) = \mathsf{var}_R(a, b)$.

First, we do the case when (a, b) is an island. Then $H(z) = H_+(z) = R(b) > R(z)$ are constant for all $z \in (a, b)$. So, $\widetilde{R} = H + H_+ - R$ has the same variation as R on (a, b).

Write $R = H + (R - H)$. Then we claim that

$$\mathsf{var}_R(a, b) = \mathsf{var}_H(a, b) + \mathsf{var}_{R-H}(a, b).$$

To see this take any sequence $a < x_0 < x_1 < \cdots < x_n < b$. Then the sum

$$\sum_{i=1}^{n} |R(x_i) - R(x_{i-1})|$$

can be broken up into parts. Let $A_1, \dots, A_m$ be the sequence of disjoint subsets of $S = \{x_0, \dots, x_n\}$ so that A_j is the intersection of S with some island (a_j, b_j). We may assume that a_j for $1 < j \leq m$ and b_j for $1 \leq j < m$ are in the set S since they lie in the interval (a, b). For $1 < j \leq m$, if x_i is the smallest element of A_j, then $x_{i-1} = a_j$ and the x_i, x_{i-1} term in the approximation of $\mathsf{var}_H(a, b) + \mathsf{var}_{H-R}(a, b)$ is

$$|H(a_j) - H(x_i)| + |(R - H)(a_j) - (R - H)(x_i)| = |R(a_j) - H(x_i)| + |H(x_i) - R(x_i)|$$

since $H(a_j) = R(a_j)$. This sum is equal to $|R(a_j) - R(x_i)|$, the corresponding term in the approximation of $\mathsf{var}_R(a, b)$, since $R(a_j) \geq H(x_i) > R(x_i)$. Similarly, $H(b_j) = R(b_j)$ by definition and $R(b_j) = H(x_k) > R(x_k)$ for any $x_k \in A_j$. So,

$$|H(b_j) - H(x_k)| + |(R - H)(b_j) - (R - H)(x_k)| = |R(b_j) - R(x_k)|.$$

If x_i, x_{i+1} both lie in A_j then $H(x_i) = H(x_{i+1})$. So,

$$|R(x_i) - R(x_{i+1})| = |(R - H)(x_i) - (R - H)(x_{i+1})| + |H(x_i) - H(x_{i+1})|.$$

This equation also holds if x_i, x_{i+1} do not lie in any A_j since, in that case, $R = H$ at both x_i and x_{i+1}. Thus every term in the sum approximating $\mathsf{var}_R(a, b)$ is equal to the sum of the corresponding terms for $\mathsf{var}_H(a, b)$ and $\mathsf{var}_{R-H}(a, b)$. Taking supremum we get the equation $\mathsf{var}_R(a, b) = \mathsf{var}_H(a, b) + \mathsf{var}_{R-H}(a, b)$ as claimed.

A similar calculation shows that

$$\mathsf{var}_{\widetilde{R}}(a, b) = \mathsf{var}_{H_+}(a, b) + \mathsf{var}_{\widetilde{R}-H_+}(a, b).$$

But this is equal to $\mathsf{var}_R(a, b) = \mathsf{var}_H(a, b) + \mathsf{var}_{R-H}(a, b)$ since $H - R = \widetilde{R} - H_+$ by definition of $\widetilde{R}$ and $\mathsf{var}_H(a, b) = H_+(a) - H_-(b) = \mathsf{var}_{H_+}(a, b)$. Thus $\mathsf{var}_R(a, b) = \mathsf{var}_{\widetilde{R}}(a, b) = \mathsf{var}_{B'}(\mathfrak{t}(b), \mathfrak{t}(a))$. ☐

For x_0 in the interior of the domain of f let

$$\mathsf{var}_f(x_0) := \lim_{\delta \to 0} \mathsf{var}_f(x_0 - \delta, x_0 + \delta) = \lim_{\delta \to 0} \mathsf{var}_f[x_0 - \delta, x_0 + \delta]$$

We call this the *local variation* of f at x_0. If $x_0 \in (a, b)$ this is equivalent to:

$$\mathsf{var}_f(x_0) = \mathsf{var}_f(a, b) - \mathsf{var}_f(a, x_0) - \mathsf{var}_f(x_0, b)$$

since this is the limit of $\mathsf{var}_f(a, b) - \mathsf{var}_f(a, x_0 - \delta] - \mathsf{var}_f[x_0 + \delta, b) = \mathsf{var}_f[x_0 - \delta, x_0 + \delta]$.

To show that B' is a useful function we need the following lemma.

Lemma 3.14. *A real valued function f of bounded variation defined in a neighborhood of x_0 is continuous at x_0 if and only if its local variation, $\mathsf{var}_f(x_0) = 0$. In particular, R is continuous at $x \in K$ if and only if B' is continuous at $\mathfrak{t}(x) \in K^*$.*

Proof. Suppose that $\mathsf{var}_f(x_0) = 0$. Then, for any $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\mathsf{var}_f(x_0 - \delta, x_0 + \delta) < \varepsilon.$$

Then $|f(x) - f(x_0)| < \varepsilon$ for all $x \in (x_0 - \delta, x_0 + \delta)$. So, f is continuous at x_0.

Conversely, suppose f is continuous at x_0. Then, for any $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x) - f(x_0)| < \varepsilon$ for $|x - x_0| < \delta$. Let $V = \mathsf{var}_f[x_0, x_0 + \delta)$. By definition of variation there exist $x_0 < x_1 < \cdots < x_n < x_0 + \delta$ so that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| > V - \varepsilon.$$

Since $|f(x_1)-f(x_0)|<\varepsilon$ this implies $\sum_{i=2}^n |f(x_i)-f(x_{i-1})| > V-2\varepsilon$. So, $\mathsf{var}_f[x_0,x_1)<2\varepsilon$. Similarly, there exists $x_{-1}<x_0$ so that $\mathsf{var}_f(x_{-1},x_0)<2\varepsilon$. So, $\mathsf{var}_f(x_{-1},x_1)<4\varepsilon$ which is arbitrarily small. ☐

For a useful function F, recall that $u_a^-=F(a)-\lim_{x\to a-}F(x)$ and $u_a^+=\lim_{x\to a+}F(x)-F(a)$ (Proposition 2.10).

Proposition 3.15. *Let F be a useful function. Then, the local variation of F at any point a is*

$$\mathsf{var}_F(a)=|u_a^-|+|u_a^+|.$$

Proof. Let $g_x=u_x^-\Delta_x^-+u_x^+\Delta_x^+$. Then $F-g_x$ is continuous at x and thus, by Lemma 3.14, has $\mathsf{var}_{F-g_x}(x)=0$. It follows from the triangle inequality that the variation of $F=g_x+(F-g_x)$ at x is equal to that of g_x. So,

$$\mathsf{var}_F(x)=\mathsf{var}_{g_x}(x)=|u_x^-|+|u_x^+|.$$ ☐

We can say slightly more for the functions R and B'. (See also Figure 3.2.)

Lemma 3.16. *For any $a\in K=[-\infty,s)$ let $b=\mathfrak{t}(a)\in K^*$. Then $v_b^-=B'(b)-B'_-(b)\leq 0$ and $v_b^+=B'_+(b)-B'(b)\geq 0$. In particular, $B'(b)=min\,B'(b)$.*

Proof. Since B' is the mirror image of $\widetilde{R}$, v_b^- and v_b^+ for B' are equal to $-v_a^+,-v_a^-$ for $\widetilde{R}$, respectively, where $v_a^-=\widetilde{R}(a)-\widetilde{R}_-(a)$ and $v_a^+=\widetilde{R}_+(a)-\widetilde{R}(a)$. Thus it suffices to show that $v_a^-\leq 0$ and $v_a^+\geq 0$.

We have $u_a^-=R(a)-R_-(a)\geq 0$. Also, $\widetilde{R}_-=(H+H_+-R)_-=2H-R_-$. So,

$$\begin{aligned}
v_a^- &= \Big(\widetilde{R}(a)-H_+(a)\Big)-\Big(\widetilde{R}_-(a)-H_+(a)\Big)\\
&=(H(a)-R(a))+R_-(a)-2H(a)+H_+(a)\\
&=-u_a^- -(H(a)-H_+(a))\leq 0
\end{aligned}$$

Similarly, we have $u_a^+=R_+(a)-R(a)\leq 0$ and $\widetilde{R}_+(a)=2H_+(a)-R_+(a)$. So,

$$\begin{aligned}
v_a^+ &= \Big(\widetilde{R}_+(a)-H_+(a)\Big)-\Big(\widetilde{R}(a)-H_+(a)\Big)\\
&=(H_+(a)-R_+(a))-(H(a)-R(a))\\
&=(H_+(a)-H(a))-u_a^+
\end{aligned}$$

To show that $v_a^+\geq 0$, there are two cases. If a lies in an island (x,y), then $H_+(a)=H(a)=R(y)$ (or $R_-(s)$ if $y=s$) and $v_a^+=-u_a^+\geq 0$. If a does not lie in an island then $H(a)=R(a)$ and $H_+(a)\geq R_+(a)$. So, $v_a^+\geq 0$. ☐

Theorem 3.17. *The new pair (R',B') is a red-blue pair for the quiver Q' and the σ'-semistable Q' modules given by this pair are the continuous tilts of the σ-semistable Q-modules given by the original pair (R,B).*

Proof. Lemma 3.13 implies that R and B' have the same local variation at the corresponding points x and $\mathfrak{t}(x)$. In particular, R and B' have discontinuities at corresponding points by Lemma 3.14 and the $B'(a)=\min B'(a)$ by Lemma 3.16.

The new red function R' is constantly equal to $R_-(s)$ on K^* and equal to the old function R on the complement Z. So, $B'(x)\geq R'(x)$ and they have the same limit as $x\to-\infty$ by Remark 3.11. Thus (R',B') form a red-blue pair for Q'.

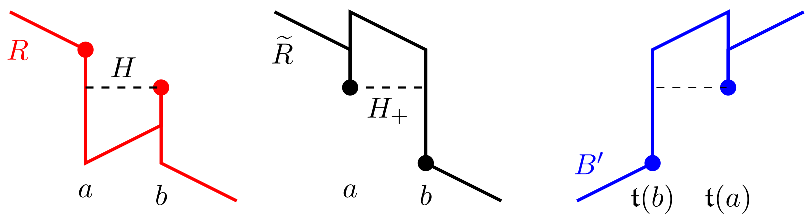

FIGURE 3.2. This red function R has a spike on the right end b of an island (a, b) and a discontinuity at the left end a. When the island is flipped, we get a downward spike at $\mathfrak{t}(a)$ and a discontinuity at $\mathfrak{t}(b)$. The function R is the maximum and the tilted functions $\widetilde{R}$ and B' are minimums on vertical lines.

Let σ, σ' be the stability conditions on Q, Q' given by the red-blue pairs (R, B) and (R', B'), resp. It remains to show that the admissible interval $I = \widehat{I}(a, b)$ is σ-semistable for Q if and only if the corresponding interval I' is σ'-semistable for Q' where $I' = \widehat{I}'(\bar{\mathfrak{t}}(a), \bar{\mathfrak{t}}(b))$ if $\bar{\mathfrak{t}}(a) < \bar{\mathfrak{t}}(b)$ in $\widehat{Q}'$ and $I' = \widehat{I}'(\bar{\mathfrak{t}}(b), \bar{\mathfrak{t}}(a))$ if $\bar{\mathfrak{t}}(b) < \bar{\mathfrak{t}}(a)$.

Consider $a < b$ in $\overline{\mathbb{R}}$. There are three cases.

(1) $a = \bar{\mathfrak{t}}(a)$ and $b = \bar{\mathfrak{t}}(b)$ both lie in Z.
(2) $-\infty \leq a < b < s$ $(a, b \in K)$ and $-\infty < \bar{\mathfrak{t}}(b) < \bar{\mathfrak{t}}(a) \leq s$ $(\bar{\mathfrak{t}}(a), \bar{\mathfrak{t}}(b) \in K^*)$.
(3) $a \in K$, $\bar{\mathfrak{t}}(a) \in K^*$ and $b = \bar{\mathfrak{t}}(b) \in Z$.

Case (1). The stability conditions σ, σ' given by the red and blue functions are the same on Z. So, $\widehat{I}(a, b)$ is σ-semistable if and only if $\widehat{I}'(a, b) = \widehat{I}'(\bar{\mathfrak{t}}(a), \bar{\mathfrak{t}}(b))$ is σ'-semistable for Q'.

Case (2). We claim that $\widehat{I}(a, b)$ at height h is σ-semistable if and only if $\widehat{I}'(\bar{\mathfrak{t}}(b), \bar{\mathfrak{t}}(a))$ is σ'-semistable at height h' where $h' = 2H(b) - h$.

An example can be visualized in Figure 3.2 by drawing horizontal lines at height $h < H$ and $h' > H_+$ under the line H on the left and over H_+ on the right.

To see this in general, note that if $\widehat{I}(a, b)$ at height h is σ-semistable then, for all $z \in (a, b)$, $R(z) \leq h$ (with equality holding for at most one value of z, call it $z = c$) and $R(a), R(b) \geq h$. Then for each $z \in [a, b]$, $H(z) \geq h$. So, for each $z \in (a, b)$, $z \neq c$, we have $H(z) > R(z)$. By Remark 3.9, z lies in the interior of an island for R. But $\widetilde{R}(z) - H_+(z) = H(z) - R(z) > 0$. So, the same values of z lie in islands for $\widetilde{R}$ and $\widetilde{R}(z) - h' = h - R(z) \geq 0$. Also, $\widetilde{R}(a), \widetilde{R}(b) \leq h'$ since:

$$\begin{aligned} h' - \widetilde{R}(b) &= 2H(b) - h - H(b) - H_+(b) + R(b) \\ &= (H(b) - H_+(b)) + (R(b) - h) \geq 0, \end{aligned}$$

and, since $H_+(a) = H(b)$ and $H(a) =$ either $H(b)$ or $R(a)$,

$$\begin{aligned} h' - \widetilde{R}(a) &= 2H(b) - h - H(a) - H_+(a) + R(a) \\ &= R(a) - h + H(b) - H(a), \\ \text{either } &= R(a) - h \geq 0, \\ \text{or } &= H(b) - h \geq 0. \end{aligned}$$

Therefore, $[a, b] \times h'$ is a chord for $\widetilde{R}$, making its mirror image $[\bar{\mathfrak{t}}(b), \bar{\mathfrak{t}}(a)] \times h'$ a chord for B' and thus $\widehat{I}'(\bar{\mathfrak{t}}(b), \bar{\mathfrak{t}}(a))$ is σ'-semistable for Q' at height h'. An analogous argument shows the converse. So, $\widehat{I}(a,b)$ at height h is σ-semistable for Q if and only if $\widehat{I}'(\bar{\mathfrak{t}}(b),\bar{\mathfrak{t}}(a))$ is σ'-semistable at height h' for Q'.

Case (3). We change notation to match Figure 3.2. Suppose we have $b\in K$, $\bar{\mathfrak{t}}(b)\in K^*$ and $c=\bar{\mathfrak{t}}(c)\in Z$. We claim that $\widehat{I}(b,c)$ is σ-semistable at height h if and only if $\widehat{I}'(\bar{\mathfrak{t}}(b),\bar{\mathfrak{t}}(c))$ is σ'-semistable at the same height h.

In Figure 3.2, the chord $[b,c]\times h$ would be a horizontal line starting at any point on the vertical red line at b and going to the right. For $\widetilde{R}$, we have $H(b)\geq h\geq H_+(b)$, so a horizontal line at height h starting anywhere on the vertical segment $b\times[H_+(b),H(b)]$ could go left without hitting the function $\widetilde{R}$ except at height $h=H_+(a)=H(b)$ where it would touch the function at $(a,H_+(a))$ then continue. For B', the horizontal line starting at $(\bar{\mathfrak{t}}(b),h)$ would go right, possibly touch the curve at $\bar{\mathfrak{t}}(a)$ and continue to the point (c,h).

The situation in general is very similar. $\widehat{I}(b,c)$ is σ-semistable at height h for some $c\in Z$ if and only if $H_+(b)\leq h\leq H(b)=R(b)$. Since $H_+(b)$ is the supremum of $R(x)$ for all $b<x<s$, this is equivalent to saying the horizontal line at h does not touch the curve R except possibly at one point (not more by the four point condition). If $h=H(b)$, this horizontal line might continue to the left of (b,h) and hit at most one point (a,h) on the curve R.

If $h<H(b)$ then the horizontal line at (b,h) on $\widetilde{R}$, would go to the left and not hit anything since, for all $x<b$, we have $\widetilde{R}(x)\geq H_+(x)\geq H(b)>h$. So, the line from $(\bar{\mathfrak{t}}(b),h)$ to $(\bar{\mathfrak{t}}(c),h)$ would not hit B'.

If $h=H(b)$, then, for all $x<b$, $\widetilde{R}(x)\geq H_+(x)\geq H(b)=h$. So, the line going left from $(b,h)=(b,H(b))$ would stay under $\widetilde{R}$ possibly touching it at most once, say at (a,h). Then (a,b) would be an island and we have the situation in Figure 3.2. By the four point condition we cannot have another point a' with the same property since $(a,h),(b,h),(c,h)$ are already on a line. The horizontal line going right from $(\bar{\mathfrak{t}}(b),h)$ would touch the curve B' at $(\bar{\mathfrak{t}}(a),h)$ and continue to $(\bar{\mathfrak{t}}(c),h)$.

So, $\widehat{I}(b,c)$ being σ-semistable at height h implies that $\widehat{I}'(\bar{\mathfrak{t}}(b),\bar{\mathfrak{t}}(c))$ is σ'-semistable at the same height h. The converse is similar since going from B' to R is analogous (change B' to $-B'$ and make it red). This concludes the proof of Theorem 3.17 in all cases. ☐

4. Measured Laminations and Stability Conditions

In this section we connect measured laminations of the hyperbolic plane to stability conditions for continuous quivers of type $\mathbb{A}$. We first define measured laminations (Definition 4.2) of the hyperbolic plane and prove some basic results we need in Section 4.1. In Section 4.2 we describe the correspondence that connects stability conditions to measured laminations. In Section 4.3 we present a candidate for continuous cluster characters. In Section 4.4 we briefly describe how all maximally $\mathbf{N}_\pi$-compatible sets come from a stability condition. In Section 4.5 we describe maps between cluster categories of type $\mathbb{A}_n$ that factor through our continuous tilting. We also give an example for type $\mathbb{A}_4$.

To clarify the argument, we give a simple example. Any continuous stability condition σ on a continuous quiver Q given by a red-blue pair (R,B), gives a measured lamination $L=\Phi(\sigma)$ in the following way. Each stability indicator for σ (horizontal line in $\widehat{Q}\times\mathbb{R}$ with endpoints on the graph $\widehat{\mathcal{G}}(R,B)$: see Definition 2.21) corresponds to the geodesic in the Poincaré disk $\mathfrak{h}^2$ with ideal endpoints corresponding to the endpoints of the indicator under the mapping $\Xi:\widehat{Q}\to\partial\mathfrak{h}^2$ (Definition 4.1). The transverse measure of a collection of geodesics in $\mathfrak{h}^2$ is given by the horizontal distance between the top and bottom indicators in the graph $\widehat{\mathcal{G}}(R, B)$. In our example, the fountain has measure 4 and the rainbow has measure 3 since these are the heights of the corresponding stack of indicators in the red-blue graph. In the lamination, there are additional geodesics spaced 1/3 units apart.

Given a measured lamination L, we explicitly construct the corresponding stability condition $\sigma = \Phi^{-1}(L)$ for Q with straight descending orientation in Theorem 4.14. For other orientations of Q we perform continuous tilting on σ to obtain $\sigma' = \Phi'^{-1}(L)$. The lamination is unchanged.

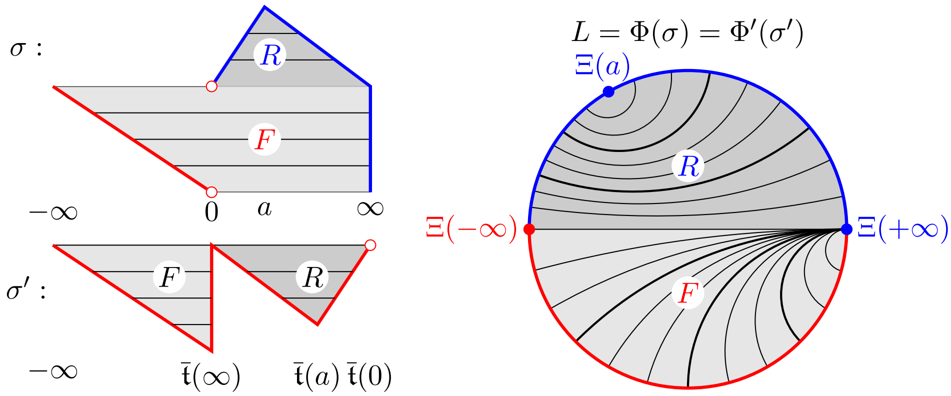

Figure 4.1. At top left we have a continuous stability condition σ on the continuous quiver Q having a source at 0 and no sinks. At bottom left we have the corresponding stability condition σ' on the straight descending quiver Q' given by continuous tilting of σ. On the right is the measured lamination L of the Poincaré disk which corresponds to both stability conditions. The geodesics of the lamination are given by the horizontal lines on σ and σ'. The endpoints of the geodesics are the points corresponding to the endpoints of these horizontal lines under the mapping Ξ (Definition 4.1). The measure of each part is given by the height of the corresponding portions of the stability condition.

Definition 4.1. For a general continuous quiver Q of type $\mathbb{A}$ with finitely many sinks and sources, we introduce a mapping $\Xi : \widehat{Q} \to \partial\mathfrak{h}^2$. Since $\mathbb{A}$ has finitely many sinks and sources, we can index the red intervals $I_1, I_2, \ldots, I_m$ and the blue intervals $I^1, I^2, \ldots, I^n$. An interval I_i is of the form $[a_i, b_i)$, for $a_i < b_i \in \mathbb{R} \cup \{-\infty\}$. An interval I^i is of the form $(a^i, b^i]$ for $a^i < b^i \in \mathbb{R} \cup \{+\infty\}$. We assume that if $i < j$ then $a_i < a_j$ and $b^i < b^j$. That is, I_0 and I^0 are the "lowest" red interval and blue interval, respectively.

First, set $\ell_0 = \ell^0 = 0$. For each $1 \leq j \leq m$, set $\ell_j = \sum_{k=1}^{j} 2(\arctan(b_k) - \arctan(a_k))$ and $\zeta_j = \pi - \ell_{j-1} + 2\arctan(a_j)$. For each $1 \leq j \leq n$, set $\ell^j = \sum_{k=1}^{j} 2(\arctan(b^k) - \arctan(a^k))$ and $\zeta^j = \pi - \ell^{j-1} + 2\arctan(a^j)$. We define maps $\Xi_j : \widehat{I}_j \to \partial\mathfrak{h}^2$ and $\Xi^j : \widehat{I}^j \to \partial\mathfrak{h}^2$ for each red interval I_j and blue interval I^j.

$$\begin{aligned} \Xi_j : \widehat{I}_j = [a_{j+}, b_j) &\longrightarrow \partial\mathfrak{h}^2 \\ x &\longmapsto e^{i(2\arctan(x) - \zeta_j)} \end{aligned} \qquad \begin{aligned} \Xi^k : \widehat{I}^k = (a^k, b^k_-] &\longrightarrow \partial\mathfrak{h}^2 \\ x &\longmapsto e^{i\left(\zeta^k - 2\arctan(x)\right)}. \end{aligned}$$

The map $\Xi : \widehat{Q} \to \partial\mathfrak{h}^2$ is the map that sends $x \in \widehat{I}_j$ and $x \in \widehat{I}^j$ to $\Xi_j(x)$ or $\Xi^j(x)$, respectively.

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The map Ξ wraps the red intervals around $\partial\mathfrak{h}^2$ starting at $e^{-i\pi} = e^{i\pi}$ and moving anticlockwise. The blue intervals are wrapped around starting at $e^{i\pi} = e^{-i\pi}$ and moving clockwise. It is straightforward to check that $\Xi : \widehat{Q} \to \partial\mathfrak{h}^2$ is a bijection.

4.1. **Measured Laminations.** We denote by $\mathfrak{h}^2$ the Poincaré disk model of the hyperbolic plane and by $\partial\mathfrak{h}^2$ the boundary of the disk such that $\partial\mathfrak{h}^2$ is the unit circle in $\mathbb{C}$. Recall a *lamination* of $\mathfrak{h}^2$ is a maximal set of noncrossing geodesics and that a geodesic in $\mathfrak{h}^2$ is uniquely determined by a distinct pair of points on $\partial\mathfrak{h}^2$.

Let L be a lamination of $\mathfrak{h}^2$. Choose two open interval subsets A and B of $\partial\mathfrak{h}^2$, each of which may be all of $\partial\mathfrak{h}^2$ or empty. Let $O_{A,B}$ be the set of geodesics with one endpoint in A and the other in B. We call $O_{A,B}$ a *basic open subset* of L. Notice that $O_{A,B} = O_{B,A}$. The basic open sets define a topology on L.

Definition 4.2. Let L be a lamination of $\mathfrak{h}^2$ and let $\mathcal{M}$ be a finite Borel measure on the σ-algebra generated by $\{O_{A,B}\}$. We say $(L, \mathcal{M})$ is a *measured lamination* if $0 < \mathcal{M}(O_{A,B}) < \infty$ for every $O_{A,B} \neq \emptyset$.

We immediately see any measured lamination $(L, \mathcal{M})$ has finite measure. That is, $0 < \mathcal{M}(L) < \infty$.

We now define some useful pieces of laminations.

Definition 4.3. Let L be a lamination of $\mathfrak{h}^2$.

(1) Let $\gamma \in L$ be a geodesic determined by $a, b \in \partial\mathfrak{h}^2$. We say γ is a *discrete arc* if there exists non-intersecting open subsets $A \ni a$ and $B \ni b$ of $\partial\mathfrak{h}^2$ such that $O_{A,B} = \{\gamma\}$.

(2) Let $a \in \partial\mathfrak{h}^2$. Let A be some interval subset of $\partial\mathfrak{h}^2$ with more than one element such that for every geodesic $\gamma \in L$ determined by some $a' \in A$ and $b \in \partial\mathfrak{h}^2$, we have $b = a$. Then we define the set K of geodesics determined by the pair a, A to be called a *fountain*. We say K is *maximal* if a fountain determined by a, A', where $A' \supseteq A$, is precisely K.

(3) Let A, B be interval subsets of $\partial\mathfrak{h}^2$ whose intersection contains at most one point. Suppose that for every geodesic $\gamma \in L$ determined by $a, b \in \partial\mathfrak{h}^2$, we have $a \in A \setminus \partial A$ if and only if $b \in B \setminus \partial B$. If there is more than one such geodesic, we call the set K of all such geodesics determined by a, b with $a \in A$ and $b \in B$ a *rainbow*. We say K is *maximal* if a rainbow determined by $A' \supseteq A$ and $B' \supseteq B$ is precisely K.

From the definitions we have a result about discrete arcs, fountains, and rainbows.

Proposition 4.4. *Let $(L, \mathcal{M})$ be a measured lamination of $\mathfrak{h}^2$ and let K be a discrete geodesic, a fountain of L, or a rainbow of L. Then $\mathcal{M}(K) > 0$.*

Proof. By definition, if $K = \{\gamma\}$ is a discrete arc then $K = O_{A,B}$ and so $\mathcal{M}(K) > 0$. Additionally, if $K = L$ then $K = O_{\partial\mathfrak{h}^2, \partial\mathfrak{h}^2}$ and so $\mathcal{M}(K) > 0$. So we will assume K is either a fountain or a rainbow and $K \neq L$; in particular K has more than one element.

First suppose K is a fountain determined by $a \in \partial\mathfrak{h}^2$ and $A \subset \partial\mathfrak{h}^2$. By definition K has more than one element and so $A \setminus \partial A \neq \emptyset$. If $a \notin A$ then let $B \ni a$ be a small open ball around a in $\partial\mathfrak{h}^2$ such that $B \cap A = \emptyset$. Now consider $O_{A \setminus \partial A, B}$. We see $O_{A \setminus \partial A, B} \subset K$ and $\mathcal{M}(O_{A \setminus \partial A, B}) > 0$. If $a \in A$ then every geodesic determined by an a' and b with $a' \in A \setminus (\{a\} \cup \partial A)$ has $b = a$. Let $A' = A \setminus (\{a\} \cup \partial A)$ and let $B \ni a$ be an open ball such that $A \setminus \partial A \not\subset B$. Now we have $O_{A',B} \subset K$ and $\mathcal{M}(O_{A',B}) > 0$. Therefore $\mathcal{M}(K) > 0$.

Now suppose K is a rainbow determined by A and B. Again we know K has more than one element so both $A \setminus \partial A$ and $B \setminus \partial B$ are nonempty. Take $A' = A \setminus \partial A$ and $B' = B \setminus \partial B$. Then $O_{A',B'} \subset K$ and $\mathcal{M}(O_{A',B'}) > 0$. Therefore, $\mathcal{M}(K) > 0$. □

4.2. **The Correspondence.** In this section, we recall the connection between $\mathbf{N}_\pi$-clusters and (unmeasured) laminations of $\mathfrak{h}^2$ for the straight descending orientation of a continuous quiver of type $\mathbb{A}$, from [15]. We then extend this connection to measured laminations and stability conditions that satisfy the four point condition, obtaining a "2-bijection" (Theorem 4.13). Then we further extend this "2-bijection" between measured laminations and stability conditions to all continuous quivers of type $\mathbb{A}$ with finitely many sinks and sources (Corollary 4.14). We conclude that section with an explicit statement that tilting a stability condition $\sigma \in \mathcal{S}_{\text{fpc}}(Q)$ to a stability condition $\sigma' \in \mathcal{S}_{\text{fpc}}(Q')$ yields the *same* measured lamination, for continuous quivers Q, Q' of type $\mathbb{A}$ (Theorem 4.15).

Theorem 4.5 ([15])**.** *There is a bijection Φ from maximally $\mathbf{N}_\pi$-compatible sets to laminations of $\mathfrak{h}^2$. For each maximally $\mathbf{N}_\pi$-compatible set T and corresponding lamination $\Phi(T)$, there is a bijection $\phi_T : T \to \Phi(T)$ that takes objects in T to geodesics in $\Phi(T)$.*

Before we proceed we introduce some notation to make some remaining definitions and proofs in this section more readable. First, we fix an indexing on $\partial\mathfrak{h}^2$ in the following way. To each point $x \in \mathbb{R}\cup\{-\infty\}$, the map Ξ assigns the value $e^{2i\arctan(x)} \in \partial\mathfrak{h}^2$ (Definition 4.1).

We now refer to points in $\partial\mathfrak{h}^2$ as points in $\mathbb{R} \cup \{-\infty\}$.

Notation 4.6. Let $(L, \mathcal{M})$ be a measured lamination of $\mathfrak{h}^2$.

- For each $\gamma \in L$ we denote by a_γ and b_γ the unique points in $\partial\mathfrak{h}^2$ that determine γ such that $a_\gamma < b_\gamma$ in $\mathbb{R} \cup \{-\infty\}$.
- For each $x \in \partial\mathfrak{h}^2$ such that $x \neq -\infty$,

$$\begin{aligned} \frac{L}{x} &:= \{\gamma \in L \mid a_\gamma < x < b_\gamma\} \\ L \cdot x &:= \{\gamma \in L \mid b_\gamma = x\} \\ x \cdot L &:= \{\gamma \in L \mid a_\gamma = x\}. \end{aligned}$$

- For $-\infty$,

$$\begin{aligned} \frac{L}{-\infty} &:= \emptyset \\ L \cdot (-\infty) &:= \emptyset \\ (-\infty) \cdot L &:= \{\gamma \in L \mid a_\gamma = -\infty\}. \end{aligned}$$

- Finally, for some interval $I \subset \mathbb{R}$,

$$\begin{aligned} I \cdot L &:= \bigcup_{x \in I} x \cdot L = \{\gamma \in L \mid a_\gamma \in I\} \\ L \cdot I &:= \bigcup_{x \in I} L \cdot x = \{\gamma \in L \mid b_\gamma \in I\}. \end{aligned}$$

We denote by $\mathcal{L}$ the set of measured laminations of $\mathfrak{h}^2$ and by $\bar{\mathcal{L}}$ the set of laminations of $\mathfrak{h}^2$ (without a measure).

Now we define how to obtain a useful function F from any measured lamination $L \in \mathcal{L}$. We will use this to define a function $\mathcal{L} \to \mathcal{S}_{\text{fpc}}(Q)$, where Q is the continuous quiver of type $\mathbb{A}$ with straight descending orientation.

Definition 4.7. Let $(L,\mathcal{M}) \in \mathcal{L}$. We will define a useful function F on $-\infty$, $+\infty$, and then all of $\mathbb{R}$. For $-\infty$, define

$$u^-_{-\infty} := 0 \qquad u^+_{-\infty} := -\mathcal{M}((-\infty)\cdot L)$$
$$F(-\infty) := 0 \qquad f(-\infty) := 0.$$

For $+\infty$, define

$$u^-_{+\infty} = u^+_{+\infty} = F(+\infty) = f(+\infty) = 0.$$

For each $a \in \mathbb{R}$, define

$$u^-_a := \mathcal{M}(L\cdot a) \qquad u^+_a := -\mathcal{M}(a\cdot L)$$
$$F(a) := -\mathcal{M}\left(\frac{L}{a}\right) \qquad f(a) := F(a) - \left(\sum_{x\le a} u^-_x\right) - \left(\sum_{x<a} u^+_x\right).$$

First, note that since $\mathcal{M}(L) < \infty$, each of the assignments is well-defined. It remains to show that F is a useful function.

Proposition 4.8. *Let $(L,\mathcal{M}) \in \mathcal{L}$ and let F be as in Definition 4.7. Then F is useful.*

Proof. Since $\mathcal{M}(L) < \infty$, we see $\sum_{x\in\mathbb{R}\cup\{+\infty\}} |u^-_x| + \sum_{x\in\mathbb{R}\cup\{-\infty\}} |u^+_x| < \infty$. Now we show f is continuous. Consider $\lim_{x\to a^-} f(x)$ for any $a \in \mathbb{R}$:

$$\begin{aligned}
\lim_{x\to a^-} f(x) &= \lim_{x\to a^-}\left[F(x) - \left(\sum_{y\le x} u^-_y\right) - \left(\sum_{y<x} u^+_y\right)\right]\\
&= \lim_{x\to a^-}\left[-\mathcal{M}\left(\frac{L}{x}\right) - \left(\sum_{y\le x}\mathcal{M}(L\cdot y)\right) - \left(\sum_{y<x}\mathcal{M}(y\cdot L)\right)\right]\\
&= -\mathcal{M}\left(\frac{L}{a}\right) - \mathcal{M}(L\cdot a) - \left(\sum_{y<a}\mathcal{M}(L\cdot y)\right) - \left(\sum_{y<a}\mathcal{M}(y\cdot L)\right)\\
&= F(a) - \left(\sum_{y\le a} u^-_y\right) - \left(\sum_{y<a} u^+_y\right)\\
&= f(a).
\end{aligned}$$

A similar computation shows $\lim_{x\to a^+} f(x) = f(a)$. Therefore, f is continuous on $\mathbb{R}$. We also note that $\lim_{x\to\pm\infty} f(x) = 0$, using similar computations.

It remains to show that f has bounded variation. Let $a < b \in \mathbb{R}$ and let $F_0 = f$. Denote by $\mathsf{var}_f([a,b))$ the variance of f over $[a,b)$. We see that

$$\mathsf{var}_f([a,b)) = \mathcal{M}\big(([a,b)\cdot L)\cup(L\cdot[a,b))\big) - \sum_{x\in[a,b)}\big(\mathcal{M}(x\cdot L) + \mathcal{M}(L\cdot x)\big).$$

That is, $\mathsf{var}_f([a,b))$ is the measure of the geodesics with endpoints in $[a,b)$ that are not discrete and do not belong to a fountain. So,

$$\mathsf{var}_f([a,b)) \le \mathcal{M}\big(([a,b)\cdot L)\cup(L\cdot[a,b))\big).$$

Then we have

$$\mathsf{var}_f(\mathbb{R}) = \sum_{i\in\mathbb{Z}}\mathsf{var}_f([i,i+1)) \le \sum_{i\in\mathbb{Z}}\mathcal{M}\big(([i,i+1)\cdot L)\cup(L\cdot[i,i+1))\big) < \infty.$$

Thus, f has bounded variation. □

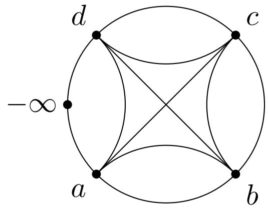

FIGURE 4.2. The geodesics a-b, a-c, a-d, b-c, b-d, and c-d used in the proof of Proposition 4.12. Notice a-b, b-c, c-d, and a-d form a quadrilateral and its diagonals, a-c and b-d, cross.

We state the following lemma without proof, since the proof follows directly from Definitions 2.14 and 4.7 and Proposition 4.8.

Lemma 4.9. *Let $(L, \mathcal{M}) \in \mathcal{L}$ and let F be as in Definition 4.7. Then $(F, 0)$ is a red-blue function pair for the continuous quiver Q of type $\mathbb{A}$ with straight descending orientation.*

Now we define the function $\mathcal{L} \to \mathcal{S}(Q)$.

Definition 4.10. Let $(L, \mathcal{M}) \in \mathcal{L}$, let F be as in Definition 4.7, and let Q be the continuous quiver of type $\mathbb{A}$ with straight descending orientation. The map $\Phi : \mathcal{L} \to \mathcal{S}(Q)$ is defined by setting $\Phi((L, \mathcal{M}))$ equal to the equivalence class of $(F, 0)$.

Lemma 4.11. *Let $(L, \mathcal{M}) \in \mathcal{L}$ and let $\partial\mathfrak{h}^2$ be indexed as $\mathbb{R} \cup \{-\infty\}$, as before. Suppose there are points $a, b \in \partial\mathfrak{h}^2$ such that for all $x \in (a, b)$ we have $\mathcal{M}(\frac{L}{x}) \geq \mathcal{M}(\frac{L}{a})$ and $\mathcal{M}(\frac{L}{x}) \geq \mathcal{M}(\frac{L}{b})$. Then the geodesic in $\mathfrak{h}^2$ uniquely determined by a and b is in L.*

Proof. For contradiction, suppose there is $\alpha \in L$ such that α is uniquely determined by c and d, where $a < c < b < d$. Then, we must have $\beta \in L$ uniquely determined by b and d and $\beta' \in L$ determined by c and b, or else there is a set K with positive measure such that $K \subset \frac{L}{b}$ but $K \not\subset \frac{L}{c}$. Similarly, we must have $\gamma \in L$ uniquely determined by a and c. Now, we cannot have a fountain at c or else we will have a set with positive measure K such that $K \subset \frac{L}{b}$ or $K \subset \frac{L}{a}$ but $K \not\subset \frac{L}{c}$. Since c has a geodesic to both the left and right, α must be discrete. Since α is discrete, $\{\alpha\}$ has positive measure, a contradiction since $\{\alpha\} \subset \frac{L}{b}$ and $\{\alpha\}$ is disjoint from $\frac{L}{c}$. Thus, there is no $\alpha \in L$ such that α is uniquely determined by c and d, where $a < c < b < d$. Similarly, there is no $\alpha \in L$ such that α is uniquely determined by c and d, where $c < a < d < b$. Therefore, since L is maximal, we must have the geodesic uniquely determined by a and b in L. $\square$

Proposition 4.12. *Let $(L, \mathcal{M}) \in \mathcal{L}$, let F be as in Definition 4.7, and let Q be the continuous quiver of type $\mathbb{A}$ with straight descending orientation. Then $\Phi((L, \mathcal{M})) \in \mathcal{S}_{\text{fpc}}(Q)$.*

Proof. For contradiction, suppose there exists a $\Phi((L, \mathcal{M}))$-semistable module M_I such that $|(\widehat{I} \times \{h\}) \cap (\widehat{Q} \times \mathbb{R})| \geq 4$. Choose 4 points $a < b < c < d$ in $\widehat{Q}$ corresponding to four intersection points.

For the remainder of this proof, write x-y to mean the geodesic in $\mathfrak{h}^2$ uniquely determined by $x \neq y \in \partial\mathfrak{h}^2$. By Lemma 4.11, we have the following geodesics in L: a-b, a-c, a-d, b-c, b-d, and c-d. However, this is a quadrilateral with *both* diagonals, as shown in Figure 4.2. Since L is a lamination, this is a contradiction. $\square$

Theorem 4.13. *Let Q be the continuous quiver of type $\mathbb{A}$ with straight descending orientation. Then $\Phi : \mathcal{L} \to \mathcal{S}_{\mathrm{fpc}}(Q)$ is a bijection. Furthermore, for a measured lamination L and stability condition $\Phi(L)$, there is a bijection ϕ_L from L to $\Phi(L)$-semistable indecomposable modules.*

Proof. By the proof of Proposition 4.12, we see that the second claim follows. Thus, we now show Φ is a bijection.

Injectivity. Consider $(L, \mathcal{M})$ and $(L', \mathcal{M}')$ in $\mathcal{L}$. Let $\sigma = \Phi(L, \mathcal{M})$ and $\sigma' = \Phi(L', \mathcal{M}')$. If $L \neq L'$ then we see that the set of σ-semistable modules is different from the set of σ'-semistable modules. Thus, $\sigma \neq \sigma'$. If $L = L'$ but $\mathcal{M} \neq \mathcal{M}'$ there must be some $x \in \mathbb{R} \cup \{-\infty\}$ such that $\mathcal{M}(\frac{L}{x}) \neq \mathcal{M}'(\frac{L'}{x})$. But the functions F and F' from L and L', respectively using Definition 4.7, both have the same limits at $\pm\infty$. Thus, $\widehat{\mathcal{G}}(F, 0)$ is not a vertical translation of $\widehat{\mathcal{G}}(F', 0)$ in $\mathbb{R}^2$. Therefore, $\sigma \neq \sigma'$.

Surjectivity. Let σ be a stability condition. Let T be the maximal $\mathbf{N}_\pi$-compatible set of indecomposable modules determined by σ (Theorem 2.28). Let L be the lamination of $\mathfrak{h}^2$ uniquely determined by T (Theorem 4.5). In particular, the indecomposable M_I corresponds to the geodesic uniquely determined by $\inf I$ and $\sup I$.

Let (R, B) be the representative of σ such that $B = 0$; that is, $(R, B) = (R, 0)$. For each $x \in \partial\mathfrak{h}^2$, let

$$\mathcal{M}(L \cdot x) = u_x^- \qquad \mathcal{M}(x \cdot L) = -u_x^+$$
$$\mathcal{M}\left(\frac{L}{x}\right) = R(x).$$

Since r must have bounded variation and $\sum_{x \in \overline{R}} |u_x^-| + |u_x^+| < \infty$, we see $\mathcal{M}(L) < \infty$.

Let $O_{A,B} \subset L$ be a basic open subset. If $O_{A,B} = \emptyset$ then we're done.

Now we assume $O_{A,B} \neq \emptyset$ and let $\gamma \in O_{A,B}$. If there exist two stability indicators (Definition 2.21) for the indecomposable M_I corresponding to γ, with heights $h_0 < h_1$, then we know $\mathcal{M}(\{\gamma\}) > |h_1 - h_0| > 0$ and so $\mathcal{M}(O_{A,B}) > 0$.

We now assume there is a unique stability indicator of height h for the indecomposable M_I corresponding to γ. Without loss of generality, since $O_{A,B} = O_{B,A}$, assume $a = a_\gamma \in A$ and $b = b_\gamma \in B$. We know that, for all $a < x < b$, we have $R(x) \leq R(a)$ and $R(x) \leq R(b)$. There are two cases:

(1) $R(x) < R(a)$ and $R(x) < R(b)$, for all $x \in (a, b)$, and
(2) there exists $x \in (a, b)$ such that $R(x) = R(a)$ or $R(x) = R(b)$.

(1). Let $e = \tan(\frac{1}{2}(\tan^{-1}(a) + \tan^{-1}(b)))$. Let $\{h_i\}_{i \in \mathbb{N}}$ be a strictly increasing sequence such that $h_0 = R(e)$ and $\lim_{i \to \infty} h_i = h$. By Lemma 2.15 (1) and our assumption that $\mathcal{M}(\{\gamma\}) = 0$, for each $i > 0$, there is a stability indicator with height h_i and endpoints a_i, b_i such that $a < a_i < b_i < b$. Then $\lim_{i \to \infty} a_i = a$ and $\lim_{i \to \infty} b_i = b$, again by Lemma 2.15 (1). Since A and B are open, there is some $N \in \mathbb{N}$ such that, for all $i \geq N$, we have $a_i \in A$ and $b_i \in B$. Let $C = (a, a_N)$ and $D = (b_N, b)$. Then, $\mathcal{M}(O_{C,D}) \geq |h - h_N|$ and so $\mathcal{M}(O_{A,B}) > 0$.

(2). Assume there exists $x \in (a, b)$ such that $R(x) = R(a)$ or $R(x) = R(b)$. Let e be this x. If $R(x) = R(a)$ and $a = -\infty$, then $R(b) = 0$ (or else $\gamma \notin O_{A,B} \subset L$). Then we use the technique from Case (1) with b and $+\infty$ to obtain some $C = (b, d)$ and $D = (c, +\infty)$ such that $\mathcal{M}(O_{C,D}) > 0$. Thus, $\mathcal{M}(O_{A,B}) > 0$.

Now we assume $a > -\infty$ and $R(x) = R(a)$ or $R(x) = R(b)$. We consider $R(x) = R(b)$ as the other case is similar. Since σ satisfies the four point condition, we know that for any $\varepsilon > 0$ such that $R(b+\varepsilon) < R(b)$ we must have $0 < \lambda < \varepsilon$ such that $R(b+\lambda) > R(b)$. Similarly, for any $\varepsilon > 0$ such that $R(a-\varepsilon) < R(b)$ we must have $0 \le \lambda < \varepsilon$ such that $R(a-\lambda) > R(b)$. Notice the strict inequality in the statement about $R(b+\lambda)$ and the weak inequality in the statement about $R(a-\lambda)$.

Let $\{h_i\}$ be a strictly decreasing sequence such that $h_0 = 0$ and $\lim_{i\to\infty} h_i = h$. By Lemma 2.15 (1) and our assumption that $\mathcal{M}(\{\gamma\}) = 0$, for each $i > 0$, there is a stability indicator with height h_i and endpoints a_i, b_i such that $a_i \le a < b < b_i$. Since σ satisfies the four point condition, and again by Lemma 2.15 (1), $\lim_{i\to\infty} b_i = b$. Since A and B are open, there is $N \in \mathbb{N}$ such that, if $i \ge N$, we have $a_i \in A$ and $b_i \in B$. If $a_i = a$ for any $i > N$, let C be a tiny epsilon ball around a that does not include b. Otherwise, let $C = (a_N, a)$. Let $D = (b, b_N)$. Then $\mathcal{M}(O_{C,D}) \ge |h_N - h|$ and so $\mathcal{M}(O_{A,B}) > 0$.

Conclusion. Since $\mathcal{M}(L) < \infty$, we know $\mathcal{M}(O_{A,B}) < \infty$ for each $O_{A,B}$. This proves $(L, \mathcal{M})$ is a measured lamination. By the definition of Φ, we see that $\Phi(L,\mathcal{M}) = \sigma$. Therefore, Φ is surjective and thus bijective. $\square$

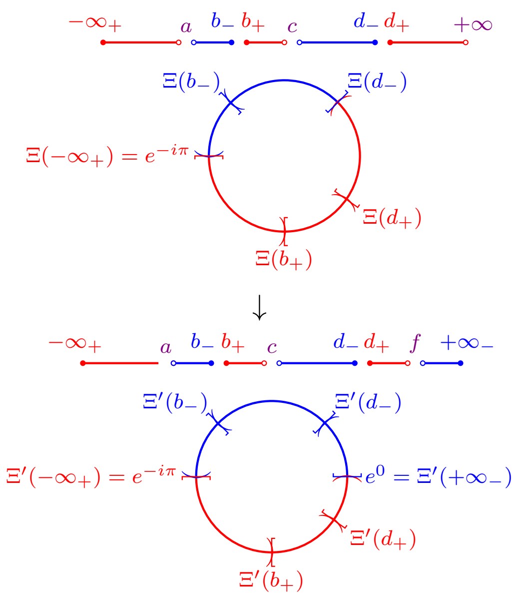

Figure 4.3. An example of two mappings $\Xi : \widehat{Q} \to \partial\mathfrak{h}^2$ and $\Xi' : \widehat{Q}' \to \partial\mathfrak{h}^2$ where we obtain Q' from Q by tilting the interval $[f, +\infty)$, for $d < f < +\infty$. At the top of each picture, we have the modified quiver. Since a and c are sources in Q and Q', they are labeled with purple. Similarly for f in Q'. We also have $+\infty$ in purple for Q since it is not acting as a sink. However, in Q', we do have $+\infty$ acting as a sink and so $+\infty_-$ has a blue label. We also have the sinks b and d, doubled in the modified quivers, where b_- and d_- are blue and b_+ and d_+ are red. Since $-\infty$ is acting as a sink we have $-\infty_+$ in red. On the bottom of each picture, we have $\partial\mathfrak{h}^2$. We indicate the values of Ξ and Ξ' applied to $-\infty_+$ and the sinks. For Q', we also show Ξ' applied to $+\infty_-$.

Corollary 4.14 (To Theorems 3.17 and 4.13)**.** *Let Q be a continuous quiver of type $\mathbb{A}$. Then there is a bijection $\Phi : \mathcal{L} \to \mathcal{S}_{\mathrm{fpc}}(Q)$. Furthermore, for a measured lamination L and stability condition $\Phi(L)$, there is a bijection ϕ_L from L to $\Phi(L)$-semistable indecomposable modules.*

Theorem 4.15. *Let $\sigma \in \mathcal{S}_{fpc}(Q)$ be the stability condition given by (R, B) and let $\sigma' \in \mathcal{S}_{fpc}(Q')$ be given by (R', B'). If σ' is obtained from σ by continuous tilting then σ, σ' give the same measured lamination on the Poincaré disk.*

To make the statement of the theorem easier to see, we have two examples of the Ξ functions, related by a continuous tilt, in Figure 4.3. Thus, there is a bijection between representable indecomposables with endpoints $a < b \in \widehat{Q}$ and geodesics in $\mathfrak{h}^2$ determined by $\Xi(a)$ and $\Xi(b)$.

Proof of Theorem 4.15. The set of geodesics going from intervals (a, b) to (x, y) has the same measure as those going from $(\mathfrak{t}(a), \mathfrak{t}(b))$ to $(\mathfrak{t}(x), \mathfrak{t}(y))$ where we may have to reverse the order of the ends. We can break up the intervals into pieces and assume that $(a, b), (x, y)$ are either both in K, both in Z or one is in K and the other in Z. The only nontrivial case is when (a, b) is in K and (x, y) is in Z. In that case, the measure of this set of geodesics for σ is equal to the variation of H on (a, b) since the islands don't "see" Z. Similarly, the measure of the same set of geodesics for σ', now parametrized as going from $(\mathfrak{t}(b), \mathfrak{t}(a))$ to (x, y) is equal to the variation of H' on $(\mathfrak{t}(b), \mathfrak{t}(a))$ where $H'(z) = H_+(\mathfrak{t}(z))$.

There is one other case that we need to settle: we need to know that the local variation of H at $\pm\infty$ is equal to the local variation of H' at $\mathfrak{t}(\pm\infty)$. But this holds by definition of H, H'. □

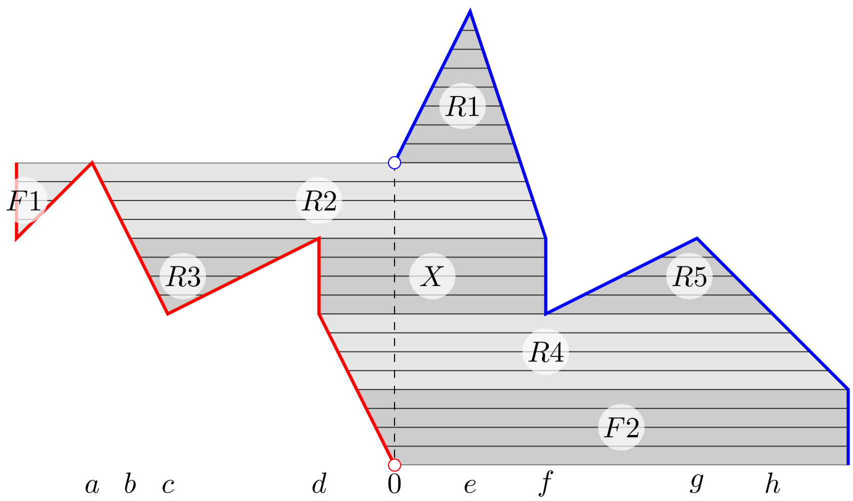

FIGURE 4.4. The modified graph of the red-blue function pair (R, B). Horizontal lines indicate semistable indecomposable representations. The rectangle labeled X represents one object with positive measure. Note that if X does not have positive measure then the stability condition violates the four point condition (Definition 2.24). The measure of a region is given by its height.

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An example of a stability condition σ and corresponding measured lamination are shown in Figures 4.4, 4.5. The continuously tilted stability condition σ' is shown in Figure 4.6.

One may obtain any continuous quiver Q of type A (with finitely-many sinks and sources) through a series of flips, where

$$\#\text{flips} = \begin{cases} (\#\text{sinks and sources in } Q) & -\infty \text{ is red} \\ (\#\text{sinks and sources in } Q) + 1 & -\infty \text{ is blue,} \end{cases}$$

and we start with a straight descending continuous quiver of type A.

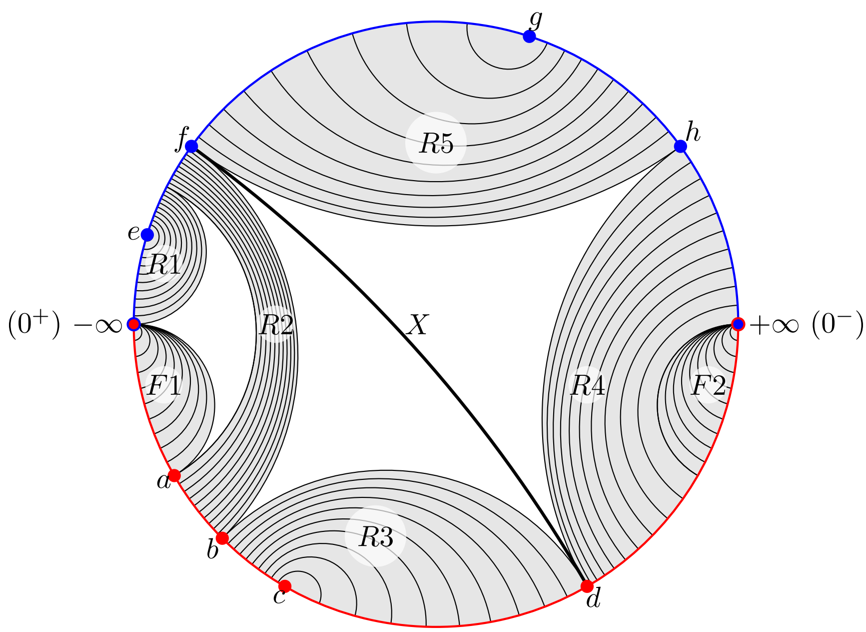

FIGURE 4.5. The lamination of the hyperbolic plane corresponding to the stability condition shown in Figure 4.4, where we label the points $\Xi(x) \in \partial\mathfrak{h}^2$ simply by x. The thick arc labeled X is an isolated geodesic with positive measure. The measure is equal to the height of the rectangle labeled X in Figure 4.4. We have labeled the boundary by sending $[-\infty, 0)$ to $\partial\mathfrak{h}^2$ by $x \mapsto e^{i2\arctan(x)}$ and sending $[0, +\infty)$ to $\partial\mathfrak{h}^2$ by $x \mapsto e^{i(\pi - 2\arctan(x))}$.

4.3. **Continuous cluster character.** We present a candidate for the continuous cluster character using the formula from [20] which applies in the continuous case. This lives in a hypothetical algebra having a variable x_t for every real number t. In this algebra which we have not defined, we give a simple formula for the cluster variable of an admissible module M_{ab} where $a < b$ and the quiver Q is oriented to the left (is red) in a region containing $(a, b]$ in its interior. In analogy with the cluster character in the finite case (1.5) or [20], replacing summation with integration, we define $\chi(M_{ab})$ to be the formal expression:

$$\chi(M_{ab}) = \int_a^b \frac{x_a x_b \,\mathrm{d}t}{x_t^2}. \tag{4.1}$$

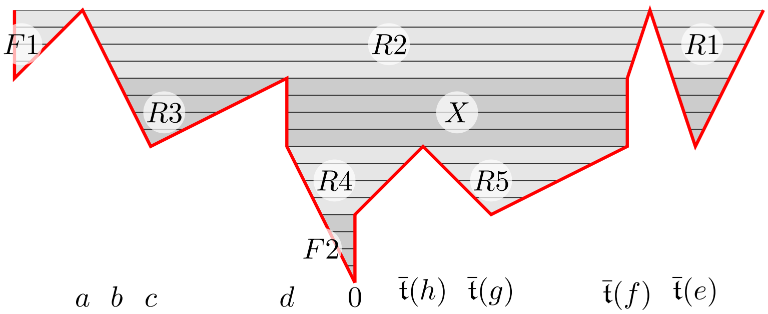

FIGURE 4.6. This is the continuous tilting of Figure 4.4. There are two islands $R1$ and $R5$ which have been flipped down. The measured lamination (Figure 4.5) is unchanged, only relabeled.

This could be interpreted as an actual integral of some function x_t. For example, if we let $x_t = t$ then we get $\chi(M_{ab}) = b - a$, the length of the support of M_{ab}. The constant function $x_t = 1$ gives the same result.

The same cluster character formula will be used for modules with support $[a, b)$ in the blue region (where the quiver is oriented to the right).

This can also be written as

$$\chi(M_{ab}) = x_a\chi(P_b) - x_b\chi(P_a)$$

where P_b is the projective module at b with cluster character

$$\chi(P_b) = \int_{-\infty}^{b} \frac{x_b \,\mathrm{d}t}{x_t^2}.$$

Then the cluster mutation equation

$$\chi(M_{ac})\chi(M_{bd}) = \chi(M_{ab})\chi(M_{cd}) + \chi(M_{bc})\chi(M_{ad})$$

follows, as in the finite A_n case, from the Plücker relation on the matrix:

$$\begin{bmatrix} x_a & x_b & x_c & x_d \\ \chi(P_a) & \chi(P_b) & \chi(P_c) & \chi(P_d) \end{bmatrix}.$$

In Figures 4.4 and 4.5, if the measure of $X = M_{df}$ is decreased to zero, the height of the rectangle in Figure 4.4 will go to zero, the four point condition will be violated and we can mutate X to $X^* = M_{bh}$. Then the cluster characters are mutated by the Ptolemy equation:

$$\chi(X)\chi(X^*) = \chi(M_{fh})\chi(M_{bd}) + \chi(M_{dh})\chi(M_{bf})$$

where $\chi(M_{bd})$ and $\chi(M_{fh})$ are given by (4.1) and the other four terms have a different equation since there is a source (0) in the middle ($d < 0 < f$):

$$\chi(X) = \chi(M_{df}) = \int_d^0 \int_0^f \frac{x_d x_0 x_f}{x_s^2 x_t^2} \,\mathrm{d}s \,\mathrm{d}t + \frac{x_d x_f}{x_0}.$$

The double integral counts the proper submodules $M_{ds} \oplus M_{tf} \subset X$ and there is one more term for the submodule $X \subseteq X$.

The continuous cluster character will be explained in more detail in another paper.

4.4. **Every $\mathbf{N}_\pi$-cluster comes from a stability condition.** Let L be a lamination of $\mathfrak{h}^2$. Then there exists a measured lamination $(L, \mathcal{M})$ in the following way. There are at most countably many discrete arcs in L. Assign each discrete arc a natural number n. Then, set $\mathcal{M}(\{\gamma_n\}) = \frac{1}{1+n^2}$, for $n \in \mathbb{N}$. Let K be the set of all discrete geodesics in L. On $L \setminus K$, give each $O_{A,B}$ its transversal measure. Thus, we have given L a finite measure satisfying Definition 4.2. Therefore, $(L, \mathcal{M})$ is a measured lamination. This means the set of measured laminations, $\mathcal{L}$, surjects on to the set of laminations, $\bar{\mathcal{L}}$, by "forgetting" the measure. Then, the set $\mathcal{S}_{\mathrm{fpc}}(Q)$, for some continuous quiver of type $\mathbb{A}$ with finitely many sinks and sources, surjects onto the set of $\mathbf{N}_\pi$-clusters, $\mathcal{T}_{\mathbf{N}_\pi}$, in the following way.

$$\begin{tikzcd}
\mathcal{S}_{\mathrm{fpc}}(Q) \arrow[r, leftrightarrow, "\Phi"] \arrow[d, dashed, two heads, "\exists"'] & \mathcal{L} \arrow[d, two heads] \\
\mathcal{T}_{\mathbf{N}_\pi} \arrow[r, leftrightarrow, "\bar{\Phi}"'] & \bar{\mathcal{L}}.
\end{tikzcd}$$

Essentially, there is a surjection $\mathcal{S}_{\mathrm{fpc}}(Q) \twoheadrightarrow \mathcal{T}_{\mathbf{N}_\pi}$ defined using the surjection $\mathcal{L} \twoheadrightarrow \bar{\mathcal{L}}$. If we follow the arrows around, we see that each stability condition σ is sent to the set of σ-semistable modules, which form an $\mathbf{N}_\pi$-cluster.

4.5. **Maps between cluster categories of type $\mathbb{A}_n$.** Let Q be a quiver of type $\mathbb{A}_n$, for $n \geq 2$. Label the vertices $1, \dots, n$ in Q such that there is an arrow between i and $i+1$ for each $1 \leq i < n$.

For each $i \in \{-1, 0, \dots, n, n+1, n+2\}$ let

$$x_i = \tan\left(\frac{i+1}{n+3}\pi - \frac{\pi}{2}\right).$$

We define a continuous quiver $\mathcal{Q}$ of type $\mathbb{A}$ based on Q, called the *continuification* of Q. If 1 is a sink (respectively, source) then $-\infty$ is a sink (respectively, source) in $\mathcal{Q}$. If n is a sink (respectively, source) then $+\infty$ is a sink (respectively, source in $\mathcal{Q}$). For all i such that $2 \leq i \leq n-1$, we have x_i is a sink (respectively, source) in $\mathcal{Q}$ if and only if i is a sink (respectively, source) in Q.

Define a map $\Omega : \mathrm{Ind}(\mathcal{C}(Q)) \to \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q})$ in Figure 4.7 on page 242.

Let $1 \leq m < n$ such that there is a path $1 \to m$ in Q or a path $m \to 1$ in Q (possibly trivial). Let Q' be obtained from Q by reversing the path between 1 and m (if $m = 1$ then $Q = Q'$). It is well known that $\mathcal{D}^b(Q)$ and $\mathcal{D}^b(Q')$ are equivalent as triangulated categories. Let $F : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q')$ be a triangulated equivalence determined by sending $P_n[0]$ to $P_n[0]$. Furthermore, we know $\tau \circ F(M) \cong F \circ \tau(M)$ for every object M in $\mathcal{D}^b(Q)$, where τ is the Auslander–Reiten translation. Then this induces a functor $\overline{F} : \mathcal{C}(Q) \to \mathcal{C}(Q')$. Overloading notation, we denote by $\overline{F} : \mathrm{Ind}(\mathcal{C}(Q)) \to \mathrm{Ind}(\mathcal{C}(Q'))$ the induced map on isomorphism classes of indecomposable objects.

Let $\mathcal{Q}'$ be the continuification of Q' and $\Omega' : \mathrm{Ind}(\mathcal{C}(Q')) \to \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}')$ the inclusion defined in the same way as Ω. Notice that orientation of $\mathcal{Q}$ and $\mathcal{Q}'$ agree above x_m. Furthermore, if $m > 1$, the interval $(-\infty, x_m)$ is blue in $\mathcal{Q}$ if and only if it is red in $\mathcal{Q}'$ and vice versa. Using Theorem 3.2, there is a map $\phi : \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}) \to \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}')$ such that

$$P_i \overset{\Omega}{\longmapsto} P_{x_i} \quad M_{[i,j]} \overset{\Omega}{\longmapsto} \begin{cases} M_{[x_i,x_j]} & i \text{ is blue, } j \text{ is red,} \\ M_{[x_i,x_{j+2})} & i \text{ is blue, } j \text{ is blue, } j+1 \text{ is not a sink or } j+1=n \\ M_{[x_i,x_\ell)} & i \text{ is blue, } j \text{ is blue, } j+1<n \text{ is a sink, } \ell = (\text{smallest source} > j)+1, \\ M_{(x_{i-2},x_j]} & i \text{ is red, } i-1 \text{ is not a sink or } i-1=1, \ j \text{ is red,} \\ M_{(x_{i-2},x_{j+2})} & i \text{ is red, } i-1 \text{ is not a sink or } i-1=1, \ j \text{ is blue, } j+1 \text{ is not a sink or } j+1=n, \\ M_{(x_{i-2},x_\ell)} & i \text{ is red, } i-1 \text{ is not a sink or } i-1=1, \ j \text{ is blue, } j+1<n \text{ is a sink,} \\ & \hfill \ell = (\text{smallest source} > j)+1 \\ M_{(x_k,x_j]} & i \text{ is red, } i-1>1 \text{ is a sink, } k = (\text{largest source}) < i) - 1, \ j \text{ is red,} \\ M_{(x_k,x_{j+2})} & i \text{ is red, } i-1>1 \text{ is a sink, } k = (\text{largest source}) < i) - 1, \\ & \hfill j \text{ is blue, } j+1 \text{ is not a sink or } j+1=n, \\ M_{(x_k,x_\ell)} & i \text{ is red, } i-1>1 \text{ is a sink, } k = (\text{largest source}) < i) - 1, \ j \text{ is blue, } j+1<n \text{ is a sink,} \\ & \hfill \ell = (\text{smallest source} > j)+1. \end{cases}$$

$$P_i[1] \overset{\Omega}{\longmapsto} \begin{cases} M_{(x_{i-1},x_{i+1})} & i \text{ is a source,} \\ M_{[x_0,x_{n+1}]} & i \text{ is the only sink,} \\ M_{(x_j,x_{n+1}]} & i \text{ is a sink, } \nexists \text{ source} > i, \ \exists \text{ sink} < i, \ j = (\text{largest source} < i) - 1, \\ M_{[0,x_j)} & i \text{ is a sink, } \nexists \text{ source} < i, \ \exists \text{ sink} > i \, j = (\text{smallest source} > i) + 1, \\ M_{(x_k,x_\ell)} & i \text{ is a sink, } \exists \text{ source} > i, \ \exists \text{ source} < i, \ j = (\text{largest source} < i) - 1, \\ & \hfill k = (\text{smallest source} > i) + 1, \\ M_{[x_0,x_{i+1})} & i \text{ is not a sink/source, } i \text{ is blue, } \nexists \text{ sink} < i, \\ M_{(x_{i-1},x_{n+1}]} & i \text{ is not a sink/source, } i \text{ is red, } \nexists \text{ sink} > i, \\ M_{(x_j,x_{i+1})} & i \text{ is not a sink/source, } i \text{ is blue, } \exists \text{ sink} < i \, j = (\text{biggest source} < i) - 1, \\ M_{(x_{i-1},x_j)} & i \text{ is not a sink/source, } i \text{ is red, } \exists \text{ sink} > i \, j = (\text{smallest source} > i) + 1. \end{cases}$$

FIGURE 4.7. The map $\Omega : \mathrm{Ind}(\mathcal{C}(Q)) \to \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q})$. In the formulae above, $M_{[i,j]}$ is assumed to be not projective. We say a discrete vertex i is red if there exists arrows $i-1 \leftarrow i \leftarrow i+1$. We say i is blue if there exist arrows $i-1 \to i \to i+1$. Additionally, for a discrete interval $[i,j]$ we say i is red if it meets the above criteria or i is a source. We say i is blue if it meets the above criteria or i is a sink. Dually, we say j is red if it satisfies the above criteria or j is a sink. We say j is blue if it satisfies the above criteria or j is a source. The phrase "largest source$<i$" means the maximal element in the set of sources s such that $s<i$. The phrase "smallest source$>i$" means the minimal element in the set of sources s such that $s<i$. When dealing with the discrete values we always mean sinks and sources in Q.

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$\{M, N\} \subset \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q})$ are $\mathbf{N}_\pi$-compatible if and only if $\{\phi(M), \phi(N)\} \subset \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}')$ are $\mathbf{N}_\pi$-compatible. Following tedious computations, we have the following commutative diagram that preserves compatibility:

$$\begin{tikzcd} \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}) \arrow[r, "\phi"] & \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}) \\ \mathrm{Ind}(\mathcal{C}(Q)) \arrow[u, "\Omega"] \arrow[r, "\overline{F}"'] & \mathrm{Ind}(\mathcal{C}(Q)). \arrow[u, "\Omega'"'] \end{tikzcd}$$

4.5.1. *An example for $\mathbb{A}_4$ quivers.* Let Q, Q' be the following quivers and let $\mathcal{Q}, \mathcal{Q}'$ be the respective continuifications defined above with functions Ω, Ω'.

$$Q = \ 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \qquad Q' = \ 1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4.$$

$$\Omega : \mathrm{Ind}(\mathcal{C}(Q)) \longrightarrow \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}) \qquad \Omega' : \mathrm{Ind}(\mathcal{C}(Q')) \longrightarrow \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}')$$

Let $\overline{F} : \mathcal{C}(Q) \to \mathcal{C}(Q')$ be defined as above. A visualization of the commutative diagram above in $\mathfrak{h}^2$ is contained in Figure 4.8 on page 244.

For $\Omega : \mathrm{Ind}(\mathcal{C}(Q)) \to \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q})$:

$$\begin{aligned} A &= \Omega(P_4) & F &= \Omega(M_{23}) & K &= \Omega(P_3[1]) \\ B &= \Omega(P_3) & G &= \Omega(I_3) & L &= \Omega(I_1) \\ C &= \Omega(P_2) & H &= \Omega(P_4[1])) & M &= \Omega(P_2[1]) \\ D &= \Omega(P_1) & I &= \Omega(S_2) & N &= \Omega(P_1[1]) \\ E &= \Omega(S_3) & J &= \Omega(I_2). \end{aligned}$$

To save space we will indicate an indecomposable module by its support interval.

In $\mathrm{mod}^{\mathrm{r}}(\mathcal{Q})$):

$$\begin{aligned} A &= [\tan(3\pi/14), +\infty) & F &= [\tan(-\pi/14), \tan(5\pi/14)) & K &= [\tan(-5\pi/14), \tan(3\pi/14)) \\ B &= [\tan(\pi/14), +\infty) & G &= [\tan(-3\pi/14), \tan(5\pi/14)) & L &= [\tan(-3\pi/14), \tan(\pi/14)) \\ C &= [\tan(-\pi/14), +\infty) & H &= [\tan(-5\pi/14), \tan(5\pi/14)) & M &= [\tan(-5\pi/14), \tan(\pi/14)) \\ D &= [\tan(-3\pi/14), +\infty) & I &= [\tan(-\pi/14), \tan(3\pi/14)) & N &= [\tan(-5\pi/14), \tan(-\pi/14)) \\ E &= [\tan(\pi/14), \tan(5\pi/14)) & J &= [\tan(-3\pi/14), \tan(3\pi/14)). \end{aligned}$$

For $\Omega' : \mathrm{Ind}(\mathcal{C}(Q')) \to \mathrm{Ind}^{\mathrm{r}}(\mathcal{Q}')$:

$$\begin{aligned} A &= \Omega'(P'_4) & F &= \Omega'(I'_2) & K &= \Omega'(P'_3[1]) \\ B &= \Omega'(P'_3) & G &= \Omega'(I'_3) & L &= \Omega'(P'_1) \\ C &= \Omega'(M'_{23}) & H &= \Omega'(P'_4[1]) & M &= \Omega'(P'_2) \\ D &= \Omega'(I'_4) & I &= \Omega'(P'_1[1]) & N &= \Omega'(S'_2) \\ E &= \Omega'(I'_1) & J &= \Omega'(P'_2[1]). \end{aligned}$$

In $\mathrm{mod}^{\mathrm{r}}(\mathcal{Q}')$):

$$\begin{aligned} A &= [\tan(3\pi/14), +\infty) & F &= (\tan(-5\pi/14), \tan(5\pi/14)) & K &= (\tan(-\pi/14), \tan(3\pi/14)) \\ B &= (-\infty, +\infty) & G &= (\tan(-3\pi/14), \tan(5\pi/14)) & L &= (-\infty, \tan(-3\pi/14)] \\ C &= (\tan(-5\pi/14), +\infty) & H &= (\tan(-\pi/14), \tan(5\pi/14)) & M &= (-\infty, \tan(\pi/14)) \\ D &= (\tan(-3\pi/14), +\infty) & I &= (\tan(-5\pi/14), \tan(3\pi/14)) & N &= (\tan(-5\pi/14), \tan(-\pi/14)] \\ E &= (-\infty, \tan(5\pi/14)) & J &= (\tan(-3\pi/14), \tan(3\pi/14)). \end{aligned}$$

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The orange highlights changes due to tilting. The purple highlights a *coincidental* fixed endpoint (but notice the change in open/closed).

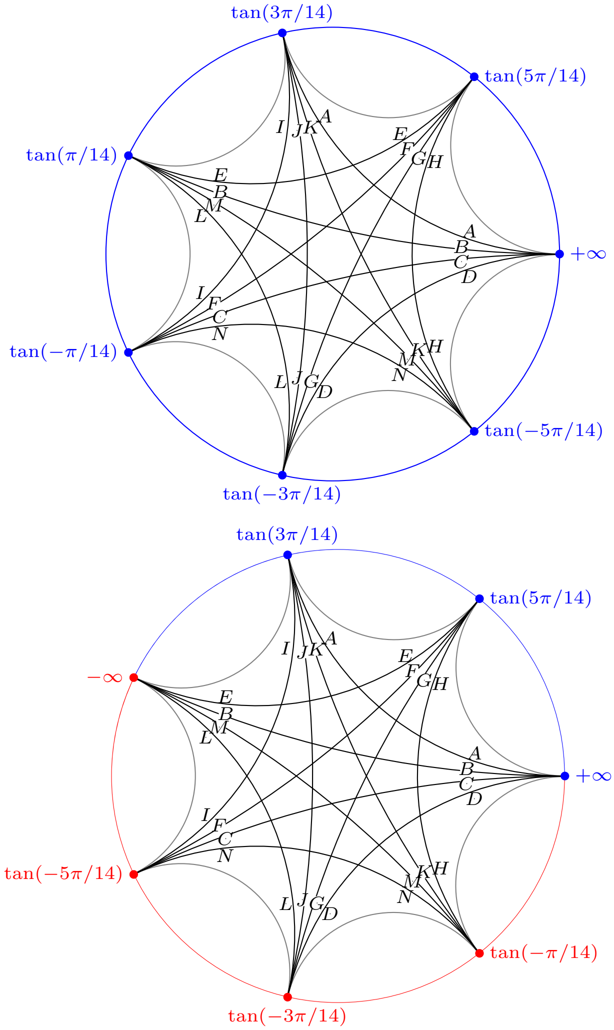

FIGURE 4.8. $\mathbb{A}_4$ example – arcs in $\mathfrak{h}^2$. Continuous tilting doesn't move arcs in the hyperbolic plane. We can see this by relabeling the boundary of $\mathfrak{h}^2$ accordingly. We also see how the diagonals of the heptagon (which models the cluster combinatorics for $\mathcal{C}(Q)$ and $\mathcal{C}(Q')$) are preserved by $\overline{F}$.

FUTURE WORK

There are a few questions that naturally arise from our results. What is the connection between our tilting and the reflection functors introduced in [19]? What if we considered *all* modules over a continuous quiver of type $\mathbb{A}$, instead of just those that are representable. Can we expand Section 4.3 and describe a continuous cluster algebra? The authors plan to explore some of these questions in future research.

There is still much work to do with general continuous stability, as well. What can we learn by studying measured laminations of other surfaces? For example, can we connect a continuous type $\mathbb{D}$ quiver to measured laminations of the punctured (Poincaré) disk? In the present paper, we consider stability conditions in the sense of King. What about other kinds of stability conditions? Furthermore, can the connections between stability conditions and moduli spaces be generalized to the continuous case?

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