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On self-orthogonal modules in Iwanaga–Gorenstein rings

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Dedicated to the memory of Jimmy

Abstract. Let A be an Iwanaga–Gorenstein ring. Enomoto conjectured that a self-orthogonal A-module has finite projective dimension. We prove this conjecture for A having the property that every indecomposable non-projective maximal Cohen–Macaulay module is periodic. This answers a question of Enomoto and shows the conjecture for monomial quiver algebras and hypersurface rings.

1. Introduction

We assume always that A is a two-sided noetherian semiperfect ring and all modules are finitely generated right modules unless otherwise stated. Recall that A is called *n-Iwanaga–Gorenstein* if the injective dimensions of A as a left and right module are equal to n. If the n does not matter we will often just say Iwanaga–Gorenstein ring instead of n-Iwanaga–Gorenstein ring. The category of *maximal Cohen–Macaulay modules* $\operatorname{CM} A$ of a n-Iwanaga–Gorenstein ring is defined as the category of n^{th} syzygy modules $\Omega^n(\operatorname{mod} A)$ consisting of modules X that are projective or direct summands of a module of the form $\Omega^n(M)$ for some $M \in \operatorname{mod} A$. A module M is called *self-orthogonal* if $\operatorname{Ext}_A^i(M, M) = 0$ for all $i \geq 1$. The definition of Iwanaga–Gorenstein rings includes the classical cases of Iwanaga–Gorenstein rings, namely the commutative local Gorenstein rings and the Iwanaga–Gorenstein–Artin algebras.

We are interested in the following problem that was stated in [2, Conjecture 4.8] for Artin algebras.

Problem 1. *Let A be Iwanaga–Gorenstein and let M be self-orthogonal. Then M has finite projective dimension.*

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A positive solution of this problem would have important consequences for the theory of tilting modules for Iwanaga–Gorenstein Artin algebras, see [2, Sections 3 and 4]. For Artin algebras the conjecture of Enomoto is a generalisation of the classical Tachikawa conjecture that states that a self-orthogonal module over a selfinjective algebra is projective. The Tachikawa conjecture can be seen as one of the most important homological conjectures for Artin algebras since a counterexample to the Tachikawa conjecture would give counterexamples to other homological conjectures such as the Nakayama conjecture, the Auslander–Reiten conjecture and the finitistic dimension conjecture, see for example [7] for a survey on those conjectures. Our main result gives a positive answer to the above problem for an important class of Iwanaga–Gorenstein algebras:

Theorem 1.1. *Let A be an n-Iwanaga–Gorenstein ring such that every indecomposable non-projective module $X \in \operatorname{CM} A$ is periodic. Assume M has the property that $\operatorname{Ext}_A^u(M,M) = 0$ for all $u > n$. Then M has finite projective dimension. In particular, all self-orthogonal modules have finite projective dimension.*

In [2] a positive solution to the above problem was proven for representation-finite Iwanaga-Gorenstein Artin algebras using the theory of generalised tilting modules. In [2, Question 4.7] a more direct proof for this case is asked for and our main result gives such a direct proof for a much larger class of Iwanaga–Gorenstein Artin algebras, which contain all CM-finite Iwanaga–Gorenstein rings and in particular the subclass of all such representation-finite algebras.

2. Proof of the main results

In this section $\underline{\operatorname{CM}} A$ will denote the category of maximal Cohen–Macaulay modules modulo projective modules.

Lemma 2.1. *Assume A is an n-Iwanaga–Gorenstein ring and $M \in \operatorname{CM} A$ and $N \in \operatorname{mod} A$.*

(1) $\underline{\operatorname{Hom}}_A(M,N) \cong \underline{\operatorname{Hom}}_A(\Omega^s(M), \Omega^s(N))$ for all $s \geq 0$.
(2) $\operatorname{Ext}_A^p(M,N) \cong \underline{\operatorname{Hom}}_A(\Omega^p(M), N)$ for all $p \geq 1$.
(3) If M is indecomposable, then also $\Omega^i(M) \in \operatorname{CM} A$ is indecomposable for all $i \geq 1$.
(4) If $M, N \in \operatorname{CM} A$ are indecomposable and satisfy $\Omega^1(M) \cong \Omega^1(N)$ then $M \cong N$.

Proof. Note that every module $X \in \operatorname{CM} A$ satisfies $\operatorname{Ext}_A^i(X, A) = 0$ for all $i > 0$, since X is a direct summand of a module of the form $\Omega^n(Y)$ and $\operatorname{Ext}_A^i(\Omega^n(Y), A) = \operatorname{Ext}_A^{n+i}(Y, A) = 0$ for all $i > 0$ since A has injective dimension n. Then (1) and (2) are a special case of [4, § 2.1]. (3) follows from [6, Corollary 3.3] and (4) is a consequence [5, Theorem 5.5]. □

Recall that a module $X \in \operatorname{mod} A$ is called *periodic* if $\Omega^l(X) \cong X$ for some $l \geq 1$.

Theorem 2.2. *Let A be an n-Iwanaga–Gorenstein ring such that every indecomposable non-projective module $X \in \operatorname{CM} A$ is periodic. Assume M has the property that $\operatorname{Ext}_A^u(M,M) = 0$ for all $u > n$. Then M has finite projective dimension. In particular, all self-orthogonal modules have finite projective dimension.*

Proof. Assume M has the property that $\operatorname{Ext}_A^{n+l}(M,M) = 0$ for all $l \geq 1$. Then

$$\operatorname{Ext}_A^{n+l}(M,M) \cong \operatorname{Ext}_A^l(\Omega^n(M), M) \cong \underline{\operatorname{Hom}}_A(\Omega^{l+n}(M), M)$$

by dimension shifting and Lemma 2.1 (2), which we are allowed to use since $\Omega^n(M) \in \operatorname{CM} A$. Using 2.1 (1) with $s = n$ and then (2) again we obtain:

$$\underline{\operatorname{Hom}}_A\left(\Omega^{l+n}(M), M\right) \cong \underline{\operatorname{Hom}}_A\left(\Omega^{l+n+n}(M), \Omega^n(M)\right) \cong \operatorname{Ext}_A^{n+l}\left(\Omega^n(M), \Omega^n(M)\right).$$

Thus $\operatorname{Ext}_A^{n+l}(\Omega^n(M), \Omega^n(M)) = 0$ for all $l \geq 1$, since we assume that $\operatorname{Ext}_A^{n+l}(M, M) = 0$ for all $l \geq 1$. Let X be an indecomposable direct summand of $\Omega^n(M)$. Then $X \in \operatorname{CM} A$ with $\operatorname{Ext}_A^{n+l}(X, X) = 0$ for all $l \geq 1$. Assume that X is non-zero in the stable module category. By assumption X is periodic. So assume that $X \cong \Omega^q(X)$ for some $q \geq 1$. Note that this also implies that $X \cong \Omega^{qm}(X)$ for all $m \geq 1$. Then for all $p \geq 1$ and $m \geq 1$ we obtain:

$$\operatorname{Ext}_A^p(X, X) \cong \operatorname{Ext}_A^p\left(\Omega^{qm}(X), X\right) \cong \operatorname{Ext}_A^{p+qm}(X, X).$$

Now choose m big enough so that $p + qm > n$. Then

$$\operatorname{Ext}_A^p(X, X) \cong \operatorname{Ext}_A^{p+qm}(X, X) = 0.$$

Thus X is self-orthogonal. But we also have by Lemma 2.1 (2)

$$\operatorname{Ext}_A^q(X, X) \cong \underline{\operatorname{Hom}}_A\left(\Omega^q(X), X\right) \cong \underline{\operatorname{Hom}}_A(X, X) \neq 0,$$

since the identity map in $\underline{\operatorname{Hom}}_A(X, X)$ is certainly non-zero. This is a contradiction and thus X must be zero in the stable module category. Thus every indecomposable direct summand of $\Omega^n(M)$ is the zero module in the stable module category and thus $\Omega^n(M)$ must be the a projective module, which implies that M has finite projective dimension. □

Recall that an Iwanaga–Gorenstein ring is called *CM-finite* if there are only finitely many indecomposable modules in $\operatorname{CM} A$.

Corollary 2.3. *Let A be a CM-finite Iwanaga–Gorenstein ring. Then every self-orthogonal module has finite projective dimension.*

Proof. We show that every indecomposable module $X \in \operatorname{CM} A$ is periodic. Then the result follows from 2.2. Let X be indecomposable. Since with X also $\Omega^i(X) \in \operatorname{CM} A$ is indecomposable for all $i \geq 0$ by Lemma 2.1 (3) and since there are only finitely many indecomposable modules in $\operatorname{CM} A$, we have that $\Omega^i(X) \cong \Omega^{i+l}(X)$ for some $i \geq 0$ and $l \geq 1$. This implies that $X \cong \Omega^l(X)$ by Lemma 2.1 (4) and thus X is periodic. □

We give two important examples. The first is for finite dimensional algebras and the second for commutative local rings.

Example 2.4. Let A be a finite dimensional quiver algebra KQ/I with admissible monomial ideal I. Then A has the property that $\Omega^2(\operatorname{mod} - A)$ is representation-finite, see [8], and thus A is CM-finite if A is Iwanaga–Gorenstein. In particular, all gentle algebras are CM-finite Iwanaga–Gorenstein algebras as gentle quiver algebras are always Iwanaga–Gorenstein by [3]. Thus for the class of monomial Iwanaga–Gorenstein algebras, every self-orthogonal module has finite projective dimension by our main result.

Example 2.5. Let R be a regular commutative local ring and $f \neq 0$. Then the hypersurface ring $A = R/(f)$ is Iwanaga–Gorenstein and every module $X \in \operatorname{CM} A$ is periodic of period at most 2 by the classical result about matrix factorisations by Eisenbud, see [1]. By our main result, every self-orthogonal A-module has finite projective dimension.

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