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Global Weyl modules for thin Lie algebras are finite-dimensional

Vladimir Dotsenko * and Sergey Mozgovoy

Abstract. The notion of Weyl modules, both local and global, goes back to Chari and Pressley in the case of affine Lie algebras, and has been extensively studied for various Lie algebras graded by root systems. We extend that definition to a certain class of Lie algebras graded by weight lattices and prove that if such a Lie algebra satisfies a natural "thinness" condition, then already the global Weyl modules are finite-dimensional. Our motivating example of a thin Lie algebra is the Lie algebra of polynomial Hamiltonian vector fields on the plane vanishing at the origin. We also introduce stratifications of categories of modules over such Lie algebras and identify the corresponding strata categories.

1. Introduction

Weyl modules $W(\lambda)$ were originally introduced by Chari and Pressley [10] for the extended affine Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}d$ associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$; they are parametrized by dominant integral weights λ. These modules were related to the Demazure modules of affine Kac–Moody algebras [9], and generalized to a plethora of contexts, an incomplete list of which includes twisted affine Lie algebras [7, 21], algebras of $\mathfrak{g}$-valued currents on arbitrary affine algebraic variety [16], Yangians [36, 37], toroidal Lie algebras [28, 32], currents valued in an arbitrary Kac–Moody Lie algebra [14], Lie superalgebras [1, 5], Borel–de Siebenthal pairs [8], and Lie algebras of TKK type [29]; generalizations for the case of non-dominant weights have also been considered [18, 19, 20]. More precisely, one can talk about global Weyl modules and local Weyl modules. In both cases, one considers the category $\mathcal{I}(\bar{\mathfrak{g}})$ of $\bar{\mathfrak{g}}$-modules that are integrable over $\mathfrak{g}$. For each dominant weight λ of $\mathfrak{g}$, the global Weyl module $W(\lambda) \in \mathcal{I}(\bar{\mathfrak{g}})$ is the maximal integrable module that possesses a cyclic vector v of weight λ and such that $xv = 0$ for every $x \in \bar{\mathfrak{g}}$

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of positive $\mathfrak{h}$-weight. Local Weyl modules are obtained from the global ones by imposing the condition that the cyclic vector v is additionally an eigenvector for the weight zero subalgebra of $\bar{\mathfrak{g}}$.

All the examples above share a common feature: the algebra $\bar{\mathfrak{g}}$ whose representations one considers is, in a sense, made of several copies of the adjoint representation of the Lie algebra $\mathfrak{g}$. Therefore $\bar{\mathfrak{g}}$ is graded by a root system [3], and as such fits into the general framework of [31]. In this paper, we consider a more general case of Lie algebras $\bar{\mathfrak{g}}$ that contain a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ in a way that the adjoint action of $\mathfrak{g}$ on $\bar{\mathfrak{g}}$ is integrable, so that $\bar{\mathfrak{g}}$ is graded by the weight lattice of $\mathfrak{g}$. We shall however assume that all nonzero weights of the adjoint action $\mathfrak{g}$ on $\bar{\mathfrak{g}}$ are either positive or negative; for instance, this way one can consider various subalgebras of the Lie algebra of vector fields on the plane that preserves the origin, or the Lie algebras $\mathfrak{sl}(\lambda)$, $\lambda \in \mathbb{C}$, defined by Feigin [15].

Our original motivation came from thinking about the Lie algebra H_2 of polynomial Hamiltonian vector fields on the plane. Several different constructions of H_2-modules emerged in representation theory and algebraic geometry in the past years. For instance, that Lie algebra acts on global sections of various vector bundles on Hilbert schemes of points on the plane [24], on the "y-ified Khovanov–Rozansky homology of links" of [23], and on vector spaces of the form $H^\bullet(X)[x, y]$, where X is a moduli space of stable parabolic Higgs bundles of certain rank and degree for a generic stability condition [25]; in this latter case, this action was instrumental in one of the proofs of the $P = W$ conjecture. It is probably reasonable to speculate that at least some of these actions emerge because the Lie algebra H_2 is very close to the Lie algebra of the group of polynomial automorphisms of the plane, see [35].

The Lie algebra H_2 has no nontrivial finite-dimensional representations; however, it has an important Lie subalgebra $L_0(\mathsf{H}_2)$ of Hamiltonian vector fields preserving the origin. That latter Lie algebra acts on finite-dimensional vector spaces of global sections of the above-mentioned vector bundles over the punctual Hilbert schemes; moreover, the vector spaces $H^\bullet(X)[x, y]$ look like the underlying vector spaces of modules that are coinduced from that subalgebra (in reality the action is more complicated and is not coinduced). This raises a question of better understanding finite-dimensional representations of the Lie algebra $L_0(\mathsf{H}_2)$, and the question of studying Weyl modules and their quotients arises naturally. While studying these modules, we realized, that, by contrast with most previously studied cases (with the exception of some of the Weyl modules of [8]), already all global Weyl modules over $L_0(\mathsf{H}_2)$ are finite-dimensional.

In fact, once this was observed, we managed to prove that the global Weyl modules are finite-dimensional for a large class of algebras that contains $L_0(\mathsf{H}_2)$. Namely, let us call the Lie algebra $\bar{\mathfrak{g}}$ *thin* (with respect to $\mathfrak{g}$) if the centralizer of $\mathfrak{g}$ in $\bar{\mathfrak{g}}$ is zero and all irreducible $\mathfrak{g}$-modules have finite multiplicity in $\bar{\mathfrak{g}}$. In the case of a thin algebra, a very general result on finite-dimensionality holds. To state that, we define the functor $\mathbf{B}_{\leq\lambda} \colon \mathcal{I}(\mathfrak{g}) \to \mathcal{I}(\bar{\mathfrak{g}})$ that sends an integrable $\mathfrak{g}$-module V to the maximal quotient of the induced $\bar{\mathfrak{g}}$-module $\operatorname{Ind}_{\mathfrak{g}}^{\bar{\mathfrak{g}}} V$ that is $\mathfrak{g}$-integrable and λ-bounded. (In particular, the module $\mathbf{B}_{\leq\lambda}(L(\lambda))$ coincides with the global Weyl module $W(\lambda)$.) The general finite-dimensionality result that we prove is the following one.

Theorem. *If the adjoint action of $\mathfrak{g}$ on $\bar{\mathfrak{g}}$ is thin, then the functor $\mathbf{B}_{\leq\lambda}$ maps finite-dimensional modules to finite-dimensional modules.*

Independently of the above result, we prove that the category $\mathcal{I}_{\mathsf{b}}(\bar{\mathfrak{g}}) \subset \mathcal{I}(\bar{\mathfrak{g}})$ of $\bar{\mathfrak{g}}$-modules that are integrable and bounded over $\mathfrak{g}$ can be equipped with a (left) stratification structure so that the global Weyl modules are the standard objects and the local Weyl modules are the proper standard objects of this stratification. (At this point we must warn the reader that the existing literature contains many closely related notions of stratification of abelian categories [11, 12, 27, 30, 39]. Additionally, standardly stratified categories [12, 30] are often referred to as stratified.) The strata categories of this stratification are parametrized by dominant weights $\lambda \in P_+$ and can be identified with the categories $\operatorname{Mod} \mathsf{A}_\lambda$ of left modules over the associative algebras $\mathsf{A}_\lambda \simeq \operatorname{End}_{\bar{\mathfrak{g}}}(W(\lambda))^{\mathrm{op}}$ originally introduced in [6] for the Lie algebras of the form $\bar{\mathfrak{g}} = \mathfrak{g} \otimes A$, where A is commutative. If $\bar{\mathfrak{g}} = \bigoplus_{i \geq 0} \bar{\mathfrak{g}}_{(i)}$ is non-negatively graded, with $\bar{\mathfrak{g}}_{(0)} = \mathfrak{g}$ and finite-dimensional components $\bar{\mathfrak{g}}_{(i)}$, then the category $\operatorname{mod}^{\mathbb{Z}}_{\mathsf{b}}(\bar{\mathfrak{g}})$ of $\mathbb{Z}$-graded $\bar{\mathfrak{g}}$-modules (bounded over $\mathfrak{g}$) with finite-dimensional components can be also equipped with a left stratification structure. We will show that this structure is a full stratification if $\bar{\mathfrak{g}}$ has an automorphism that restricts to $-\operatorname{id}$ on $\mathfrak{h}$ (this is true, in particular, for $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2)$).

The paper is organized as follows. In Section 2, we fix the notational conventions concerning semisimple Lie algebras and prove an adjunction result involving maximal integrable quotients of $\mathfrak{g}$-modules. In Section 3, we introduce the functor $\mathbf{B}_{\leq\lambda}$, prove the finite-dimensionality results, and give several examples of situations where this result applies. In Section 4, we introduce left recollements and left stratifications of abelian categories. We introduce the algebras A_λ, generalize to our context the categorical approach of [6, 31], and equip the category $\mathcal{I}_{\mathsf{b}}(\bar{\mathfrak{g}})$ with a left stratification structure with strata categories $\operatorname{Mod} \mathsf{A}_\lambda$. Moreover, we show for $\mathbb{Z}$-graded Lie algebras $\bar{\mathfrak{g}} = \bigoplus_{i \geq 0} \bar{\mathfrak{g}}_{(i)}$ that the category $\operatorname{mod}^{\mathbb{Z}}_{\mathsf{b}}(\bar{\mathfrak{g}})$ has a full stratification structure if $\bar{\mathfrak{g}}$ satisfies some additional conditions. In Section 5, we discuss appropriate local Weyl modules, particularly those arising from augmentations $\bar{\mathfrak{g}} \to \mathfrak{g}$. We conclude with a conjecture motivated by experimental data on local Weyl modules for the Lie algebra $L_0(\mathsf{H}_2)$: in particular, it appears that the socle of every local Weyl module for that Lie algebra is isomorphic to the defining two-dimensional $\mathfrak{sl}_2$-module.

2. Integrable and bounded modules

2.1. **Notation.** Throughout the paper, we use the following notation. The symbol $\subset$ denotes a possibly non-strict inclusion. Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, root system $\Delta = \Delta_+ \sqcup \Delta_- \subset \mathfrak{h}^*$, coroots $\alpha^\vee \in \mathfrak{h}$ for $\alpha \in \Delta$, and Chevalley generators $(e_i, f_i, h_i)_{i=1}^r$. We let $Q = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha$ and $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \;\; \forall\, \alpha \in \Delta\}$ be, respectively, the root lattice and the weight lattice of $\mathfrak{g}$. Furthermore, let $Q_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geq 0}\alpha$ and $P_+ = \{\lambda \in P \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \;\; \forall\, \alpha \in \Delta_+\}$ be, respectively, the positive root monoid and the dominant weight monoid. The monoid P_+ is contained in the positive root cone $Q_+^{\mathbb{Q}} = \sum_{\alpha \in \Delta_+} \mathbb{Q}_{\geq 0}\alpha$. We shall consider the partial order on P defined by $\lambda \leq \mu$ if $\mu - \lambda \in Q_+^{\mathbb{Q}}$. This poset is locally finite since the intervals

$$P_{\lambda,\mu} = \{\nu \in P \mid \lambda \leq \nu \leq \mu\} = P \cap (\lambda + Q_+^{\mathbb{Q}}) \cap (\mu - Q_+^{\mathbb{Q}})$$

are finite. In particular, the set $P_+^{\leq\lambda} = \{\mu \in P_+ \mid \mu \leq \lambda\} \subset P_{0,\lambda}$ is finite. We may have $P_+^{\leq\lambda} \neq \varnothing$ for $\lambda \notin P_+$; for instance, if we consider $\mathfrak{g} = \mathfrak{sl}_3$, so that $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ with $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$, $(\alpha_1, \alpha_2) = -1$ and $P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, where $\omega_1 = \frac{1}{3}(2\alpha_1 + \alpha_2)$ and $\omega_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$, we have $\omega_1 \in P_+^{\leq\lambda}$ for $\lambda = 3\omega_1 - \omega_2$.

2.2. **Integrable modules.** A $\mathfrak{g}$-module V is called *weighted* if $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V \mid hv = \lambda(h)v \ \ \forall\, h \in \mathfrak{h}\}$ is the weight λ subspace. A weighted $\mathfrak{g}$-module V is called *integrable* [26, Section 3.6] if all e_i and f_i are locally-nilpotent on V. One can show [26, Section 3.8] that integrability is equivalent to either of the following properties:

- every vector of V is contained in a finite-dimensional $\mathfrak{g}$-submodule.
- V is a direct sum of finite-dimensional $\mathfrak{g}$-modules.

Note that under these conditions we have $P(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\} \subset P$.

For a weight $\lambda \in P$, a weighted $\mathfrak{g}$-module V is called λ*-bounded* if $P(V) \subset \lambda - Q_+^{\mathbb{Q}}$. Furthermore, such module is said to be *bounded* if $P(V) \subset \bigcup_{i=1}^n (\lambda_i - Q_+^{\mathbb{Q}})$ for finitely many weights $\lambda_i \in P$. Note that an integrable $\mathfrak{g}$-module V is λ-bounded if and only if $\mu \leq \lambda$ for every irreducible summand $L(\mu)$ of V.

Lemma 2.1. *For a weighted $\mathfrak{g}$-module V the following conditions are equivalent:*

(1) $P(V)$ is finite.
(2) V is integrable and bounded.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Assuming (2), we consider the action of the Weyl group W on $P(V)$. Let $w_0 \in W$ be the longest element so that $w_0(Q_+) = -Q_+$. Then $P(V) \subset \bigcup_{i,j}(w_0\lambda_i + Q_+^{\mathbb{Q}}) \cap (\lambda_j - Q_+^{\mathbb{Q}})$ and the sets $P_{\lambda,\mu} = P \cap (\lambda + Q_+^{\mathbb{Q}}) \cap (\mu - Q_+^{\mathbb{Q}})$ are finite. □

In what follows, we denote by $\mathcal{I}(\mathfrak{g})$ the category of integrable $\mathfrak{g}$-modules, by $\mathcal{B}(\mathfrak{g})$ the category of bounded $\mathfrak{g}$-modules, and by $\mathcal{I}_\mathsf{b}(\mathfrak{g})$ the category of integrable and bounded $\mathfrak{g}$-modules. The BGG category $\mathcal{O} \subset \mathcal{B}(\mathfrak{g})$ consists of modules having finite-dimensional weight spaces. The intersection $\mathcal{O} \cap \mathcal{I}(\mathfrak{g})$ consists of finite-dimensional $\mathfrak{g}$-modules. For $V \in \mathcal{B}(\mathfrak{g})$, let $\mathbf{I}(V)$ denote the maximal integrable quotient of V. In view of Lemma 2.1, this module may be constructed as follows. Since V is bounded, we have $P(V) \subset \bigcup_i(\lambda_i - Q_+^{\mathbb{Q}})$ for finitely many $\lambda_i \in P$, and we may consider the finite set $\Omega = \bigcup_{i,j} P_{w_0\lambda_i,\lambda_j}$ and define $\mathbf{I}(V)$ as the quotient of V by the $\mathfrak{g}$-submodule generated by $\bigoplus_{\mu\notin\Omega} V_\mu$. If $f\colon V \to U$ is a morphism in $\mathcal{B}(\mathfrak{g})$, consider the composition $\bar{f}\colon V \to U \to \mathbf{I}(U)$. Then $\bar{f}(V) \subset \mathbf{I}(U)$ is integrable and bounded, hence the map $\bar{f}$ factors through the maximal integrable quotient of V and we obtain the morphism $\mathbf{I}(f)\colon \mathbf{I}(V) \to \mathbf{I}(U)$.

Lemma 2.2. *The maximal integrable quotient $V \mapsto \mathbf{I}(V)$ defines a functor $\mathbf{I}\colon \mathcal{B}(\mathfrak{g}) \to \mathcal{I}_\mathsf{b}(\mathfrak{g})$ that is left adjoint to the embedding functor $\imath\colon \mathcal{I}_\mathsf{b}(\mathfrak{g}) \to \mathcal{B}(\mathfrak{g})$.*

Proof. Let V be a bounded module and U be an integrable and bounded module. Consider a morphism $f\colon V \to U$. The module $V' = f(V) \subset U$ is integrable and bounded, hence the map f factors through the maximal integrable quotient of V. Thus, we have

$$\operatorname{Hom}_{\mathcal{I}_\mathsf{b}}(\mathbf{I}(V), U) \simeq \operatorname{Hom}_{\mathcal{B}}(V, \imath(U)) = \operatorname{Hom}_{\mathcal{B}}(V, U),$$

as desired. □

For $\lambda \in P_+$, let $L(\lambda) \in \mathcal{I}(\mathfrak{g})$ denote the irreducible $\mathfrak{g}$-module of highest weight λ. Every $V \in \mathcal{I}(\mathfrak{g})$ can be written in the form

$$V = \bigoplus_{\lambda\in P_+} V^{(\lambda)} \otimes L(\lambda), \qquad V^{(\lambda)} = \operatorname{Hom}_{\mathfrak{g}}(L(\lambda), V). \tag{2.1}$$

The number $[V : L(\lambda)] = \dim V^{(\lambda)}$ is called the *multiplicity* of $L(\lambda)$ in V.

3. Finite-dimensionality of global Weyl modules

We now fix the following set-up that will be assumed throughout the paper. Let $\bar{\mathfrak{g}}$ be a Lie algebra containing a semi-simple Lie algebra $\mathfrak{g}$ as a subalgebra and assume that the adjoint action of $\mathfrak{g}$ on $\bar{\mathfrak{g}}$ is integrable. Let

$$\bar{\mathfrak{h}} = \bar{\mathfrak{g}}_0, \qquad \bar{\mathfrak{n}}_- = \bigoplus_{\lambda<0} \bar{\mathfrak{g}}_\lambda, \qquad \bar{\mathfrak{n}}_+ = \bigoplus_{\lambda>0} \bar{\mathfrak{g}}_\lambda,$$

and assume that $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$.

Example 3.1. As indicated in the introduction, our original motivating example was the Lie algebra $L_0(\mathsf{H}_2)$ of polynomial Hamiltonian vector fields on the plane vanishing at the origin. Consider the graded algebra $R = \bigoplus_{i\geq 0} R_i = \mathbb{C}[x,y]$, where $\deg x = \deg y = 1$, and set $R_{\geq k} = \bigoplus_{i\geq k} R_i$ for $k \geq 0$. The algebra H_2 is the Lie algebra (with $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$, $f_x = \partial_x f$, $f_y = \partial_y f$)

$$\mathsf{H}_2 = \{X_f = f_x\partial_y - f_y\partial_x \mid f \in R\} \subset \operatorname{Der}(R) = R\partial_x \oplus R\partial_y.$$

The map $f \mapsto X_f$ has a one-dimensional kernel and induces an isomorphism of vector spaces $R_{\geq 1} \xrightarrow{\sim} \mathsf{H}_2$. Under this identification, the Lie bracket of vector fields corresponds to the Poisson bracket

$$\{f, g\} = f_x g_y - f_y g_x, \qquad f, g \in R.$$

The Lie algebra $\operatorname{Der}(R)$ inherits a grading from R, where $\delta \in \operatorname{Der}(R)$ has degree k if $\delta(R_i) \subset R_{i+k}$ for all $i \geq 0$. The subalgebra H_2 is also graded, with $\deg X_f = \deg f - 2$. Let $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2) \subset \mathsf{H}_2$ be the Lie subalgebra of elements of nonnegative degree. Under the isomorphism $R_{\geq 1} \xrightarrow{\sim} \mathsf{H}_2$, it corresponds to $R_{\geq 2}$. The degree zero component of the Lie algebra $\bar{\mathfrak{g}}$ is identified with R_2 and is isomorphic to the Lie algebra $\mathfrak{sl}_2$. Using the standard basis of $\mathfrak{sl}_2$, we consider the isomorphism

$$e \longmapsto X_{x^2/2} = x\partial_y, \quad f \longmapsto X_{-y^2/2} = y\partial_x, \quad h \longmapsto X_{-xy} = x\partial_x - y\partial_y.$$

The Lie algebra $\bar{\mathfrak{g}}$ admits a basis $e_{ij} = X_{x^{i+1}y^{j+1}}$, for $i, j \geq -1$ and $i + j \geq 0$, with bracket

$$[e_{ij}, e_{kl}] = \begin{vmatrix} i+1 & j+1 \\ k+1 & l+1 \end{vmatrix} e_{i+k,j+l}.$$

The adjoint action of $\mathfrak{sl}_2$ on $\bar{\mathfrak{g}}$ is integrable. In fact, we have an isomorphism of $\mathfrak{sl}_2$-modules $\bar{\mathfrak{g}}_{(i)} \simeq L(i+2)$ for all $i \geq 0$. In particular, $h = -e_{00}$ and $[e_{00}, e_{ij}] = (j-i)e_{ij}$, hence e_{ij} has weight $i - j$ and degree $i + j$. Note that $\bar{\mathfrak{h}} = \bigoplus_{i\geq 0} \mathbb{C}e_{ii}$ is an abelian subalgebra.

Remark 3.2. In the case of a Lie algebra $\bar{\mathfrak{g}}$ integrable over $\mathfrak{g} = \mathfrak{sl}_2$ (like $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2)$), the assumption $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ is automatically satisfied since $P \simeq \mathbb{Z}$. This phenomenon does not persist for other Lie algebras of vector fields from the Cartan classification [33]. In particular, it is easy to check that for each of the two natural generalizations of the Lie algebra H_2, namely the Lie algebra $\bar{\mathfrak{g}} = \mathsf{H}_{2n}$ of polynomial Hamiltonian vector fields on the $2n$-dimensional affine space (with its subalgebra $\mathfrak{g} = \mathfrak{sp}_{2n}$) and the Lie algebra $\bar{\mathfrak{g}} = \mathsf{S}_{n+1}$ of polynomial vector fields of zero divergence on the $(n+1)$-dimensional affine space (with its subalgebra $\mathfrak{g} = \mathfrak{sl}_{n+1}$), the direct sum $\bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ is a proper subalgebra of $\bar{\mathfrak{g}}$ for $n > 1$. It is tempting to find a way to separate weights into positive and negative, for instance to fix a vector $x \in \mathfrak{h}$ for which $\langle \alpha, x\rangle > 0$ for all $\alpha \in \Delta_+$ and $\langle \lambda, x\rangle = 0$ for $\lambda \in P$ implies $\lambda = 0$, consider the decomposition

$$\bar{\mathfrak{g}} = \left(\bigoplus_{\langle \lambda,x\rangle<0} \bar{\mathfrak{g}}_\lambda\right) \oplus \bar{\mathfrak{g}}_0 \oplus \left(\bigoplus_{\langle \lambda,x\rangle>0} \bar{\mathfrak{g}}_\lambda\right)$$

and then modify the partial order on the weight lattice using x-positive weights appearing in $\bar{\mathfrak{g}}$ (cf. [27]). However, already for the Lie algebra H_4 the corresponding partial order is not locally finite, so a generalization of our results to that case is far from straightforward; we do not know if our main finite-dimensionality result (Theorem 3.11) admits an analogue in that case.

The following definition of the global Weyl module usually appears in the literature.

Definition 3.3. For $\lambda \in P$, the *global Weyl module* $W(\lambda)$ is the maximal $\mathfrak{g}$-integrable $\bar{\mathfrak{g}}$-module generated by the vector v_λ of weight λ such that $\bar{\mathfrak{n}}_+ v_\lambda = 0$.

For example, if A is a unital commutative algebra, we may consider the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g} \otimes A$, for which the global Weyl modules $W(\lambda)$ are the modules introduced in [16, Definition 3]. Note that if $W(\lambda) \neq 0$, then λ is dominant. Indeed, $W(\lambda)$ is integrable and $\lambda \in P(W(\lambda))$. We have $\lambda + \alpha_i \notin P(V)$, hence $\langle \lambda, \alpha_i^\vee\rangle \geq 0$ by [26, Corollary 3.6].

3.1. **Truncations.** We shall now define functors that give a categorical context for Weyl modules. For $V \in \mathcal{I}(\mathfrak{g})$, we consider the decomposition (2.1) and define the λ-*truncation* $\tau_{\leq\lambda}(V)$ and the *complementary* λ-*truncation* $\overline{\tau}_{\leq\lambda}(V)$ by the formulas

$$\tau_{\leq\lambda}(V) = \bigoplus_{\mu\leq\lambda} V^{(\mu)} \otimes L(\mu), \qquad \overline{\tau}_{\leq\lambda}(V) = \bigoplus_{\mu\not\leq\lambda} V^{(\mu)} \otimes L(\mu).$$

Let $\mathcal{I}(\bar{\mathfrak{g}})$ denote the category of $\bar{\mathfrak{g}}$-modules that are integrable over $\mathfrak{g}$. The forgetful functor $\mathcal{I}(\bar{\mathfrak{g}}) \to \mathcal{I}(\mathfrak{g})$ has the left adjoint

$$\mathrm{Ind}_{\mathfrak{g}}^{\bar{\mathfrak{g}}} \colon \mathcal{I}(\mathfrak{g}) \longrightarrow \mathcal{I}(\bar{\mathfrak{g}}), \qquad V \longmapsto U(\bar{\mathfrak{g}}) \otimes_{U(\mathfrak{g})} V.$$

Let $\mathcal{I}_{\leq\lambda}(\bar{\mathfrak{g}})$ denote the full subcategory of $\mathcal{I}(\bar{\mathfrak{g}})$ consisting of modules V such that $V = \tau_{\leq\lambda}V$. The embedding functor $\mathcal{I}_{\leq\lambda}(\bar{\mathfrak{g}}) \hookrightarrow \mathcal{I}(\bar{\mathfrak{g}})$ has the left adjoint

$$\tau_{\leq\lambda}^{\bar{\mathfrak{g}}} \colon \mathcal{I}(\bar{\mathfrak{g}}) \longrightarrow \mathcal{I}_{\leq\lambda}(\bar{\mathfrak{g}}), \qquad V \longmapsto V/U(\bar{\mathfrak{g}})\overline{\tau}_{\leq\lambda}V.$$

The composition

$$\mathbf{B}_{\leq\lambda} = \tau_{\leq\lambda}^{\bar{\mathfrak{g}}} \mathrm{Ind}_{\mathfrak{g}}^{\bar{\mathfrak{g}}} \colon \mathcal{I}(\mathfrak{g}) \longrightarrow \mathcal{I}_{\leq\lambda}(\bar{\mathfrak{g}})$$

is left adjoint to the forgetful functor $\mathcal{I}_{\leq\lambda}(\bar{\mathfrak{g}}) \to \mathcal{I}(\mathfrak{g})$.

Notice that the functor $\mathbf{B}_{\leq\lambda}$ depends only on the set $\Omega = P_+^{\leq\lambda}$. As we will see below, all our arguments use only the fact that $P_+^{\leq\lambda}$ is finite. In fact, the proofs become somewhat shorter and cleaner if we adopt that level of generality and define, for any finite subset $\Omega \subset P_+$,

$$\tau_\Omega(V) = \bigoplus_{\mu\in\Omega} V^{(\mu)} \otimes L(\mu), \qquad \overline{\tau}_\Omega(V) = \bigoplus_{\mu\notin\Omega} V^{(\mu)} \otimes L(\mu). \tag{3.1}$$

We denote by $\mathcal{I}_\Omega(\bar{\mathfrak{g}}) \subset \mathcal{I}(\bar{\mathfrak{g}})$ the category of objects $V \in \mathcal{I}(\bar{\mathfrak{g}})$ such that $V = \tau_\Omega(V)$. The embedding functor $\mathcal{I}_\Omega(\bar{\mathfrak{g}}) \hookrightarrow \mathcal{I}(\bar{\mathfrak{g}})$ has the left adjoint

$$\tau_\Omega^{\bar{\mathfrak{g}}} \colon \mathcal{I}(\bar{\mathfrak{g}}) \longrightarrow \mathcal{I}_\Omega(\bar{\mathfrak{g}}), \qquad V \longmapsto V/U(\bar{\mathfrak{g}})\overline{\tau}_\Omega V. \tag{3.2}$$

We consider the composition

$$\mathbf{B}_\Omega = \tau_\Omega^{\bar{\mathfrak{g}}} \operatorname{Ind}_{\mathfrak{g}}^{\bar{\mathfrak{g}}} \colon \mathcal{I}(\mathfrak{g}) \longrightarrow \mathcal{I}_\Omega(\bar{\mathfrak{g}}). \tag{3.3}$$

Note that a left adjoint functor maps projective objects to projective objects. Therefore all objects in the image of $\mathbf{B}_\Omega$ (in particular, in the image of $\mathbf{B}_{\leq\lambda}$) are projective. The following result, generalizing the known property of global Weyl modules in the case $\bar{\mathfrak{g}} = \mathfrak{g} \otimes A$ for a commutative algebra A [6, Section 3.3], relates the classical definition of Weyl modules to the functor $\mathbf{B}_{\leq\lambda}$.

Proposition 3.4. *There is an isomorphism of $\bar{\mathfrak{g}}$-modules*

$$W(\lambda) \simeq \mathbf{B}_{\leq\lambda}(L(\lambda)).$$

Proof. Consider the Lie algebras $\bar{\mathfrak{b}} = \mathfrak{h} \oplus \bar{\mathfrak{n}}_+$ and $\bar{\mathfrak{b}}_- = \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_-$. Let $\bar{\mathfrak{b}}$ act on $\mathbb{C}v_\lambda$ through $\bar{\mathfrak{b}} \to \mathfrak{h} \xrightarrow{\lambda} \mathbb{C}$ and let $N(\lambda) = \operatorname{Ind}_{\bar{\mathfrak{b}}}^{\bar{\mathfrak{g}}}(\mathbb{C}v_\lambda)$. We have $\bar{\mathfrak{b}} + \bar{\mathfrak{b}}_- = \bar{\mathfrak{g}}$, hence $N(\lambda)$ is generated by v_λ under the action of $U(\bar{\mathfrak{b}}_-)$. Therefore $N(\lambda)$ is λ-bounded, and we have $W(\lambda) = \mathbf{I}(N(\lambda))$ by Definition 3.3. On the other hand, we note that the module $W(\lambda)$ is integrable and λ-bounded, hence its irreducible summands $L(\mu)$ satisfy $\mu \leq \lambda$. The highest weight vector $v_\lambda \in N(\lambda)$ induces a morphism $L(\lambda) \to N(\lambda)$ of $\mathfrak{g}$-modules, hence a surjective morphism

$$\mathbf{B}_{\leq\lambda}(L(\lambda)) = \tau_{\leq\lambda}^{\bar{\mathfrak{g}}} \operatorname{Ind}_{\mathfrak{g}}^{\bar{\mathfrak{g}}} L(\lambda) \longrightarrow W(\lambda).$$

of $\bar{\mathfrak{g}}$-modules. The module $\mathbf{B}_{\leq\lambda}(L(\lambda))$ is integrable, generated by v_λ and is λ-bounded, hence $\bar{\mathfrak{n}}_+ v_\lambda = 0$. By the maximality of $W(\lambda)$, we conclude that the surjective map $\mathbf{B}_{\leq\lambda}(L(\lambda)) \to W(\lambda)$ is an isomorphism. ☐

Lemma 3.5. *For all $V \in \mathcal{I}_{\leq\lambda}(\bar{\mathfrak{g}})$, we have an isomorphism of vector spaces*

$$\operatorname{Hom}_{\bar{\mathfrak{g}}}(W(\lambda), V) \simeq V_\lambda.$$

Proof. Indeed, $\operatorname{Hom}_{\bar{\mathfrak{g}}}(W(\lambda), V) = \operatorname{Hom}_{\bar{\mathfrak{g}}}(\mathbf{B}_{\leq\lambda} L(\lambda), V) \simeq \operatorname{Hom}_{\mathfrak{g}}(L(\lambda), V) \simeq V_\lambda$. ☐

The following result is known in the case $\bar{\mathfrak{g}} = \mathfrak{g} \otimes A$ for a commutative algebra A [6, Proposition 4]; it gives an explicit construction of global Weyl modules, which is useful for doing explicit computations with those modules. In particular, it is this definition that we implemented in `Magma` to compute various examples throughout the paper. However, we shall see that for theoretical purposes, one can entirely bypass this construction and only use the universal properties of the Weyl modules.

Lemma 3.6. *The global Weyl module $W(\lambda)$ is isomorphic to the cyclic module with a cyclic vector v_λ such that*

$$\bar{\mathfrak{n}}_+ v_\lambda = 0, \qquad h(v_\lambda) = \lambda(h) v_\lambda \ \ \forall\, h \in \mathfrak{h}, \qquad f_i^{\lambda(h_i)+1} v_\lambda = 0 \ \ \forall\, i = 1, \dots, r.$$

Proof. The proof is completely identical to that of [6, Proposition 4]; essentially, it is an immediate consequence of the formula $W(\lambda) \simeq \mathbf{B}_{\leq\lambda}(L(\lambda))$ of Proposition 3.4, combined with the standard fact that $L(\lambda)$ is the quotient of $U(\mathfrak{g})$ by the left ideal generated by

$$\mathfrak{n}_+ v_\lambda = 0, \qquad h(v_\lambda) = \lambda(h) v_\lambda \ \ \forall\, h \in \mathfrak{h}, \qquad f_i^{\lambda(h_i)+1} v_\lambda = 0 \ \ \forall\, i = 1, \dots, r. \qquad ☐$$

3.2. **Global Weyl modules for $L_0(\mathsf{H}_2)$.** For the graded Lie algebra $\bar{\mathfrak{g}} = \bigoplus_{i\geq 0} \bar{\mathfrak{g}}_{(i)} = L_0(\mathsf{H}_2)$, the explicit formula of Lemma 3.6 implies that if we set the degree of v_λ to be equal to zero, the global Weyl $W(\lambda)$ obtains a $\mathbb{Z}$-grading compatible with the grading of $\bar{\mathfrak{g}}$. The following are $\mathfrak{sl}_2$-decompositions of global Weyl modules for the Lie algebra $L_0(\mathsf{H}_2)$ for $\lambda = 0, \ldots, 7$, listed degree by degree: the top row is the degree zero component of $W(\lambda) \simeq \mathbf{B}_{\leq\lambda}(L(\lambda))$, which is always $L(\lambda)$, the next row is the degree one component, etc.

$$W(0) = L(0), \qquad W(1) = L(1), \qquad W(2) = \begin{matrix} L(2) \\ L(1) \end{matrix}, \qquad W(3) = \begin{matrix} L(3) \\ L(2) \oplus L(0) \\ L(3) \oplus L(1) \end{matrix},$$

$$W(4) = \begin{matrix} L(4) \\ L(3) \oplus L(1) \\ L(4) \oplus 2L(2) \oplus L(0) \\ L(3) \oplus 2L(1) \end{matrix}, \qquad W(5) = \begin{matrix} L(5) \\ L(4) \oplus L(2) \\ L(5) \oplus 2L(3) \oplus 2L(1) \\ 2L(4) \oplus 3L(2) \oplus 2L(0) \\ L(5) \oplus 2L(3) \oplus 3L(1) \end{matrix},$$

$$W(6) = \begin{matrix} L(6) \\ L(5) \oplus L(3) \\ L(6) \oplus 2L(4) \oplus 2L(2) \oplus L(0) \\ 2L(5) \oplus 4L(3) \oplus 3L(1) \\ 2L(6) \oplus 3L(4) \oplus 6L(2) \oplus 3L(0) \\ 2L(5) \oplus 4L(3) \oplus 4L(1) \end{matrix}, \qquad W(7) = \begin{matrix} L(7) \\ L(6) \oplus L(4) \\ L(7) \oplus 2L(5) \oplus 2L(3) \oplus L(1) \\ 2L(6) \oplus 4L(4) \oplus 4L(2) \oplus L(0) \\ 2L(7) \oplus 4L(5) \oplus 7L(3) \oplus 6L(1) \\ 3L(6) \oplus 7L(4) \oplus 9L(2) \oplus 5L(0) \\ L(7) \oplus 4L(5) \oplus 7L(3) \oplus 7L(1). \end{matrix}$$

In particular, all these modules are finite-dimensional. As we shall see below, this is always the case for $L_0(\mathsf{H}_2)$, as well as a large class of other Lie algebras.

3.3. **Main finite-dimensionality result.** The category $\mathcal{I}(\mathfrak{g})$ is closed under tensor product, hence it makes sense to talk about algebras in this category. For an associative algebra A in $\mathcal{I}(\mathfrak{g})$, let $\mathrm{Mod}(A, \mathcal{I}(\mathfrak{g}))$ be the category of left A-modules in $\mathcal{I}(\mathfrak{g})$. Similarly, for a finite subset $\Omega \subset P_+$, let $\mathrm{Mod}(A, \mathcal{I}_\Omega(\mathfrak{g})) \subset \mathrm{Mod}(A, \mathcal{I}(\mathfrak{g}))$ denote the full subcategory of left A-modules whose underlying objects lie in $\mathcal{I}_\Omega(\mathfrak{g})$. The embedding functor $\mathrm{Mod}(A, \mathcal{I}_\Omega(\mathfrak{g})) \hookrightarrow \mathrm{Mod}(A, \mathcal{I}(\mathfrak{g}))$ has the left adjoint

$$\tau_\Omega^A \colon \mathrm{Mod}(A, \mathcal{I}(\mathfrak{g})) \longrightarrow \mathrm{Mod}(A, \mathcal{I}_\Omega(\mathfrak{g})), \qquad M \longmapsto M/A\bar{\tau}_\Omega(M).$$

For example, if $\bar{\mathfrak{g}} = \mathfrak{g} \oplus I$, where I is an ideal of the Lie algebra $\bar{\mathfrak{g}}$, then the universal enveloping algebra $A = U(I)$ is an associative algebra in the category $\mathcal{I}(\mathfrak{g})$. We have an equivalence of categories $\mathrm{Mod}(A, \mathcal{I}(\mathfrak{g})) \simeq \mathcal{I}(\bar{\mathfrak{g}})$ since every $\bar{\mathfrak{g}}$-module can be identified with a $\mathfrak{g}$-module M equipped with a morphism $I \otimes M \to M$ of $\mathfrak{g}$-modules that makes M a module over I. The functor τ_Ω^A coincides with the truncation functor $\tau_\Omega^{\bar{\mathfrak{g}}}$ defined in (3.2).

Proposition 3.7. *If V is a finite-dimensional $\mathfrak{g}$-module such that $[V : L(0)] = 0$, then the algebra $S_\Omega(V) = (SV)/(SV \cdot \bar{\tau}_\Omega(SV))$ is finite-dimensional.*

Proof. The $\mathfrak{g}$-module $A = S_\Omega(V)$ has only irreducible summands $L(\mu)$ with $\mu \in \Omega$. Therefore it has a weight decomposition $A = \bigoplus_{\mu \in P} A_\mu$ such that $P(A) = \{\mu \in P \mid A_\mu \neq 0\}$ is finite. Consider the weight decomposition $V = \bigoplus_{\mu \in P} V_\mu$ and let $A_{(\mu)}$ be the image of $S(V_\mu)$ in A. If $\mu \neq 0$, then $S^k(V_\mu)$ is mapped to $A_{k\mu}$ which is zero for $k \gg 0$. Therefore $A_{(\mu)}$ is finite-dimensional. We have $A_{(\mu)} = \mathbb{C}$ for all but a finite number of $\mu \in P$, hence $B = \prod_{\mu \neq 0} A_{(\mu)} \subset A$ is finite-dimensional.

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The algebra A is generated over the algebra B by the image of V_0. Let us show that the elements of this image are integral over B. The assumption $[V\colon L(0)] = 0$ implies that V_0 is spanned by elements of the form $u = fv$, where $f \in \mathfrak{g}_{-\alpha}$ and $v \in V_\alpha$ for some $\alpha \in \Delta$. The endomorphism f of V induces a derivation of $S(V)$. This derivation preserves $\overline{\tau}_\Omega(SV)$ and the ideal $SV \cdot \overline{\tau}_\Omega(SV)$ (which are both $\mathfrak{g}$-modules). Therefore f induces a derivation of A. Let $\bar{u}, \bar{v} \in A$ be the images of $u, v \in V$. The element $f^m(\bar{v}^m)$ is a linear combination of products $\prod_{i=1}^m f^{k_i}(\bar{v})$, where $k_i \geq 0$ and $\sum_i k_i = m$. We have $f^k(\bar{v}) \in B$ if $k \neq 1$, hence $\prod_{i=1}^m f^{k_i}(\bar{v}) \in B\bar{u}^r$, where $r = \#\{i \,|\, k_i = 1\}$. For $m \gg 0$ we have $\bar{v}^m = 0$, hence $0 = f^m(\bar{v}^m) \in m! \cdot \bar{u}^m + \sum_{r=0}^{m-1} B\bar{u}^r$. Therefore $\bar{u}^m \in \sum_{r=0}^{m-1} B\bar{u}^r$ and $\bar{u}$ is integral over B. This implies that the algebra A is generated over B by finitely many integral elements, hence A is finite over B. □

Remark 3.8. Consider the case $\mathfrak{g} = \mathfrak{sl}_2$ and suppose that $V = V^{(2)} \otimes L(2)$ is the sum of several copies of the adjoint module $L(2)$. In this case, the algebra $S_{\leq\lambda}(V)$ for $\lambda = 2$ is precisely the algebra $\mathrm{Com}^{\mathfrak{T}}(V)$ from [13], and the finite-dimensionality of that algebra (in fact, a precise description of its underlying vector space) is established in [13, Proposition 3.2].

The previous result motivates the following definition.

Definition 3.9. An integrable $\mathfrak{g}$-module V is said to be *thin* if the multiplicities of irreducible $\mathfrak{g}$-modules in it are finite and $[V : L(0)] = 0$.

Corollary 3.10. *If V is a thin $\mathfrak{g}$-module, then the algebra $S_\Omega(V)$ is finite-dimensional.*

Proof. Consider the decomposition $V = V' \oplus V''$, where $V' = \tau_\Omega(V)$ and $V'' = \overline{\tau}_\Omega(V)$. Then $V'' \subset \overline{\tau}_\Omega(SV)$, hence $\bigoplus_{k\geq 1} S^k(V'') \subset S(V'')$ is mapped to zero in $S_\Omega(V)$. Therefore the map $S_\Omega(V') \to S_\Omega(V)$ is surjective. By Proposition 3.7, the algebra $S_\Omega(V')$ is finite-dimensional. □

Theorem 3.11. *If the adjoint action of $\mathfrak{g}$ on $\bar{\mathfrak{g}}$ is thin, then the functor $\mathbf{B}_\Omega\colon \mathcal{I}(\mathfrak{g}) \to \mathcal{I}_\Omega(\bar{\mathfrak{g}})$ maps finite-dimensional modules to finite-dimensional modules.*

Proof. Let V be a finite-dimensional $\mathfrak{g}$-module. We have $\mathbf{B}_\Omega(V) = N'/A(\overline{\tau}_\Omega N')$, where $A = U(\bar{\mathfrak{g}})$ and $N' = A \otimes_{U(\mathfrak{g})} V$. For $N = A \otimes V$ there is a surjective morphism $\overline{N} = N/A(\overline{\tau}_\Omega N) \to N'/A(\overline{\tau}_\Omega N') = \mathbf{B}_\Omega(V)$. Thus, to show that $\mathbf{B}_\Omega(V)$ is finite-dimensional, it suffices to prove that $\overline{N}$ is finite-dimensional. Let $A = \bigoplus_{\mu\in P_+} A^{(\mu)} \otimes L(\mu)$. The image of $A^{(\mu)} \otimes L(\mu) \otimes V$ in $\overline{N}$ can be nonzero only if there is a summand $L(\nu)$ of $L(\mu) \otimes V$ such that $\nu \in \Omega$. We have

$$\mathrm{Hom}_{\mathfrak{g}}(L(\mu) \otimes V, L(\nu)) \simeq \mathrm{Hom}_{\mathfrak{g}}(L(\mu), \mathrm{Hom}(V, L(\nu))),$$

hence $L(\mu)$ has to be a summand of $\mathrm{Hom}(V, L(\nu))$ for some $\nu \in \Omega$. The set Ω' of such $\mu \in P_+$ is finite. We conclude that $A\overline{\tau}_{\Omega'}(A) \otimes V$ is mapped to zero in $\overline{N}$, hence $\overline{N}$ is a quotient of $(A/A\overline{\tau}_{\Omega'}(A)) \otimes V$ and it is enough to show that the A-module $M = A/A(\overline{\tau}_{\Omega'}A)$ is finite-dimensional. The PBW filtration $(F_i)_{i\geq 0}$ on A induces the filtration $(F_iM)_{i\geq 0}$ on M. The epimorphism $A \to M$ induces the epimorphism between associated graded modules $\mathrm{Gr}\, A \simeq S\bar{\mathfrak{g}} \to \mathrm{Gr}\, M$. We have $\overline{\tau}_{\Omega'} M = 0$, hence $\overline{\tau}_{\Omega'} \mathrm{Gr}\, M = 0$ and the above map factorises through $S\bar{\mathfrak{g}} \to S_{\Omega'}(\bar{\mathfrak{g}}) = (S\bar{\mathfrak{g}})/(S\bar{\mathfrak{g}} \cdot \overline{\tau}_{\Omega'}(S\bar{\mathfrak{g}}))$. According to Proposition 3.7, $S_{\Omega'}(\bar{\mathfrak{g}})$ is finite-dimensional, completing the proof. □

3.4. **Applications of the main result.** Let us give some examples of situations where our result applies. We begin with a class of examples including that of $L_0(\mathsf{H}_2)$. Let us denote by W_2 the Lie algebra of all polynomial vector fields on the plane, and by $L_0(\mathsf{W}_2)$ the Lie subalgebra of that algebra consisting of all vector fields that vanish at the origin.

Corollary 3.12. *For every Lie subalgebra $\bar{\mathfrak{g}} \subset L_0(\mathsf{W}_2)$ that contains*

$$\mathfrak{g} = \mathbb{C}\{x\partial_y, y\partial_x, x\partial_x - y\partial_y\} \simeq \mathfrak{sl}_2$$

and does not contain the Euler vector field $x\partial_x + y\partial_y$, all global Weyl modules are finite-dimensional.

Proof. A vector field vanishing at the origin is of the form $f(x,y)\partial_x + g(x,y)\partial_y$, where f and g belong to the ideal of $\mathbb{C}[x,y]$ generated by x and y. It easily follows that, if we consider $L_0(\mathsf{W}_2)$ as an $\mathfrak{sl}_2$-module with respect to the adjoint action, we have an isomorphism of $\mathfrak{sl}_2$-modules

$$L_0(\mathsf{W}_2) \simeq L(1) \otimes \bigoplus_{k\geq 1} L(k) \simeq \bigoplus_{k\geq 1} (L(k-1) \oplus L(k+1)).$$

We note that each irreducible $\mathfrak{sl}_2$-module appears in this decomposition with multiplicity at most two. Moreover, this decomposition has exactly one summand $L(0)$; it appears for $k = 1$, that is, for linear vector fields, and corresponds precisely to the Euler vector field. It follows that the adjoint action of $\mathfrak{g} = \mathfrak{sl}_2$ on $\bar{\mathfrak{g}}$ is thin, and Theorem 3.11 applies. □

Another class of examples where our results apply is that where we consider a finite-dimensional Lie algebra containing $\mathfrak{g}$.

Corollary 3.13. *For every finite-dimensional Lie algebra $\bar{\mathfrak{g}}$ containing $\mathfrak{g}$ as a Lie subalgebra and graded by the root system of $\mathfrak{g}$, if the centralizer of $\mathfrak{g}$ in $\bar{\mathfrak{g}}$ is zero, then all global Weyl modules are finite-dimensional.*

Proof. The root grading defines the necessary decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$. Moreover, finite-dimensionality of $\bar{\mathfrak{g}}$ together with the zero centralizer condition assures that the adjoint action of $\mathfrak{g}$ on $\bar{\mathfrak{g}}$ is thin, so Theorem 3.11 applies. □

To give one final example, let us recall the definition of the Lie algebra $\mathfrak{sl}(\lambda)$ introduced by Feigin [15]. Given $\lambda \in \mathbb{C}$, consider the quotient of the universal enveloping algebra $U(\mathfrak{sl}_2)$ by the two-sided ideal generated by $\Delta - \frac{\lambda^2-1}{2}$, where $\Delta = ef + fe + \frac{h^2}{2} \in U(\mathfrak{sl}_2)$ is the Casimir element. Let $\mathfrak{gl}(\lambda)$ be the Lie algebra obtained from that associative algebra by considering the usual Lie bracket $[a,b] = ab - ba$. If $\lambda = n \in \mathbb{Z}_{\geq 1}$, the Casimir element acts by $\frac{n^2-1}{2}$ on the irreducible $\mathfrak{sl}_2$-module $V = L(n-1)$ of dimension n. Therefore we have a surjective algebra homomorphism $\mathfrak{gl}(\lambda) \to \mathfrak{gl}(V) \simeq \mathfrak{gl}_n$, explaining the name $\mathfrak{gl}(\lambda)$. Let $\mathfrak{sl}(\lambda) = \mathfrak{gl}(\lambda)/\mathbb{C}1$ be the quotient of $\mathfrak{gl}(\lambda)$ by the center. Being a quotient of $U(\mathfrak{sl}_2)$, the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{sl}(\lambda)$ contains $\mathfrak{g} = \mathfrak{sl}_2$ as a Lie subalgebra.

Corollary 3.14. *For the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{sl}(\lambda)$ and its Lie subalgebra $\mathfrak{g} = \mathfrak{sl}_2$, all global Weyl modules are finite-dimensional.*

Proof. We have a decomposition with respect to the adjoint action of $\mathfrak{sl}_2$ [15]

$$\mathfrak{sl}(\lambda) = \bigoplus_{n\geq 1} L(2n),$$

so the adjoint action of $\mathfrak{g} = \mathfrak{sl}_2$ on $\bar{\mathfrak{g}}$ is thin, and Theorem 3.11 applies. □

4. Stratifications

In this section we will equip the category $\mathcal{A} = \mathcal{I}_{\mathfrak{b}}(\bar{\mathfrak{g}})$ of $\bar{\mathfrak{g}}$-modules that are integrable and bounded over $\mathfrak{g}$ with a (left) stratification structure over the poset P_+. It turns out that the strata categories $\mathcal{A}_\lambda$ of this stratification (for $\lambda \in P_+$) are the categories of left modules over certain algebras A_λ originally introduced in [6] for the Lie algebras of the form $\bar{\mathfrak{g}} = \mathfrak{g} \otimes A$, where A is commutative. These algebras were further studied in [31] for the root-graded Lie algebras $\bar{\mathfrak{g}}$, where it was shown that the algebras A_λ are isomorphic to the algebras originally introduced in [34] in the context of λ-admissible modules. We will see that $\mathsf{A}_\lambda^{\mathrm{op}}$ is isomorphic to the endomorphism algebra of the Weyl module $W(\lambda)$. The Weyl functors $\mathbf{W}_\lambda\colon \operatorname{Mod}\mathsf{A}_\lambda \to \mathcal{I}(\bar{\mathfrak{g}})$ (for the algebras $\bar{\mathfrak{g}} = \mathfrak{g} \otimes A$) have been used to produce and study new $\bar{\mathfrak{g}}$-modules, but their interpretation as building blocks of a stratification structure seems to be missing in the literature. Using this point of view, we can interpret global Weyl modules as standard objects and local Weyl modules as proper standard objects of our stratified category (cf. [27]).

4.1. **Left recollement.** If $j^*\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories, then the category $\mathcal{C} = \operatorname{Ker} j^*$ is a Serre subcategory of $\mathcal{A}$ and the induced functor $\mathcal{A}/\mathcal{C} \to \mathcal{B}$ is faithful and exact [38, Lemma 12.10.7] . The functor $\mathcal{A}/\mathcal{C} \to \mathcal{B}$ can be full even if $j^*\colon \mathcal{A} \to \mathcal{B}$ was not.

Lemma 4.1. *Let $j^*\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories having a fully faithful left adjoint functor $j_!$ (equivalently, the adjunction morphism* $\mathrm{id} \to j^* j_!$ *is an isomorphism). Then the induced functor $\mathcal{A}/\mathcal{C} \to \mathcal{B}$, for $\mathcal{C} = \operatorname{Ker} j^*$, is an equivalence (cf.* [22, Proposition III.2.5]*) and the embedding functor $i_*\colon \mathcal{C} \to \mathcal{A}$ has a left adjoint i^*.*

Proof. Consider the induced functor $\mathbf{j}^*\colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$ and the composition $\mathbf{j}_!\colon \mathcal{B} \xrightarrow{j_!} \mathcal{A} \to \mathcal{A}/\mathcal{C}$. Then $\mathbf{j}^*\mathbf{j}_! \simeq \mathrm{id}$ and we need to show that $\mathbf{j}_!\mathbf{j}^* \to \mathrm{id}$ is an isomorphism. For every $X \in \mathcal{A}$, the morphism $j_! j^* X \to X$ is an isomorphism in $\mathcal{A}/\mathcal{C}$, since $j^*(j_! j^* X) \to j^*(X)$ is an isomorphism in $\mathcal{B}$ and $\mathcal{A}/\mathcal{C} \to \mathcal{B}$ is faithful and exact. This implies that $\mathbf{j}_!\mathbf{j}^* \to \mathrm{id}$ is an isomorphism.

For $X \in \mathcal{A}$, let $i^*(X) = X' = \operatorname{Coker}(j_! j^* X \to X)$. We have an exact sequence

$$j^* j_! j^* X \longrightarrow j^* X \longrightarrow j^* X' \longrightarrow 0,$$

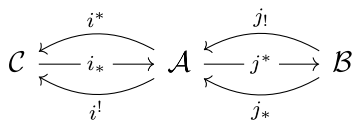
implying that $j^* X' = 0$ and $X' \in \mathcal{C}$. For $Y \in \mathcal{C}$, we have an exact sequence $0 \to \operatorname{Hom}(X', Y) \to \operatorname{Hom}(X, Y) \to \operatorname{Hom}(j_! j^* X, Y)$.

Since $\operatorname{Hom}(j_! j^* X, Y) \simeq \operatorname{Hom}(j^* X, j^* Y) = 0$, we conclude that

$$\operatorname{Hom}(i^* X, Y) = \operatorname{Hom}(X', Y) \simeq \operatorname{Hom}(X, Y). \qquad \square$$

A *recollement* of abelian categories (cf. [2, Section 1.4]) is a sequence of exact functors $\mathcal{C} \xrightarrow{i_*} \mathcal{A} \xrightarrow{j^*} \mathcal{B}$ between abelian categories such that:

(1) $j^* i_* = 0$ and $i_*\colon \mathcal{C} \to \operatorname{Ker} j^*$ is an equivalence.
(2) j^* has left and right fully faithful adjoints $j_!, j_*$.



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By Lemma 4.1, the functor i_* has left and right adjoints $i^*, i^!$ and the induced functor $\mathcal{A}/\mathcal{C} \to \mathcal{B}$ is an equivalence. The adjunction morphisms

$$i^* i_* \longrightarrow \mathrm{id} \longrightarrow i^! i_* \quad \text{and} \quad j^* j_* \longrightarrow \mathrm{id} \longrightarrow j^* j_!$$

are isomorphisms. We will say that we have a *left recollement* if only the fully faithful left adjoint functor $j_!$ is required to exist.

Example 4.2. Let $(X, \mathcal{O})$ be a ringed space, $U \subset X$ be an open subset, $Z = X \backslash U$ and $j \colon U \hookrightarrow X$, $i \colon Z \hookrightarrow X$ be the embeddings. Let $\mathcal{A} = M(X, \mathcal{O})$ be the abelian category of (left) $\mathcal{O}$-modules over X and similarly for $\mathcal{B} = M(U, \mathcal{O})$ and $\mathcal{C} = M(Z, \mathcal{O})$. Then there is a recollement

$$M(Z,\mathcal{O}) \xrightarrow{\;i_*\;} M(X,\mathcal{O}) \xrightarrow{\;j^*\;} M(U,\mathcal{O}), \qquad i^*, i^! \colon M(X,\mathcal{O}) \to M(Z,\mathcal{O}), \qquad j_!, j_* \colon M(U,\mathcal{O}) \to M(X,\mathcal{O})$$

where

$$j^* \colon M(X, \mathcal{O}) \longrightarrow M(U, \mathcal{O}) \quad \text{and} \quad i^* \colon M(X, \mathcal{O}) \longrightarrow M(Z, \mathcal{O})$$

are restriction functors,

$$j_* \colon M(U, \mathcal{O}) \longrightarrow M(X, \mathcal{O}) \quad \text{and} \quad i_* \colon M(Z, \mathcal{O}) \longrightarrow M(X, \mathcal{O})$$

are direct image functors, $j_! \colon M(U, \mathcal{O}) \to M(X, \mathcal{O})$ is extension by zero, and $i^! \colon M(X, \mathcal{O}) \to M(Z, \mathcal{O})$ is the functor of sections supported on Z. For $F \in M(X, \mathcal{O})$ there are exact sequences $0 \to j_! j^* F \to F \to i_* i^* F \to 0$ and $0 \to i_* i^! F \to F \to j_* j^* F$. For an arbitrary recollement we don't have exactness on the left in the first sequence in general (but we have exactness on the right by Lemma 4.1).

Assume that we have a left recollement $\mathcal{C} \xrightarrow{i_*} \mathcal{A} \xrightarrow{j^*} \mathcal{B}$.

Lemma 4.3 (cf. [6, Theorem 1])**.** *An object $X \in \mathcal{A}$ lies in the essential image of $j_!$ if and only if*

$$\mathrm{Hom}(X, \mathcal{C}) = \mathrm{Ext}^1(X, \mathcal{C}) = 0.$$

Proof. Let $X = j_! Z$ for $Z \in \mathcal{B}$ and let $Y \in \mathcal{C}$. Then $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Z, j^* Y) = 0$. If $0 \to Y \to X' \to X \to 0$ is an exact sequence, then $j^* X' \simeq j^* X = j^* j_! Z \simeq Z$ and this isomorphism induces $X = j_! Z \to X'$ which splits the above exact sequence.

Conversely, assume that $X \in \mathcal{A}$ satisfies $\mathrm{Hom}(X, \mathcal{C}) = \mathrm{Ext}^1(X, \mathcal{C}) = 0$. We need to show that $j_! j^* X \to X$ is an isomorphism. Consider an exact sequence $0 \to Y_1 \to j_! j^* X \to X \to Y_2 \to 0$. Applying j^*, we obtain an isomorpism $j^* j_! j^* X \xrightarrow{\sim} j^* X$, hence $Y_1, Y_2 \in \mathcal{C}$. As $\mathrm{Hom}(X, \mathcal{C}) = 0$, we obtain $Y_2 = 0$. As $\mathrm{Ext}^1(X, \mathcal{C}) = 0$, we obtain $j_! j^* X = Y_1 \oplus X$. But $\mathrm{Hom}(j_! j^* X, Y_1) \simeq \mathrm{Hom}(j^* X, j^* Y_1) = 0$, hence $Y_1 = 0$ and $j_! j^* X \simeq X$. ☐

Lemma 4.4. *If $\mathcal{B}$ has enough projectives, then the functor $j_! \colon \mathcal{B} \to \mathcal{A}$ is exact if and only if* $\mathrm{Ext}^2(j_! \mathcal{B}, \mathcal{C}) = 0$.

Proof. Let $j_!$ be exact. Consider an exact sequence $0 \to X' \to P \to X \to 0$ in $\mathcal{B}$ with projective P. For $Y \in \mathcal{C}$, we have an exact sequence $\mathrm{Ext}^1(j_! X', Y) \to \mathrm{Ext}^2(j_! X, Y) \to \mathrm{Ext}^2(j_! P, Y)$. The first term is zero by the previous lemma. The last term is zero since $j_!$ is left adjoint, hence maps projectives to projectives.

Conversely, given an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{B}$, we have an exact sequence $j_!X' \xrightarrow{f} j_!X \to j_!X'' \to 0$ and we need to show that f is a monomorphism. Let $Y_1 = \operatorname{Ker} f$ and $Y_2 = \operatorname{Im} f$. We have an exact sequence

$$0 \longrightarrow j^*Y_1 \longrightarrow j^*j_!X' \longrightarrow j^*j_!X.$$

Since $j^*j_! \simeq \mathrm{id}$, we conclude that $j^*j_!X' \to j^*j_!X$ is a monomorphism, hence $j^*Y_1 = 0$ and $Y_1 \in \mathcal{C}$. Using the short exact sequence $0 \to Y_2 \to j_!X \to j_!X'' \to 0$, we obtain an exact sequence

$$0 = \operatorname{Ext}^1(j_!X, Y_1) \longrightarrow \operatorname{Ext}^1(Y_2, Y_1) \longrightarrow \operatorname{Ext}^2(j_!X'', Y_1) = 0,$$

hence $\operatorname{Ext}^1(Y_2, Y_1) = 0$. Therefore $j_!X' = Y_1 \oplus Y_2$. But $\operatorname{Hom}(j_!X', Y_1) = 0$, hence $Y_1 = 0$. □

Lemma 4.5. *If $\mathcal{A}' \subset \mathcal{A}$ is a full subcategory such that $\operatorname{Hom}(\mathcal{C}, \mathcal{A}') = \operatorname{Hom}(\mathcal{A}', \mathcal{C}) = 0$, then $j^*\colon \mathcal{A}' \to \mathcal{B}$ is fully faithful. This is true if $\mathcal{A}'$ is a Serre subcategory such that $\mathcal{A}' \cap \mathcal{C} = 0$.*

Proof. For $X, Y \in \mathcal{A}$, we have

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(X, Y) = \varinjlim \operatorname{Hom}_{\mathcal{A}}(X', Y/Y')$$

where the limit is taken over $X' \subset X$ with $X/X' \in \mathcal{C}$ and $Y' \subset Y$ with $Y' \in \mathcal{C}$. If X, $Y \in \mathcal{A}'$, then $X/X' = 0$ and $Y' = 0$, hence $\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{A}}(X, Y) = \operatorname{Hom}_{\mathcal{A}'}(X, Y)$. This implies that $j^*\colon \mathcal{A}' \to \mathcal{B}$ is fully faithful.

Let $\mathcal{A}' \subset \mathcal{A}$ be a Serre subcategory such that $\mathcal{A}' \cap \mathcal{C} = 0$ and let $X \in \mathcal{C}$, $Y \in \mathcal{A}'$. For every morphism $f\colon X \to Y$, we have $\operatorname{Im} f \in \mathcal{A}' \cap \mathcal{C} = 0$, hence $f = 0$. This implies $\operatorname{Hom}(\mathcal{C}, \mathcal{A}') = 0$. Similarly $\operatorname{Hom}(\mathcal{A}', \mathcal{C}) = 0$. □

Lemma 4.6. *If $L \in \mathcal{A}\backslash\mathcal{C}$ is simple, then $M = j^*L$ is simple and L is a quotient of $j_!M$. If $L, L' \in \mathcal{A}\backslash\mathcal{C}$ are simple and non-isomorphic, then j^*L, j^*L' are non-isomorphic. If $\mathcal{A}$ is of finite length, then there is a 1-1 correspondence between isomorphism classes of simple objects in $\mathcal{A}\backslash\mathcal{C}$ and isomorphism classes of simple objects in $\mathcal{B}$.*

Proof. For a nonzero subobject $N \hookrightarrow M = j^*L$, the corresponding map $j_!N \to L$ is nonzero, hence an epimorphism. Therefore $N \simeq j^*j_!N \to j^*L$ is an epimorphism and $N = M$. This implies that M is simple. We have seen that $j_!M \to L$ is an epimorphism. Let $\mathcal{A}' \subset \mathcal{A}$ be the Serre subcategory generated by L, L'. Then $\mathcal{A}' \cap \mathcal{C} = 0$, hence $\operatorname{Hom}(L, L') \simeq \operatorname{Hom}(j^*L, j^*L')$. If j^*L, j^*L' are isomorphic, then $\operatorname{Hom}(L, L') \neq 0$, hence L, L' are isomorphic.

Assume that $\mathcal{A}$ is of finite length. For a simple object $M \in \mathcal{B}$, let L be a simple quotient of $j^!M$. A nonzero map $j^!M \to L$ corresponds to a nonzero map $M \to j^*L$. We have seen that j^*L is simple, hence $j^*L \simeq M$. This implies that j^* induces a bijection between the corresponding sets of isomorphism classes. □

4.2. **Left stratifications.** We say that a poset Λ is *lower finite* if, for every $\lambda \in \Lambda$, the set $\Lambda_{\le\lambda} = \{\mu \in \Lambda \mid \mu \le \lambda\}$ is finite. A *lower subset* of Λ is a subset $\Omega \subset \Lambda$ such that $\lambda \in \Omega$ and $\mu \le \lambda$ implies $\mu \in \Omega$ (equivalently, $\Lambda_{\le\lambda} \subset \Omega$ for all $\lambda \in \Omega$). We define a *left stratification* of an abelian category $\mathcal{A}$ by a lower finite poset Λ to be a collection of Serre subcategories $(\mathcal{A}_\Omega)_{\Omega\subset\Lambda}$ for finite lower subsets $\Omega \subset \Lambda$ and a collection of abelian categories $\mathcal{A}_\lambda$ (called *strata categories*) for $\lambda \in \Lambda$ such that the collection $(\mathcal{A}_\Omega)_{\Omega\subset\Lambda}$ is order preserving (if $\Omega \subset \Omega'$, then $\mathcal{A}_\Omega \subset \mathcal{A}_{\Omega'}$), $\mathcal{A}_\varnothing = 0$, $\bigcup_\Omega \mathcal{A}_\Omega = \mathcal{A}$, and for all finite lower subsets $\Omega \subset \Lambda$ and $\lambda \in \max\Omega$ we have compatible left recollements of abelian categories

$$\mathcal{A}_{\Omega\backslash\lambda} \xrightarrow{i_*} \mathcal{A}_\Omega \xrightarrow{j^*} \mathcal{A}_\lambda$$

The above data is called a *stratification* if all sequences are recollements (cf. [39, Section 2.1]).

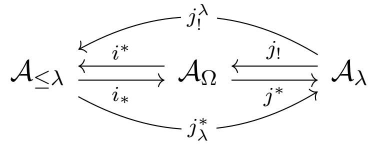
Remark 4.7. The above conditions on a left stratification imply that, for a finite lower subset $\Omega \subset \Lambda$, the embedding $i_*\colon \mathcal{A}_\Omega \to \mathcal{A}$ has a left adjoint $i^*\colon \mathcal{A} \to \mathcal{A}_\Omega$ (such that $i^*i_* \simeq \mathrm{id}$). For compatibility we require that

$$\mathcal{A}_\lambda \xrightarrow{j_!} \mathcal{A}_\Omega$$

is isomorphic to

$$\mathcal{A}_\lambda \xrightarrow{j_!^\lambda} \mathcal{A}_{\leq\lambda} \xrightarrow{i_*} \mathcal{A}_\Omega,$$

for $\lambda \in \max\Omega$ (note that $\Lambda_{\leq\lambda} \subset \Omega$ since $\Omega \subset \Lambda$ is a lower subset). This implies $i^*j_! \simeq j_!^\lambda$ which is equivalent to $j^*i_* \simeq j_\lambda^*$.



Our notion of a left stratified category is related to the notion of a highest weight category in [27] and a (standardly) stratified category in [30], although in these papers one only requires existence of the left adjoint $j_!^\lambda\colon \mathcal{A}_\lambda \to \mathcal{A}_{\leq\lambda}$ for the projection $j_\lambda^*\colon \mathcal{A}_{\leq\lambda} \to \mathcal{A}_\lambda \coloneqq \mathcal{A}_{\leq\lambda}/\mathcal{A}_{<\lambda}$. Note that in [30] the authors require $j_!^\lambda$ to be exact while we require it to be fully faithful.

Remark 4.8. If $\lambda \not\leq \mu$, consider $\Omega = \Lambda_{\leq\lambda} \cup \Lambda_{\leq\mu}$ with $\lambda \in \max\Omega$. Then $j_!\mathcal{A}_\mu \subset \mathcal{A}_{\Omega\backslash\lambda}$, hence

$$\mathrm{Hom}(j_!\mathcal{A}_\lambda, j_!\mathcal{A}_\mu) = 0, \qquad \mathrm{Ext}^1(j_!\mathcal{A}_\lambda, j_!\mathcal{A}_\mu) = 0, \qquad \forall\, \lambda \not\leq \mu. \tag{4.1}$$

Example 4.9. Let $(X, \mathcal{O})$ be a ringed space and $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$ be a partition of X for a finite poset Λ such that the closures of the parts are $\overline{X}_\lambda = \bigsqcup_{\mu\leq\lambda} X_\mu$. As in Example 4.2, we consider $\mathcal{A} = M(X, \mathcal{O})$, $\mathcal{A}_\Omega = M(X_\Omega, \mathcal{O})$ for a lower subset $\Omega \subset \Lambda$ and the closed set $X_\Omega = \bigsqcup_{\lambda\in\Omega} X_\lambda$, and $\mathcal{A}_\lambda = M(X_\lambda, \mathcal{O})$. For $\lambda \in \max\Omega$, the subset $X_\lambda \subset X_\Omega$ is open in X_Ω and $X_{\Omega\backslash\lambda} \subset X_\Omega$ is closed in X_Ω. The recollements

$$\mathcal{A}_{\Omega\backslash\lambda} \xrightarrow{i_*} \mathcal{A}_\Omega \xrightarrow{j^*} \mathcal{A}_\lambda$$

form a stratification of $\mathcal{A}$. Considering

$$j\colon X_\lambda \xrightarrow{j^\lambda} X_{\leq\lambda} \xrightarrow{i} X_\Omega,$$

we obtain $i_*j_!^\lambda = i_!j_!^\lambda = (ij^\lambda)_! = j_!$ which proves compatibility.

Lemma 4.10. *For a simple object $L \in \mathcal{A}$, there exists a unique $\lambda \in \Lambda$ such that $L \in \mathcal{A}_{\leq\lambda}\backslash\mathcal{A}_{<\lambda}$.*

Proof. Let $\Omega \subset \Lambda$ be a minimal lower subset such that $L \in \mathcal{A}_\Omega$ and let $\lambda \in \max\Omega$. Consider the left recollement

$$\mathcal{A}_{\Omega\backslash\lambda} \hookrightarrow \mathcal{A}_\Omega \xrightarrow{j^*} \mathcal{A}_\lambda.$$

If $j^*L = 0$, then $L \in \mathcal{A}_{\Omega\backslash\lambda}$, a contradiction. Therefore $j^*L \neq 0$, hence $j_!j^*L \to L$ is nonzero. Since $j_!j^*L \in j_!(\mathcal{A}_\lambda) \subset \mathcal{A}_{\leq\lambda}$ and $\mathcal{A}_{\leq\lambda}$ is a Serre subcategory, we obtain $L \in \mathcal{A}_{\leq\lambda}$ and $L \notin \mathcal{A}_{<\lambda}$.

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Assume that $L \in \mathcal{A}_{\leq\mu} \backslash \mathcal{A}_{<\mu}$ for some $\mu \neq \lambda$. If $\lambda \leq \mu$, then $L \in \mathcal{A}_{\leq\lambda} \subset \mathcal{A}_{<\mu}$, a contradiction. This implies that λ is maximal in $\Omega = \Lambda_{\leq\lambda} \cup \Lambda_{\leq\mu}$. Then $j^*\colon \mathcal{A}_\Omega \to \mathcal{A}_\lambda$ satisfies $j^*(L) = 0$ since $\mathcal{A}_{\leq\mu} \subset \mathcal{A}_{\Omega\backslash\lambda}$. Therefore $j^*_\lambda L = 0$ and $L \in \mathcal{A}_{<\lambda}$, a contradiction. ☐

Let $\mathrm{Irr}(\mathcal{A})$ denote the set of isomorphism classes of simple objects in $\mathcal{A}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}(\mathcal{A}) & \xrightarrow{\quad\phi\quad} & \bigsqcup_{\lambda\in\Lambda} \mathrm{Irr}(\mathcal{A}_\lambda) \\ & {\scriptstyle\rho}\searrow \quad \swarrow & \\ & \Lambda & \end{array} \tag{4.2}$$

where, for $L \in \mathrm{Irr}(\mathcal{A})$ contained in $\mathcal{A}_{\leq\lambda} \backslash \mathcal{A}_{<\lambda}$, we define $\rho(L) = \lambda$ and $\phi(L) = j^*_\lambda(L) \in \mathrm{Irr}(\mathcal{A}_\lambda)$. It follows from Lemma 4.6 that the map ϕ is injective and that it is bijective if $\mathcal{A}$ is of finite length. Let Θ be a set parameterizing isomorphism classes of simple objects in $\mathcal{A}$ (so that we have a bijection $L\colon \Theta \to \mathrm{Irr}(\mathcal{A})$). For $\theta \in \Theta$, we denote $\lambda = \rho L(\theta)$ by $\rho(\theta)$ and we denote $\phi L(\theta) \in \mathcal{A}_\lambda$ by $L_\lambda(\theta)$. We define the *proper standard object* (cf. [27, 30, 39])

$$\overline{\Delta}(\theta) = j_! L_\lambda(\theta) \in \mathcal{A}_{\leq\lambda}.$$

Assuming that $L_\lambda(\theta) \in \mathcal{A}_\lambda$ has a projective cover $P_\lambda(\theta) \in \mathcal{A}_\lambda$, we define the *standard object*

$$\Delta(\theta) = j_! P_\lambda(\theta) \in \mathcal{A}_{\leq\lambda}.$$

We say that a left stratified category is *standardly stratified* if the projective cover $P(\theta)$ of $L(\theta) \in \mathcal{A}$ has a filtration such that the top factor is $\Delta(\theta)$ and all other factors are of the form $\Delta(\theta')$ with $\rho(\theta') > \rho(\theta)$. Note that under certain mild restrictions, every stratified category is standardly stratified [39, Section 2.4], but this will be not our case.

4.3. **The algebra A_λ.**

Lemma 4.11. *There is a uniquely defined surjective algebra homomorphism*

$$\phi\colon U(\bar{\mathfrak{h}})^{\mathrm{op}} \longrightarrow \mathrm{End}_{\bar{\mathfrak{g}}}(W(\lambda))$$

such that $\phi_a(v_\lambda) = av_\lambda$ for $a \in U(\bar{\mathfrak{h}})$.

Proof. We have $\mathrm{End}_{\bar{\mathfrak{g}}}(W(\lambda)) \simeq W(\lambda)_\lambda$ by Lemma 3.5. For every $a \in U(\bar{\mathfrak{h}})$ we obtain $\phi_a \in \mathrm{End}_{\bar{\mathfrak{g}}}(W(\lambda))$ specified by $\phi_a(v_\lambda) = av_\lambda \in W(\lambda)_\lambda$. For $a, b \in U(\bar{\mathfrak{h}})$, we have

$$\phi_a\phi_b(v_\lambda) = \phi_a(bv_\lambda) = bav_\lambda = \phi_{ba}(v_\lambda),$$

hence $\phi\colon U(\bar{\mathfrak{h}})^{\mathrm{op}} \to \mathrm{End}_{\bar{\mathfrak{g}}}(W(\lambda))$ is an algebra homomorphism. The composition

$$U(\bar{\mathfrak{h}}) \xrightarrow{\phi} \mathrm{End}_{\bar{\mathfrak{g}}}(W(\lambda)) \xrightarrow{\sim} W(\lambda)_\lambda$$

is given by $a \mapsto av_\lambda$ which is a surjective map. Therefore ϕ is also surjective. ☐

We consider the ideal $\mathsf{J}_\lambda = \mathrm{Ker}\,\phi = \mathrm{Ann}_{U(\bar{\mathfrak{h}})}(v_\lambda)$ and define the algebra

$$\mathsf{A}_\lambda \coloneqq U(\bar{\mathfrak{h}})/\mathsf{J}_\lambda \simeq \mathrm{End}_{\bar{\mathfrak{g}}}(W(\lambda))^{\mathrm{op}}.$$

We see that the Weyl module $W(\lambda)$ is a $U(\bar{\mathfrak{g}})$-A_λ-bimodule (with $(uv_\lambda)a = \phi_a(uv_\lambda) = uav_\lambda$ for $u \in U(\bar{\mathfrak{g}})$ and $a \in \mathsf{A}_\lambda$) and each weight space $W(\lambda)_\mu$ is an A_λ-bimodule. Moreover, we have an isomorphism of A_λ-bimodules

$$\mathsf{A}_\lambda \xrightarrow{\sim} W(\lambda)_\lambda, \qquad a \longmapsto av_\lambda.$$

Let $\operatorname{Mod} \mathsf{A}_\lambda$ denote the category of left modules over A_λ. By analogy with [6, Section 3.4], we define the *Weyl functor*

$$\mathbf{W}_\lambda \colon \operatorname{Mod} \mathsf{A}_\lambda \longrightarrow \mathcal{I}_{\leq\lambda}(\bar{\mathfrak{g}}), \qquad M \longmapsto W(\lambda) \otimes_{\mathsf{A}_\lambda} M.$$

The following result generalizes [6, Section 3].

Lemma 4.12. *Let $\Omega \subset P_+$ be a finite lower subset and $\lambda \in \max \Omega$. Then the functor*

$$\mathbf{W}_\lambda \colon \operatorname{Mod} \mathsf{A}_\lambda \longrightarrow \mathcal{I}_{\Omega}(\bar{\mathfrak{g}}), \qquad M \longmapsto W(\lambda) \otimes_{\mathsf{A}_\lambda} M,$$

is left adjoint to the functor

$$\mathbf{R}_\lambda \colon \mathcal{I}_{\Omega}(\bar{\mathfrak{g}}) \longrightarrow \operatorname{Mod} \mathsf{A}_\lambda, \qquad V \longmapsto V_\lambda \simeq \operatorname{Hom}_{\bar{\mathfrak{g}}}(W(\lambda), V).$$

The adjunction morphism $\mathrm{id} \to \mathbf{R}_\lambda \mathbf{W}_\lambda$ *is an isomorphism, meaning that* $\mathbf{W}_\lambda$ *is fully faithful.*

Proof. Recall that $W(\lambda) = \mathbf{B}_{\leq\lambda} L(\lambda)$ by Proposition 3.4. For all $V \in \mathcal{I}_\Omega(\bar{\mathfrak{g}})$, we have (cf. Lemma 3.5)

$$\operatorname{Hom}_{\bar{\mathfrak{g}}}(W(\lambda), V) = \operatorname{Hom}_{\bar{\mathfrak{g}}}(\mathbf{B}_{\leq\lambda} L(\lambda), V) \simeq \operatorname{Hom}_{\mathfrak{g}}(L(\lambda), V) \simeq V_\lambda.$$

Therefore $\mathbf{R}_\lambda(V) = V_\lambda \simeq \operatorname{Hom}_{\bar{\mathfrak{g}}}(W(\lambda), V)$ is a left module over $\mathsf{A}_\lambda \simeq \operatorname{End}_{\bar{\mathfrak{g}}}(W(\lambda))^{\mathrm{op}}$. Explicitly, for $a \in \mathsf{A}_\lambda$ and $f \in \operatorname{Hom}_{\bar{\mathfrak{g}}}(W(\lambda), V)$ with $v = f(v_\lambda)$, we have $(af)(v_\lambda) = (f\phi_a)(v_\lambda) = f(av_\lambda) = av$. The claim on the adjunction follows from

$$\operatorname{Hom}_{\bar{\mathfrak{g}}}(\mathbf{W}_\lambda(M), V) \simeq \operatorname{Hom}_{\mathsf{A}_\lambda}(M, \operatorname{Hom}_{\bar{\mathfrak{g}}}(W(\lambda), V)) \simeq \operatorname{Hom}_{\mathsf{A}_\lambda}(M, \mathbf{R}_\lambda(V)).$$

We have $\mathbf{R}_\lambda \mathbf{W}_\lambda(M) = (W(\lambda) \otimes_{\mathsf{A}_\lambda} M)_\lambda = W(\lambda)_\lambda \otimes_{\mathsf{A}_\lambda} M \simeq M$ since $W(\lambda)_\lambda \simeq \mathsf{A}_\lambda$. □

Using the experimental data presented in Section 3.2, one can obtain some information about the algebras A_λ in the case $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2)$. In particular, it is easy to see that, contrary to the case of affine algebras [10], the global Weyl module $W(\lambda)$ is generally not a free A_λ-module.

Example 4.13. Let us consider the global Weyl module $W(4)$ for the algebra $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2)$. As indicated in Section 3.2, this is a finite-dimensional vector space of dimension 31. Since the multiplicity of weight 4 in this module is equal to 2, the algebra A_4 is two-dimensional, and the A_4-module $W(4)$ is not free.

Theorem 4.14. *The category $\mathcal{A} = \mathcal{I}_{\mathsf{b}}(\bar{\mathfrak{g}})$ is left stratified by the poset $\Lambda = P_+$ with the Serre subcategories $\mathcal{A}_\Omega = \mathcal{I}_\Omega(\bar{\mathfrak{g}})$ for finite lower subsets $\Omega \subset \Lambda$, the strata categories $\mathcal{A}_\lambda = \operatorname{Mod} \mathsf{A}_\lambda$ for $\lambda \in \Lambda$, and the left recollement for $\lambda \in \max \Omega$ (where i_* is the embedding)*

$$\mathcal{A}_{\Omega\setminus\lambda} \xrightarrow{i_*} \mathcal{A}_\Omega \xrightarrow{j^* = \mathbf{R}_\lambda} \mathcal{A}_\lambda = \operatorname{Mod} \mathsf{A}_\lambda$$

Proof. We have seen that $\mathbf{R}_\lambda$ has the left adjoint $j_! = \mathbf{W}_\lambda$ which is fully faithful. The functor i_* has the left adjoint $i^* = \tau^{\bar{\mathfrak{g}}}_{\Omega\setminus\lambda}$ (3.2). The category $\mathcal{A}_{\Omega\setminus\lambda}$ can be identified with the subcategory of objects $V \in \mathcal{A}_\Omega$ such that $\mathbf{R}_\lambda(V) = V_\lambda = 0$. □

4.4. **Graded case.** Let $\bar{\mathfrak{g}} = \bigoplus_{i\geq 0} \bar{\mathfrak{g}}_{(i)}$ be a graded Lie algebra with the semisimple degree zero component $\mathfrak{g} = \bar{\mathfrak{g}}_{(0)}$. We say that a $\bar{\mathfrak{g}}$-module V is $\mathbb{Z}$-graded if it is equipped with a grading $V = \bigoplus_{i\in\mathbb{Z}} V_{(i)}$ compatible with the grading of $\bar{\mathfrak{g}}$, meaning that $\bar{\mathfrak{g}}_{(i)} V_{(j)} \subset V_{(i+j)}$ for all i, j. We define the shifted module $V[n]$ with $V[n]_{(i)} = V_{(i-n)}$. Let $\mathcal{I}^{\mathbb{Z}}(\bar{\mathfrak{g}})$ denote the category of $\mathbb{Z}$-graded $\bar{\mathfrak{g}}$-modules that are $\mathfrak{g}$-integrable (with morphisms of degree zero) and let $\mathcal{A} = \mathcal{I}^{\mathbb{Z}}_{\mathsf{b}}(\bar{\mathfrak{g}}) \subset \mathcal{I}^{\mathbb{Z}}(\bar{\mathfrak{g}})$ be the subcategory of modules that are bounded as $\mathfrak{g}$-modules. For a finite (lower) subset $\Omega \subset P_+$, let $\mathcal{A}_\Omega = \mathcal{I}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}}) \subset \mathcal{A}$ consist of $V \in \mathcal{I}^{\mathbb{Z}}_{\mathsf{b}}(\bar{\mathfrak{g}})$ such that $V = \tau^{\bar{\mathfrak{g}}}_\Omega(V)$.

The simple objects of $\mathcal{I}^{\mathbb{Z}}(\bar{\mathfrak{g}})$ are objects $L(\lambda, n) = L(\lambda)[n]$ indexed by pairs $(\lambda, n) \in \Theta = P_+ \times \mathbb{Z}$; such an object is simply $L(\lambda)$ placed in degree n, on which $\bar{\mathfrak{g}}$ acts via the pull-back along the projection $\pi\colon \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_{(0)}$. The functor W_Ω defined in (3.3) can be lifted to $W_\Omega\colon \mathcal{I}^{\mathbb{Z}}(\mathfrak{g}) \to \mathcal{I}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}})$. In particular, the global Weyl module $W(\lambda) = \mathbf{B}_{\leq\lambda}(L(\lambda)) \in \mathcal{I}^{\mathbb{Z}}_{\leq\lambda}(\bar{\mathfrak{g}})$ is a graded $\bar{\mathfrak{g}}$-module (with the degree zero component equal to $L(\lambda)$). We define $W(\lambda, n) = W(\lambda)[n]$ for $n \in \mathbb{Z}$.

Example 4.15. At this point, we may already see that the category $\mathcal{I}^{\mathbb{Z}}_{\mathsf{b}}(\bar{\mathfrak{g}})$ is not generally a standardly stratified category. If that were the case, it would imply that projective modules $P(\lambda)$ have a filtration by standard modules, which are exactly the global Weyl modules. Let $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2)$. We have $P(\lambda) \simeq \operatorname{Ind}^{\bar{\mathfrak{g}}}_{\mathfrak{sl}_2} L(\lambda)$, and using this formula, we can immediately compute the first few graded components of $P(0)$:

$$P(0) = \begin{matrix} L(0) \\ L(3) \\ L(6) \oplus L(4) \oplus L(2) \\ \cdots \end{matrix}$$

If the module $P(0)$ had a filtration by standard modules, the above decomposition would have a copy of $W(3)$ with degrees shifted by one, and examining the decomposition of $W(3)$ above, we see that this would imply presence of $L(0)$ in the degree 2 component of $P(0)$. The resulting contradiction shows that the category $\mathcal{I}^{\mathbb{Z}}_{\mathsf{b}}(L_0(\mathsf{H}_2))$ is not standardly stratified.

The algebra $\mathsf{A}_\lambda \simeq \underline{\operatorname{End}}_{\bar{\mathfrak{g}}}(W(\lambda))^{\mathrm{op}}$ is $\mathbb{Z}$-graded (here we consider endomorphisms of all degrees). Its degree zero component is $U(\mathfrak{h})/(h - \lambda(h) : h \in \mathfrak{h}) \simeq \mathbb{C}$. Let $\mathcal{A}_\lambda = \operatorname{Mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)$ be the category of $\mathbb{Z}$-graded left A_λ-modules. Its simple objects are of the form $\mathbb{C}_\lambda[n]$, for $n \in \mathbb{Z}$, where $\mathbb{C}_\lambda$ is the module corresponding to the projection $\mathsf{A}_\lambda \to (\mathsf{A}_\lambda)_0 = \mathbb{C}$. Similarly to Theorem 4.14 we obtain the following result.

Theorem 4.16. *The category $\mathcal{A} = \mathcal{I}^{\mathbb{Z}}_{\mathsf{b}}(\bar{\mathfrak{g}})$ is left stratified by the poset $\Lambda = P_+$ with the Serre subcategories $\mathcal{A}_\Omega = \mathcal{I}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}})$ for finite lower subsets $\Omega \subset \Lambda$, the strata categories $\mathcal{A}_\lambda = \operatorname{Mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)$ for $\lambda \in \Lambda$, and the left recollement for $\lambda \in \max\Omega$ (where i_* is the embedding)*

$$\mathcal{A}_{\Omega\setminus\lambda} \xrightarrow{i_*} \mathcal{A}_\Omega \xrightarrow{j^* = \mathbf{R}_\lambda} \mathcal{A}_\lambda = \operatorname{Mod}^{\mathbb{Z}}(\mathsf{A}_\lambda).$$

The functor $\mathbf{R}_\lambda\colon \mathcal{A}_{\leq\lambda} \to \mathcal{A}_\lambda$ induces a 1-1 correspondence between isomorphism classes of simple objects $L(\lambda, n) \in \mathcal{A}_{\leq\lambda} \setminus \mathcal{A}_{<\lambda}$ and isomorphism classes of simple objects $\mathbb{C}_\lambda[n] \in \mathcal{A}_\lambda$.

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We obtain a commutative diagram (cf. (4.2))

$$\begin{array}{ccc} \Theta = \Lambda \times \mathbb{Z} & \xrightarrow{\ \phi\ } & \bigsqcup_{\lambda\in\Lambda} \mathrm{Irr}(\mathcal{A}_\lambda) \\ & {\scriptstyle\rho}\searrow \quad \swarrow & \\ & \Lambda & \end{array} \tag{4.3}$$

where ϕ is the bijection that maps $(\lambda, n) \in \Theta$ to the irreducible module $\mathbf{R}_\lambda L(\lambda, n) = \mathbb{C}_\lambda[n] \in \mathcal{A}_\lambda$ and $\rho(\lambda, n) = \lambda$. The projective cover of $\mathbb{C}_\lambda[n]$ is $\mathsf{A}_\lambda[n]$, hence the corresponding standard object is

$$\Delta(\lambda, n) = \mathbf{W}_\lambda(\mathsf{A}_\lambda[n]) = W(\lambda)[n], \tag{4.4}$$

the shifted global Weyl module. The corresponding proper standard object is

$$\overline{\Delta}(\lambda, n) = \mathbf{W}_\lambda(\mathbb{C}_\lambda[n]) = \mathbf{W}_\lambda(\mathbb{C}_\lambda)[n]. \tag{4.5}$$

Later we will identify these objects with local Weyl modules.

4.5. **Full stratification.** Let $\bar{\mathfrak{g}}$ be a graded Lie algebra as before, with finite-dimensional graded components. Let $\mathrm{mod}^{\mathbb{Z}}(\bar{\mathfrak{g}})$ be the category of $\mathbb{Z}$-graded $\bar{\mathfrak{g}}$-modules that are finite-dimensional at every degree and let $\mathcal{A} = \mathrm{mod}^{\mathbb{Z}}_{\mathfrak{b}}(\bar{\mathfrak{g}}) = \mathrm{mod}^{\mathbb{Z}}(\bar{\mathfrak{g}}) \cap \mathcal{I}^{\mathbb{Z}}_{\mathfrak{b}}(\bar{\mathfrak{g}})$ and $\mathcal{A}_\Omega = \mathrm{mod}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}}) = \mathrm{mod}^{\mathbb{Z}}(\bar{\mathfrak{g}}) \cap \mathcal{I}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}})$ for finite lower subsets $\Omega \subset \Lambda = P_+$. Similarly, let $\mathcal{A}_\lambda = \mathrm{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)$ be the category of $\mathbb{Z}$-graded A_λ-modules that are finite-dimensional at every degree. We have $W(\lambda) \in \mathrm{mod}^{\mathbb{Z}}_{\le\lambda}(\bar{\mathfrak{g}})$ and the induced functor $\mathbf{W}_\lambda \colon \mathrm{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda) \to \mathrm{mod}^{\mathbb{Z}}_{\le\lambda}(\bar{\mathfrak{g}})$ which is left adjoint to the functor $\mathbf{R}_\lambda \colon \mathrm{mod}^{\mathbb{Z}}_{\le\lambda}(\bar{\mathfrak{g}}) \to \mathrm{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)$. We would like to construct the right adjoint of the functor $\mathbf{R}_\lambda$ which will be a part of the (full) stratification structure on the category $\mathcal{A} = \mathrm{mod}^{\mathbb{Z}}_{\mathfrak{b}}(\bar{\mathfrak{g}})$. Specifically, we shall now show that this can be done if there exists a (degree-preserving) Lie algebra automorphism $\sigma \colon \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ such that $\sigma|_{\mathfrak{h}} = -\mathrm{id}$. The restriction of such automorphism σ to $\mathfrak{g} = \bar{\mathfrak{g}}_{(0)}$ sends every root space $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$, so, up to a choice of a Chevalley basis, it is given by the Chevalley involution of $\mathfrak{g}$, so that $\sigma(e_i) = -f_i$, $\sigma(f_i) = -e_i$ and $\sigma(h_i) = -h_i$.

Example 4.17. As in Example 3.1, let $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2)$ be the Lie algebra of Hamiltonian vector fields of degree ≥ 0. We consider the automorphism of $\bar{\mathfrak{g}}$ given by $\sigma(f\partial_x + g\partial_y) = f(y, -x)\partial_y - g(y, -x)\partial_x$ (corresponding to the symplectomorphism of $\mathbb{C}^2$ given by $(x, y) \mapsto (y, -x)$). Recall that $\mathfrak{sl}_2 \simeq \bar{\mathfrak{g}}_{(0)}$ with $e = x\partial_y$, $f = y\partial_x$, $h = x\partial_x - y\partial_y$. Note that though the restriction to $\mathfrak{sl}_2$ is the Chevalley involution, on the whole Lie algebra $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2)$ we only have $\sigma^4 = \mathrm{id}$.

In what follows, for any algebra homomorphism $\phi \colon A \to B$, we denote the induced functor $\mathrm{Mod}\, B \to \mathrm{Mod}\, A$ again by ϕ. Consider the duality functor

$$D \colon \mathrm{mod}^{\mathbb{Z}}(\bar{\mathfrak{g}})^{\mathrm{op}} \longrightarrow \mathrm{mod}^{\mathbb{Z}}(\bar{\mathfrak{g}}), \qquad (DV)_{(i)} = V^*_{(-i)},$$

and similarly $D \colon \mathrm{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)^{\mathrm{op}} \to \mathrm{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda^{\mathrm{op}})$. Note that $(DV)_\lambda = V^*_{-\lambda}$ for $V \in \mathrm{mod}^{\mathbb{Z}}(\bar{\mathfrak{g}})$ and $\lambda \in P$. Taking the pullback along σ, we obtain the module $\overline{D}V = D\sigma V$ such that $(\overline{D}V)_\lambda = V^*_\lambda$. This implies that $\overline{D}$ restricts to an equivalence

$$\overline{D} \colon \mathrm{mod}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}})^{\mathrm{op}} \longrightarrow \mathrm{mod}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}})$$

Lemma 4.18. *For a finite lower subset $\Omega \subset \Lambda$, $\lambda \in \max\Omega$ and $V \in \mathrm{mod}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}})$, we have an isomorphism of $U(\mathfrak{h})$-modules*

$$D\mathbf{R}_\lambda \overline{D}(V) \simeq \sigma S \mathbf{R}_\lambda(V)$$

where $S\colon U(\bar{\mathfrak{h}})^{\mathrm{op}} \to U(\bar{\mathfrak{h}})$ *is the antipode (defined by* $S(a) = -a$ *for* $a \in \bar{\mathfrak{h}}$*). In particular, the ideal* $\mathsf{J}_\lambda = \mathrm{Ker}(U(\bar{\mathfrak{h}}) \to \mathsf{A}_\lambda)$ *is stable under the action of* σS*.*

Proof. The left action of $U(\bar{\mathfrak{h}})$ on $\mathbf{R}_\lambda \overline{D} V = V_\lambda^*$ is given by $(af)(v) = -f(\sigma(a)v)$ for $a \in \bar{\mathfrak{h}}$, $f \in V_\lambda^*$ and $v \in V_\lambda$. Therefore the right action of $U(\bar{\mathfrak{h}})$ on $D\mathbf{R}_\lambda \overline{D} V = V_\lambda$ is given by $va = -\sigma(a)v$ for $a \in \bar{\mathfrak{h}}$. This coincides with the right action of $U(\bar{\mathfrak{h}})$ on $\sigma S \mathbf{R}_\lambda(V)$. The last assertion follows from the fact that $D\mathbf{R}_\lambda \overline{D}(V)$ is a module over A_λ. □

The above result implies that we have an equivalence

$$\widehat{D} = D\sigma S\colon \operatorname{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)^{\mathrm{op}} \longrightarrow \operatorname{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)$$

such that $\mathbf{R}_\lambda \overline{D} \simeq \widehat{D} \mathbf{R}_\lambda$.

Lemma 4.19. *For a finite lower subset* $\Omega \subset \Lambda$ *and* $\lambda \in \max \Omega$*, the functor*

$$\mathbf{R}_\lambda\colon \operatorname{mod}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}}) \longrightarrow \operatorname{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)$$

has the fully faithful right adjoint $\overline{D}^{-1} \mathbf{W}_\lambda \widehat{D}$*.*

Proof. For $M \in \operatorname{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)$, $V \in \operatorname{mod}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}})$ we have

$$\begin{aligned}
\mathrm{Hom}(\mathbf{R}_\lambda V, M) &\simeq \mathrm{Hom}(\widehat{D}M, \widehat{D}\mathbf{R}_\lambda V) \\
&\simeq \mathrm{Hom}(\widehat{D}M, \mathbf{R}_\lambda \overline{D} V) \\
&\simeq \mathrm{Hom}\big(\mathbf{W}_\lambda \widehat{D} M, \overline{D} V\big) \\
&\simeq \mathrm{Hom}(V, \overline{D}^{-1} \mathbf{W}_\lambda \widehat{D} M).
\end{aligned}$$

□

This result together with Theorem 4.16 implies:

Theorem 4.20. *For a Lie algebra* $\bar{\mathfrak{g}}$ *that has a degree-preserving automorphism* σ *such that* $\sigma|_{\mathfrak{h}} = -\mathrm{id}$*, the category* $\mathcal{A} = \operatorname{mod}^{\mathbb{Z}}_{\mathfrak{b}}(\bar{\mathfrak{g}})$ *is stratified by the poset* $\Lambda = P_+$ *with the Serre subcategories* $\mathcal{A}_\Omega = \operatorname{mod}^{\mathbb{Z}}_\Omega(\bar{\mathfrak{g}})$ *for finite lower subsets* $\Omega \subset \Lambda$*, the strata categories* $\mathcal{A}_\lambda = \operatorname{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda)$ *for* $\lambda \in \Lambda$*, and the recollement for* $\lambda \in \max \Omega$ *(where* i_* *is the embedding)*

$$\mathcal{A}_{\Omega \setminus \lambda} \xrightarrow{i_*} \mathcal{A}_\Omega \xrightarrow{j^* = \mathbf{R}_\lambda} \mathcal{A}_\lambda = \operatorname{mod}^{\mathbb{Z}}(\mathsf{A}_\lambda).$$

5. Local Weyl modules

In the case of affine algebras, map algebras, and other root-graded algebras, there is an important notion of a local Weyl module; for instance, for algebras $\bar{\mathfrak{g}}$ of the form $\mathfrak{g} \otimes A$ with A commutative, local Weyl modules help to capture aspects of representation theory of $\bar{\mathfrak{g}}$ at different points of $\mathrm{Spec}(A)$. In this section, we discuss a general version of this definition and relate local Weyl modules to proper standard modules introduced earlier.

5.1. Definition and basic properties.

Definition 5.1. For $z \in (\bar{\mathfrak{h}})^*$, the *local Weyl module* $W(z)$ is the maximal $\mathfrak{g}$-integrable $\bar{\mathfrak{g}}$-module generated by the vector v_z such that

$$\bar{\mathfrak{n}}_+ v_z = 0, \qquad h v_z = z(h) v_z \quad \forall\, h \in \bar{\mathfrak{h}}.$$

The standard example of a local Weyl module is as follows (see e.g. [16, Definition 4]).

Example 5.2. Let $X = \operatorname{Spec} A$ be an affine scheme over $\mathbb{C}$ and $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i$ be a dominant weight. Let $\bar{\mathfrak{g}} = \mathfrak{g} \otimes A$ so that $\bar{\mathfrak{h}} = \mathfrak{h} \otimes A$. Let $(z_{ij})_{1 \le j \le \lambda_i}$ be a collection of points in X. We can interpret them as algebra homomorphisms $z_{ij} \colon A \to \mathbb{C}$. Consider the linear maps $z_i = \sum_{j=1}^{\lambda_i} z_{ij} \colon A \to \mathbb{C}$ and the linear map

$$z \colon \bar{\mathfrak{h}} = \mathfrak{h} \otimes A \longrightarrow \mathbb{C}, \qquad h_i \otimes a \longmapsto z_i(a).$$

We have $z(h_i) = z(h_i \otimes 1) = \sum_{j=1}^{\lambda_i} z_{ij}(1) = \lambda_i$, hence $z|_{\mathfrak{h}} = \lambda$. The local Weyl module $W(z)$ is the maximal $\mathfrak{g}$-integrable $\bar{\mathfrak{g}}$-module generated by the vector v_z such that

$$(\mathfrak{n}_+ \otimes A)v_z = 0, \qquad (h_i \otimes a)v_z = \sum_{j=1}^{\lambda_i} z_{ij}(a) \cdot v_z \quad \forall\, 1 \le i \le r,\, a \in A.$$

The local Weyl modules can be computed using the functor $\mathbf{W}_\lambda$ of Section 4.3.

Lemma 5.3. *Let $z \in (\bar{\mathfrak{h}})^*$ and $\lambda = z|_{\mathfrak{h}} \in \mathfrak{h}^*$. If $W(z) \neq 0$, then λ is a dominant weight. Moreover, we have*

$$W(z) \simeq \mathbf{W}_\lambda(\mathbb{C}_z),$$

where $\mathbb{C}_z$ is the one-dimensional $\bar{\mathfrak{h}}$-module corresponding to z.

Proof. Note that $hv_z = \lambda(h)v_z$ for $h \in \mathfrak{h}$, so there is a surjective $\bar{\mathfrak{g}}$-module morphism $W(\lambda) \to W(z)$ sending v_λ to v_z. Since $W(\lambda)$ is non-zero only for dominant λ, the first assertion follows. We have $W(z)_\lambda \simeq \mathbb{C}_z$ as a module over $U(\bar{\mathfrak{h}})$ (and A_λ). This isomorphism induces a surjective map $\mathbf{W}_\lambda(\mathbb{C}_z) \to W(z)$. The generator of $\mathbf{W}_\lambda(\mathbb{C}_z)$ satisfies the conditions of a local Weyl module, hence $\mathbf{W}_\lambda(\mathbb{C}_z) \to W(z)$ is an isomorphism by the maximality of $W(z)$. ☐

The main class of examples of local Weyl modules we are interested in is as follows.

Definition 5.4. Let λ be a dominant weight and $\pi \colon \bar{\mathfrak{g}} \to \mathfrak{g}$ be a left inverse of the embedding of Lie algebras $\mathfrak{g} \hookrightarrow \bar{\mathfrak{g}}$ such that $\pi(\bar{\mathfrak{h}}) = \mathfrak{h}$. The *local augmentation Weyl module* $W_\pi(\lambda)$ is the local Weyl module corresponding to the linear function $\lambda\pi \colon \bar{\mathfrak{h}} \to \mathbb{C}$.

Example 5.5. For a point $p \in X = \operatorname{Spec} A$ (identified with the morphism $p \colon A \to \mathbb{C}$), we may consider $\pi = \mathrm{id} \otimes p \colon \bar{\mathfrak{g}} = \mathfrak{g} \otimes A \to \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}$. The corresponding local augmentation Weyl module for $\lambda = \sum_i \lambda_i \omega_i$ is precisely the module from Example 5.2 with $z_{ij} = p$ for all $1 \le j \le \lambda_i$. Indeed, we have

$$z(h_i \otimes a) = z_i(a) = \lambda_i p(a) = (\lambda \otimes p)(h_i \otimes a) = \lambda\pi(h_i \otimes a)$$

for $a \in A$.

For a dominant weight λ, the irreducible $\mathfrak{g}$-module $L(\lambda)$ can be made into a $\bar{\mathfrak{g}}$-module using the pullback along $\pi \colon \bar{\mathfrak{g}} \to \mathfrak{g}$. This module satisfies all the conditions of $W_\pi(\lambda)$ except maximality. Therefore, local augmentation Weyl modules are always nonzero.

Let us consider now the situation of Section 4.4, where we have extra grading. Using the projection $\pi \colon \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_{(0)} = \mathfrak{g}$, we may define the corresponding local augmentation Weyl modules $W_\pi(\lambda)$. By Definition 5.1 this is the maximal $\mathfrak{g}$-integrable $\bar{\mathfrak{g}}$-module generated by a vector v_z such that $\bar{\mathfrak{n}}_+ v_z = 0$, $hv_z = \lambda(h)v_z$ for $h \in \mathfrak{h}$ and $\bar{\mathfrak{h}}_{(i)} v_z = 0$ for $i > 0$, where we consider the decomposition $\bar{\mathfrak{h}} = \bigoplus_{i \ge 0} \bar{\mathfrak{h}}_i$ with $\bar{\mathfrak{h}}_{(0)} = \mathfrak{h}$. By Lemma 5.3, we have $W_\pi(\lambda) = W(\lambda\pi) = \mathbf{W}_\lambda(\mathbb{C}_{\lambda\pi})$ and this is exactly the proper standard object $\overline{\Delta}(\lambda, 0) \in \mathcal{I}_{\mathfrak{b}}^{\mathbb{Z}}(\bar{\mathfrak{g}})$ introduced in (4.5).

5.2. **Local Weyl modules for $L_0(\mathsf{H}_2)$.** The following are $\mathfrak{sl}_2$-decompositions of local augmentation Weyl modules for the Lie algebra $\bar{\mathfrak{g}} = L_0(\mathsf{H}_2)$ for $\lambda = 0, \ldots, 7$, listed degree by degree.

$$W_\pi(0) = L(0), \quad W_\pi(1) = L(1), \qquad W_\pi(2) = \begin{matrix} L(2) \\ L(1) \end{matrix}, \quad W_\pi(3) = \begin{matrix} L(3) \\ L(2) \oplus L(0) \\ L(1) \end{matrix},$$

$$W_\pi(4) = \begin{matrix} L(4) \\ L(3) \oplus L(1) \\ 2L(2) \oplus L(0) \\ L(1) \end{matrix}, \qquad W_\pi(5) = \begin{matrix} L(5) \\ L(4) \oplus L(2) \\ 2L(3) \oplus 2L(1) \\ L(4) \oplus 2L(2) \oplus 2L(0) \\ L(1) \end{matrix},$$

$$W_\pi(6) = \begin{matrix} L(6) \\ L(5) \oplus L(3) \\ 2L(4) \oplus 2L(2) \oplus L(0) \\ L(5) \oplus 3L(3) \oplus 3L(1) \\ L(4) \oplus 4L(2) \oplus 2L(0) \\ L(1) \end{matrix}, \qquad W_\pi(7) = \begin{matrix} L(7) \\ L(6) \oplus L(4) \\ 2L(5) \oplus 2L(3) \oplus L(1) \\ L(6) \oplus 3L(4) \oplus 4L(2) \oplus L(0) \\ 2L(5) \oplus 5L(3) \oplus 5L(1) \\ 2L(4) \oplus 5L(2) \oplus 4L(0) \\ L(1) \end{matrix}.$$

Examining this data and computing basis elements in these modules, we arrived at the following conjecture.

Conjecture 5.6. For all $n \geq 1$,

(1) the module $W_\pi(n)$ is concentrated precisely in degrees $0, 1, \ldots, n-1$,
(2) the socle of the module $W_\pi(n)$ is isomorphic to $L(1)$,
(3) for the Hamiltonian vector field $u = 2xy\partial_x - y^2\partial_y$ (the basis element of the weight -1 subspace in $\bar{\mathfrak{g}}_1$), the submodule $U(\bar{\mathfrak{g}})uv_\pi$ of $W_\pi(n+1)$ is isomorphic to $W_\pi(n)$.

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