Annals of Representation Theory

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Volume 3, issue 1 (2026), p. 27-97
https://doi.org/10.5802/art.34

Communicated by Kevin Coulembier.

Annals of Representation Theory is published by the Norwegian University of Science and Technology and is a member of the Centre Mersenne for Open Scientific Publishing

e-ISSN: 2704-2081

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Representation type of higher level cyclotomic quiver Hecke algebras in affine type C

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Abstract. We determine the representation type of cyclotomic quiver Hecke algebras of affine type C. In the tame cases, we explicitly describe their basic algebras under the assumption char $\Bbbk \neq 2$, relying on the Morita invariance of cellularity.

Contents

Manuscript received 2024-07-14, revised 2025-11-10 and accepted 2025-12-15.

Keywords. Cyclotomic KLR algebras, cyclotomic quiver Hecke algebras, representation type, Brauer graph algebras, silting theory, derived equivalence.

2020 *Mathematics Subject Classification.* 20C08, 16G60, 17B65, 16G20.

The first author is supported in part by JSPS Kakenhi (Grant No. 21K03163). The third author is supported partially by NSFC (Grant No. 12071346) and Natural Science Foundation of Shanghai (Grant No. 25ZR1401352). The fourth author is supported partially by NSFC (Grant No. 12401048) and Fundamental Research Funds for the Central Universities (Grant No. DUT25RC(3)132).

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1. Introduction

Representation type serves as a fundamental tool in the representation theory of finite-dimensional algebras, especially, over an algebraically closed field $\Bbbk$. Here, we consider the category of finitely generated left modules, so that all modules are assumed to be finite-dimensional. Namely, representation type gives us criteria whether we can study the module category in depth or we must be content with either, study of better behaved subcategories, or, study on the Grothendieck group of the module category, such as character formulas for irreducible modules, etc.

A finite-dimensional $\Bbbk$-algebra A is said to be *representation-finite* if it admits only finitely many indecomposable modules up to isomorphism; otherwise, A is said to be *representation-infinite*. A representation-infinite $\Bbbk$-algebra A is said to be *tame* if all but finitely many d-dimensional indecomposable A-modules can be organized in finitely many one-parameter families, for each dimension d, and it is called *wild* if there is an exact $\Bbbk$-linear functor sending modules over the free associative algebra $\Bbbk\langle x, y\rangle$ to modules over A which preserves indecomposability and respects isomorphism classes. It is known as the famous (Finite-)Tame-Wild Trichotomy([26]) that the representation type of any finite-dimensional algebra over $\Bbbk$ is exactly one of representation-finite, tame[1] and wild.

It is a natural desire to find such criteria for well-known classes of algebras. The class of path algebras is the most famous class of algebras, and Dynkin quivers of finite ADE and affine ADE types appear beautifully in the criteria. Another important class of algebras is the class of group algebras such as those of the symmetric groups.

The modular representation theory of the symmetric group has a long history. Class of algebras which the group algebras of the symmetric group belong started with the class of the group algebras of finite Coxeter groups. Then, the class was expanded to their q-deformation, that is, the class of Iwahori–Hecke algebras, and then to the class of cyclotomic Hecke algebras ([12, 18]) associated with complex reflection groups, in which the algebras associated with complex reflection groups $G(m, 1, n)$, so-called Ariki–Koike algebras, received detailed study (e.g., [20, 25, 30, 40]). Currently, we study algebras in the much wider class of cyclotomic quiver Hecke algebras ([35, 48]), which are associated with Lie theoretic data: the Lie type determined by a symmetrizable (generalized) Cartan matrix A, an element β in the positive cone Q_+ of the root lattice, and a dominant integral weight Λ in the weight lattice. Those data come from categorification theorems which categorify weight spaces $V(\Lambda)_{\Lambda-\beta}$ of the integrable highest weight module $V(\Lambda)$ over the Kac–Moody Lie algebra $\mathfrak{g}(\mathsf{A})$ of the symmetrizable Cartan matrix. In our setting, the module category over the cyclotomic quiver Hecke algebra $R^{\Lambda}(\beta)$ categorifies the weight space. For example, the group algebras of the symmetric group in positive characteristics and Hecke algebras of type A at roots of unity are associated with level one dominant integral weights of type $A_\ell^{(1)}$, and Hecke algebras of type B at roots of unity are associated with level two dominant integral weights of type $A_\ell^{(1)}$. The cyclotomic quiver Hecke algebras are also called cyclotomic Khovanov–Lauda–Rouquier algebras, cyclotomic KLR algebras for short.

Cyclotomic quiver Hecke algebras are graded algebras. In particular, the group algebras of the symmetric group are graded algebras. This finding, due to Brundan and Kleshchev [19], could not be seen by using Coxeter generators: their deep insight led

[1]Following Erdmann [27], our tame representation type, tame for short, excludes representation-finite algebras.

them to the finding of Khovanov–Lauda–Rouquier generators in the group algebras of the symmetric group.

Recently, cyclotomic quiver Hecke algebras of affine type other than $A_\ell^{(1)}$ attracts researchers in this field. For example, Park, Speyer and the first author [14] introduced Specht modules for type $C_\ell^{(1)}$, Evseev and Mathas [29] proved and Mathas and Tubbenhauer [42] reproved that the cyclotomic quiver Hecke algebras of type $C_\ell^{(1)}$ are graded cellular algebras[2]. Some experimental calculations of the decomposition numbers have been carried out by Chung, Mathas and Speyer [23].

In this article, we determine representation type for all cyclotomic quiver Hecke algebras $R^\Lambda(\beta)$ of type $C_\ell^{(1)}$, where $\ell \geq 2$. Since we already know representation type of $R^\Lambda(\beta)$ when Λ is a fundamental weight, we assume that the level k of the dominant integral weight Λ is greater than or equal to 2. We denote the set of weights of $V(\Lambda)$ by $P(\Lambda)$. Recall that $R^\Lambda(\beta)$ and $R^\Lambda(\Lambda - w\Lambda + w\beta)$, for $w \in W$, where W is the (affine) Weyl group, have the same representation type, so that it suffices to consider those $\beta \in Q_+$ such that $\Lambda - \beta$ are dominant integral weights. Furthermore, $\Lambda - \beta$ is not a maximal weight if and only if there exists $w \in W$ such that $w(\Lambda - \beta)$ is dominant but not maximal.

Main Theorem A. *Suppose that the level of Λ is $k \geq 2$ and we write*

$$\Lambda = m_0\Lambda_0 + m_1\Lambda_1 + \cdots + m_\ell\Lambda_\ell,$$

where $m_0, m_1, \ldots, m_\ell \in \mathbb{Z}_{\geq 0}$ and $m_0 + m_1 + \cdots + m_\ell = k$.

- *(1) If $\Lambda - \beta$ is not a maximal weight, then $R^\Lambda(\beta)$ is wild.*
- *(2) Suppose that $\Lambda - \beta$ is a dominant maximal weight in $P(\Lambda)$.*
  - *(a) $R^\Lambda(\beta)$ is of finite representation type if one of the following holds.*
    - *(f1) $\beta = \alpha_a$, for $0 \leq a \leq \ell$, and $m_a \geq 2$.*
    - *(f2) $\beta = \alpha_0 + \alpha_1$, and $m_0 \geq 1$, $m_1 = 0$ or $m_0 = m_1 = 1$.*
    - *(f3) $\beta = \alpha_{\ell-1} + \alpha_\ell$, and $m_{\ell-1} = 0$, $m_\ell \geq 1$ or $m_{\ell-1} = m_\ell = 1$.*
    - *(f4) $\beta = \alpha_a + \cdots + \alpha_b$, for $1 \leq a < b \leq \ell - 1$, and $m_i = \delta_{ai} + \delta_{bi}$, for $a \leq i \leq b$.*
    - *(f5) $\beta = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_a + \alpha_{a+1}$, for $0 \leq a \leq \ell - 2$, and $m_i = \delta_{ai}$, for $0 \leq i \leq a + 1$.*
    - *(f6) $\beta = \alpha_{b-1} + 2\alpha_b + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$, for $2 \leq b \leq \ell$, and $m_i = \delta_{bi}$, for $b - 1 \leq i \leq \ell$.*
  - *(b) $R^\Lambda(\beta)$ is of tame representation type if one of the following holds.*
    - *(t1) $\beta = \alpha_0 + 2\alpha_1$, $m_0 = 0$ and $m_1 = 2$.*
    - *(t2) $\beta = 2\alpha_{\ell-1} + \alpha_\ell$, $m_{\ell-1} = 2$ and $m_\ell = 0$.*
    - *(t3) $\beta = \alpha_0 + \alpha_1$, $m_0 \geq 2$ and $m_1 = 1$.*
    - *(t4) $\beta = \alpha_{\ell-1} + \alpha_\ell$, $m_{\ell-1} = 1$ and $m_\ell \geq 2$.*
    - *(t5) $\beta = \alpha_0 + \cdots + \alpha_a$, for $1 \leq a \leq \ell - 1$, $m_0 \geq 1$ and $m_i = \delta_{ia}$, for $1 \leq i \leq a$, except for the case $a = 1$ and $m_0 = 1$, which is* (f2).
    - *(t6) $\beta = \alpha_a + \cdots + \alpha_\ell$, for $1 \leq a \leq \ell - 1$, $m_\ell \geq 1$ and $m_i = \delta_{ai}$, for $a \leq i \leq \ell - 1$, except for $a = \ell - 1$ and $m_\ell = 1$, which is* (f3).
    - *(t7) $\beta = \alpha_0 + \alpha_1$, $m_0 = 1$ and $m_1 = 2$.*
    - *(t8) $\beta = \alpha_{\ell-1} + \alpha_\ell$, $m_{\ell-1} = 2$ and $m_\ell = 1$.*
    - *(t9) $\beta = \alpha_a + \cdots + \alpha_b$, for $1 \leq a < b \leq \ell - 1$, either $m_a \geq 2$ and $m_i = \delta_{ib}$, for $a < i \leq b$, or $m_b \geq 2$ and $m_i = \delta_{ai}$, for $a \leq i < b$.*

[2]For the recent progress on cyclotomic quiver Hecke algebras of finite type, see [41].

(t10) $\beta = \alpha_0 + \alpha_i$, *for* $2 \le i \le \ell$, $m_0 = m_i = 2$.
(t11) $\beta = \alpha_i + \alpha_\ell$, *for* $0 \le i \le \ell - 2$, $m_i = m_\ell = 2$.
(t12) $\beta = \alpha_0 + \alpha_1 + \alpha_{\ell-1} + \alpha_\ell$ *where* $\ell \ge 4$, $m_0 = m_\ell = 1$ *and* $m_1 = m_{\ell-1} = 0$.
(t13) $\beta = \alpha_0 + \alpha_1 + \alpha_i$, *for* $3 \le i \le \ell$, $m_0 = 1$, $m_1 = 0$ *and* $m_i = 2$.
(t14) $\beta = \alpha_i + \alpha_{\ell-1} + \alpha_\ell$, *for* $0 \le i \le \ell - 3$, $m_i = 2$ *and* $m_{\ell-1} = 0$, $m_\ell = 1$.
(t15) $\beta = \alpha_{a-1} + 2\alpha_a + \alpha_{a+1}$, *for* $2 \le a \le \ell - 2$, $m_a = 2$, $m_{a\pm1} = 0$, *and* $\operatorname{char} \Bbbk \ne 2$.
(t16) $\beta = 2\alpha_a + \alpha_{a+1}$, *for* $1 \le a \le \ell - 2$, $m_a = 3$,$m_{a+1} = 0$ *and* $\operatorname{char} \Bbbk \ne 3$.
(t17) $\beta = \alpha_{a-1} + 2\alpha_a$, *for* $2 \le a \le \ell - 1$, $m_a = 3$, $m_{a-1} = 0$ *and* $\operatorname{char} \Bbbk \ne 3$.
(t18) $\beta = \alpha_a + \alpha_b$, *for* $1 \le a < b \le \ell - 1$ *where* $a \le b - 2$, $m_a = m_b = 2$.
(t19) $\beta = 2\alpha_a$, *for* $1 \le a \le \ell - 1$, $m_a = 4$ *and* $\operatorname{char} \Bbbk \ne 2$.
(t20) $\beta = 2\alpha_0 + 2\alpha_1$, $m_0 = 2$, $m_1 = 0$ *and* $\operatorname{char} \Bbbk \ne 2$.
(t21) $\beta = 2\alpha_{\ell-1} + 2\alpha_\ell$, $m_{\ell-1} = 0$, $m_\ell = 2$ *and* $\operatorname{char} \Bbbk \ne 2$.

(c) $R^\Lambda(\beta)$ *is of wild representation type otherwise.*

The proof of Main Theorem A uses the idea to introduce quiver structure on the set of dominant maximal weights $\max^+(\Lambda)$, which was found and applied to type $A_\ell^{(1)}$ in [15]. However, we choose a different strategy than the [loc. cit.] after introducing the quiver of dominant maximal weights. While we first fixed a certain neighborhood of the weight Λ, which was found by consideration on the coefficients of β, and started with showing that those weights outside the neighborhood give us wild cyclotomic KLR algebras in [15], we start with investigating dominant maximal weights Λ' which can be reached by at most one step, two steps, three steps from Λ one by one first, and determine representation type of the associated cyclotomic KLR algebras $R^\Lambda(\beta_{\Lambda'})$. Then, we reach the conclusion that algebras which cannot be reached by less than or equal to three steps are wild. See Section 4 for the details.

In the course of the proof, we obtain explicit presentations of non-wild algebras, see Sections 6 and 7. In type $A_\ell^{(1)}$, all tame $R^\Lambda(\beta_{\Lambda'})$ associated with dominant maximal weights Λ' are Brauer graph algebras. It implies that all tame cyclotomic KLR algebras of type $A_\ell^{(1)}$ are Brauer graph algebras, and this fact allowed us to determine the Morita equivalence classes[3] of tame cyclotomic KLR algebras of type $A_\ell^{(1)}$. In type $C_\ell^{(1)}$, there are tame cyclotomic KLR algebras $R^\Lambda(\beta)$ which are not Brauer graph algebras. One already appeared in [22, Lemma 3.1] as a level one cyclotomic KLR algebra, which is the algebra (5) in [11, Theorem 1]. The other tame algebras appear as level three cyclotomic KLR algebras in this paper, i.e., (t7) and (t8). For the former case, we need to recall Skowroński's classification of standard domestic symmetric algebras ([51]). However, since $R^\Lambda(\beta)$ is cellular (see [29]), it is natural to assume that $\operatorname{char} \Bbbk \ne 2$ and utilize Morita invariance of the cellularity. Then, the cyclotomic KLR algebras that are derived equivalent to the algebra from [22] must appear in the list [11, Theorem 1], and one can check that other algebras in the list do not appear as cyclotomic KLR algebras of type $C_\ell^{(1)}$ by excluding Brauer graph algebras and those with a different number of simple modules in the list. For the latter case, we may use silting theory to find Morita equivalence classes in the derived equivalence class of the algebra (t7) (or equivalently, (t8)). See Theorem 2.23 for the method, and see Proposition 2.26 for the Morita equivalence classes which are in the derived equivalence class of (t7). Otherwise, tame cyclotomic KLR algebras of type $C_\ell^{(1)}$ are Brauer graph algebras. As was shown in [15], their Brauer graphs are straight lines

[3] Precisely speaking, we need either $\operatorname{char} \Bbbk \ne 2$ or the cyclotomic KLR algebra being a basic algebra.

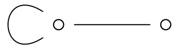
except for one Brauer graph (i.e., the cases (t1) and (t2)), and we may read off the set of multiplicities of vertices. Then, we assign the multiplicities to vertices. In the following, we give Morita equivalence classes of finite and tame algebras $R^\Lambda(\beta)$ in explicit forms[4].

Theorem B (Finite cases). *Let $R^\Lambda(\beta)$ be a cyclotomic KLR algebra of type $C_\ell^{(1)}$ and suppose that $R^\Lambda(\beta)$ is of finite representation type. If* $\operatorname{char}\Bbbk \neq 2$*, then $R^\Lambda(\beta)$ is Morita equivalent to one of the following algebras*[5]*.*

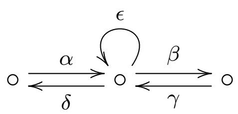
(a) *Symmetric local algebra $\Bbbk[X]/(X^m)$, for $m \geq 2$.*
(b) *Brauer tree algebra whose Brauer tree is a straight line.*

Theorem C (Tame cases). *Let $R^\Lambda(\beta)$ be a cyclotomic KLR algebra of type $C_\ell^{(1)}$ and suppose that $R^\Lambda(\beta)$ is of tame representation type. If* $\operatorname{char}\Bbbk \neq 2$*, then $R^\Lambda(\beta)$ is Morita equivalent to one of the following algebras.*

(a) *Symmetric local algebras (2), (3), (4) in* [15, 8.2].
(b) *Brauer graph algebra whose Brauer graph is a straight line and the multiset of the multiplicities of vertices is $\{1, t, 2t, \dots, 2t\}$, for $t \geq 1$, $\{4, 2, 2\}$ or Brauer graph algebras (5), (7) in* [15, 8.2]*, or the Brauer graph algebra without an exceptional vertex whose Brauer graph is as follows.*



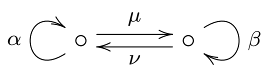
(c) *The algebra $\Bbbk Q/J$, where the quiver Q is*



and the relations given by the admissible ideal J are

$$\alpha\beta = \gamma\delta = 0, \quad \alpha\epsilon = \epsilon\beta = \gamma\epsilon = \epsilon\delta = 0, \quad \delta\alpha = \epsilon^2 = \beta\gamma.$$

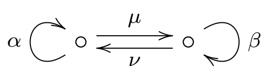
(d) *The algebra $\Bbbk Q/J$, where the quiver Q is*



and the relations given by the admissible ideal J are

$$\alpha^2 = 0, \quad \beta^2 = \nu\mu, \quad \alpha\mu = \mu\beta, \quad \beta\nu = \nu\alpha.$$

(e) *The algebra $\Bbbk Q/J$, where the quiver Q is*



and the relations given by the admissible ideal J are

$$\alpha^2 = \mu\nu, \quad \beta^2 = \nu\mu, \quad \alpha\mu = \mu\beta, \quad \beta\nu = \nu\alpha, \quad \mu\nu\mu = \nu\mu\nu = 0.$$

As we mentioned, in general it is difficult to study the category of all finite-dimensional modules and instead, we try to find nice subcategories. One such example is the representation theory of quantum affine algebras, in which field researchers found good subcategories to study such as the Hernandez–Leclerc categories: these categories have been

[4] We do not know whether all the possible assignment of the given multiset of multiplicities to vertices actually appear.

[5] These algebras already appeared in [15, 8.1].

 https://doi.org/10.5802/art.34

actively studied by cluster algebra techniques in recent years. We claim that the subcategories of modules over tame $R^\Lambda(\beta)$'s are also such nice subcategories, for which we have more chance to tackle difficult problems like finding a dimension formula for irreducible modules or decomposition numbers. Besides, in affine type A they are related to the classical subject of affine Hecke algebras in type A: if we consider the Serre subcategory consisting of modules whose composition factors belong to a given finite set of irreducible modules, then one obtains a filtration of the Serre subcategory over the affine Hecke algebra by the Serre subcategories over cyclotomic Hecke algebras which share the same set of irreducible modules. Then one may use grading and results from [15].

Another fascinating aspect of this paper is that we connect the recently emerging theory of Brauer graph algebras, τ-tilting theory and silting theory with the representation theory of cyclotomic quiver Hecke algebras: in affine type A, all tame blocks are Brauer graph algebras and we applied results by Opper and Zvonareva which they obtained by using a version of Fukaya category, and, as we have explained in the previous page, we utilize τ-tilting theory to build a complete framework (see Theorem 2.23) for finding Morita equivalence classes in the derived equivalence class of a given symmetric algebra. This will benefit not only the study in other types, but also the research of symmetric algebras in general.

Conventions. Set $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$. For $m, m' \in \mathbb{Z}$, we write $m \equiv_2 m'$ if $m - m'$ is even, and $m \not\equiv_2 m'$ otherwise. We use left modules throughout the paper. Hence, the basic algebra of an algebra A is $\mathrm{End}_A(P)^{\mathrm{op}}$, where P is a progenerator which is basic.

2. Preliminaries

We review some background materials which we need in this paper, including the definition of cyclotomic KLR algebras. Additionally, we provide several lemmas in this section for later use.

2.1. **Cartan datum in affine type C.** Set $I = \{0, 1, 2, \dots, \ell\}$ with $\ell \geq 2$. The *affine Cartan matrix* A of type $C_\ell^{(1)}$ is defined by

$$\mathsf{A} = (a_{ij})_{i,j \in I} := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -2 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix},$$

where the rows and the columns are labeled by $0, 1, \dots, \ell$ in this order. If we drop the first row and the first column of A, we obtain the Cartan matrix A' of type C_ℓ; in this case, the simple roots are realized in the lattice $\mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \oplus \dots \oplus \mathbb{Z}\epsilon_\ell$ as

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \dots, \quad \alpha_{\ell-1} = \epsilon_{\ell-1} - \epsilon_\ell, \quad \alpha_\ell = 2\epsilon_\ell,$$

and the root system is given by

$$\{\pm 2\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}.$$

We denote by $\Delta_{\mathrm{fin}}^{\pm}$ the set of positive or negative roots of the finite root system of type C_ℓ. Note that $\Delta_{\mathrm{fin}}^{-} = -\Delta_{\mathrm{fin}}^{+}$. Since the highest root $\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ (of type C_ℓ) and $\alpha_0 = \delta - \theta$, the null root in type $C_\ell^{(1)}$ is

$$\delta = \alpha_0 + 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

Then, the positive real root system Δ_{re}^{+} of type $C_\ell^{(1)}$ is given by

$$\Delta_{\mathrm{re}}^{+} = \left\{\beta + m\delta \,\middle|\, m \geq 0, \beta \in \Delta_{\mathrm{fin}}^{+} \quad \text{or} \quad \Delta_{\mathrm{fin}}^{-} + \delta\right\}.$$

We denote by $\Pi := \{\alpha_i \mid i \in I\}$ the set of *simple roots* of type $C_\ell^{(1)}$.

Let $\Pi^\vee := \{\alpha_i^\vee \mid i \in I\}$ be the set of simple coroots such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$, for $i, j \in I$. We may set a scaling element d by $\langle d, \alpha_0 \rangle = 1$ and $\langle d, \alpha_i \rangle = 0$ for $i \in I/\{0\}$. Then, $\{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee, d\}$ forms a basis of the Cartan subalgebra of the Kac–Moody Lie algebra $\mathfrak{g}$ (associated with the Cartan datum of type $C_\ell^{(1)}$). The canonical central element of $\mathfrak{g}$ is $c = \alpha_0^\vee + \alpha_1^\vee + \cdots + \alpha_\ell^\vee$. Moreover, we have $\langle d, \delta \rangle = 1$, and $\langle \alpha_i^\vee, \delta \rangle = 0$, for $i \in I$.

The *fundamental weight* Λ_j ($j \in I$) is defined by $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$ and $\langle d, \Lambda_j \rangle = 0$. Then, the weight lattice is $P := \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_\ell \oplus \mathbb{Z}\delta$. A weight $\lambda \in P$ is said to be *dominant* if $\langle \alpha_i^\vee, \lambda \rangle \geq 0$, for $i \in I$. Then, the set of dominant (integral) weights is given by $P^+ := \mathbb{Z}_{\geq 0}\Lambda_0 \oplus \mathbb{Z}_{\geq 0}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\Lambda_\ell \oplus \mathbb{Z}\delta$. Note that P contains the root lattice Q spanned by all simple roots, i.e., $Q := \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_\ell$. We denote the positive cone of the root lattice by $Q_+ := \mathbb{Z}_{\geq 0}\alpha_0 \oplus \mathbb{Z}_{\geq 0}\alpha_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_\ell$. For any $\beta \in Q_+$, the *height* of $\beta = \sum_{i \in I} m_i \alpha_i \in Q_+$ is defined by $|\beta| := \sum_{i \in I} m_i$.

We define, for a natural number $k \geq 1$,

$$P_{cl,k}^{+} := \left\{ \sum_{i=0}^{\ell} m_i \Lambda_i \,\middle|\, m_i \geq 0, \quad \sum_{i=0}^{\ell} m_i = k \right\} \subseteq P^+.$$

Here, the word *cl* stands for the classical dominant integral weights. The value $\langle c, \Lambda \rangle = k$, for $\Lambda \in P_{cl,k}^{+}$, is called the *level* of Λ. Set $\varpi_i := \Lambda_i - \Lambda_0$ ($i \in I \setminus \{0\}$) as in Kac's book [32, (12.4.3)]; these are fundamental weights of $\mathfrak{sp}(2\ell, \mathbb{C})$. Fix $\Lambda = \sum_{i=0}^{\ell} m_i \Lambda_i \in P_{cl,k}^{+}$. Then, Young-Hun Kim, Se-jin Oh and Young-Tak Oh introduced in [36, Proposition 2.1] the set

$$\mathcal{D}(\Lambda) := \left\{ \sum_{i=1}^{\ell} p_i \varpi_i \,\middle|\, p_i \geq 0, \quad \sum_{i=1}^{\ell} p_i \leq k, \quad \sum_{i=1}^{\ell} (p_i - m_i)(\mathsf{A}')^{-1} u_i \in \mathbb{Z}^\ell \right\},$$

where u_i's are unit vectors. The inverse $(\mathsf{A}')^{-1}$ is easy to calculate:

$$(\mathsf{A}')^{-1} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 1 & 2 & 2 & \ldots & 2 & 2 & 2 \\ 1 & 2 & 3 & \ldots & 3 & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \ldots & \ell-2 & \ell-1 & \ell-1 \\ 1/2 & 1 & 3/2 & \ldots & \ell/2-1 & (\ell-1)/2 & \ell/2 \end{pmatrix}.$$

We say that $\Lambda, \Lambda' \in P_{cl,k}^{+}$ are *equivalent* if $\mathcal{D}(\Lambda) = \mathcal{D}(\Lambda')$, and we denote $\Lambda \sim \Lambda'$.

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2.2. **Dominant maximal weight.** Let $U_v(\mathfrak{g})$ be the quantum group of $\mathfrak{g}$. Given a $\Lambda \in P^+$, we denote by $V(\Lambda)$ the integrable highest weight module with the highest weight Λ and by $P(\Lambda)$ the set of weights of $V(\Lambda)$. A weight $\lambda \in P(\Lambda)$ is said to be *maximal* if $\lambda + \delta \notin P(\Lambda)$. Let $\max(\Lambda)$ be the set of maximal weights in $P(\Lambda)$. It is known that

$$P(\Lambda) = \bigsqcup_{\lambda \in \max(\Lambda)} \{\lambda - m\delta \mid m \in \mathbb{Z}_{\geq 0}\}. \tag{2.1}$$

The set of all dominant maximal weights of $V(\Lambda)$ is defined as

$$\max{}^+(\Lambda) := \max(\Lambda) \cap P^+.$$

Let W be the Weyl group generated by $\{r_i\}_{i\in I}$ acting on P by $r_i\mu = \mu - \langle \alpha_i^\vee, \mu\rangle \alpha_i$, for $\mu \in P$ and $i \in I$. Then, it is known (e.g., [32, Proposition 11.2 (a)]) that any element in $\max(\Lambda)$ is W-conjugate to an element in $\max^+(\Lambda)$.

2.3. **Cyclotomic KLR algebra.** Let $\Bbbk$ be an algebraically closed field. For any $i, j \in I$, we take a family $Q_{i,j}(u,v) \in \Bbbk[u,v]$ of polynomials such that $Q_{i,i}(u,v) = 0$, $Q_{i,j}(u,v) = Q_{j,i}(v,u)$, and for any $i < j$,

$$Q_{i,j}(u,v) = \begin{cases} u - v^2 & \text{if } i = 0, j = 1, \\ u - v & \text{if } i \neq 0, j = i+1, j \neq \ell, \\ u^2 - v & \text{if } i = \ell - 1, j = \ell, \\ 1 & \text{otherwise.} \end{cases}$$

We denote by $\mathfrak{S}_n$ the symmetric group generated by elementary transpositions $\{s_i \mid 1 \leq i \leq n-1\}$. Then, the action of $\mathfrak{S}_n$ on I^n is given by

$$s_i \cdot (\nu_1, \nu_2, \dots, \nu_i, \nu_{i+1}, \dots, \nu_n) = (\nu_1, \nu_2, \dots, \nu_{i+1}, \nu_i, \dots, \nu_n).$$

Recall that, a finite-dimensional $\Bbbk$-algebra A is said to be $\mathbb{Z}$-graded if it is equipped with a $\Bbbk$-vector space decomposition $A = \bigoplus_{m\in\mathbb{Z}} A_m$ satisfying $A_m A_n \subseteq A_{m+n}$. Here, elements in A_m are called *homogeneous* of degree $m \in \mathbb{Z}$. Let q be an indeterminate. Then, the graded dimension $\dim_q A$ of A is defined by

$$\dim_q A := \sum_{m\in\mathbb{Z}} (\dim A_m) q^m \in \mathbb{Z}_{\geq 0}[q, q^{-1}].$$

Definition 2.1. Fix $\Lambda \in P^+_{cl,k}$. Let $R^\Lambda(n)$ be the $\mathbb{Z}$-graded $\Bbbk$-algebra generated by

$$\{e(\nu) \mid \nu = (\nu_1, \nu_2, \dots, \nu_n) \in I^n\}, \quad \{x_i \mid 1 \leq i \leq n\}, \quad \{\psi_j \mid 1 \leq j \leq n-1\},$$

subject to

(1) $e(\nu)e(\nu') = e(\nu)\delta_{\nu,\nu'}$, $\sum_{\nu\in I^n} e(\nu) = 1$, $x_i x_j = x_j x_i$, $x_i e(\nu) = e(\nu) x_i$,
(2) $\psi_i e(\nu) = e(s_i(\nu))\psi_i$, $\psi_i\psi_j = \psi_j\psi_i$ if $|i-j| > 1$, $\psi_i x_j = x_j \psi_i$ if $j \neq i, i+1$,
(3) $\psi_i^2 e(\nu) = Q_{\nu_i,\nu_{i+1}}(x_i, x_{i+1}) e(\nu)$,
(4) $(\psi_i x_{i+1} - x_i \psi_i) e(\nu) = (x_{i+1}\psi_i - \psi_i x_i) e(\nu) = e(\nu)\delta_{\nu_i,\nu_{i+1}}$,
(5) $(\psi_{i+1}\psi_i\psi_{i+1} - \psi_i\psi_{i+1}\psi_i) e(\nu)$

$$= \begin{cases} \frac{Q_{\nu_i,\nu_{i+1}}(x_i,x_{i+1}) - Q_{\nu_i,\nu_{i+1}}(x_{i+2},x_{i+1})}{x_i - x_{i+2}} e(\nu) & \text{if } \nu_i = \nu_{i+2}, \\ 0 & \text{otherwise,} \end{cases}$$

(6) $x_1^{\langle \alpha_{\nu_1}^\vee, \Lambda\rangle} e(\nu) = 0$,

and the $\mathbb{Z}$-grading on $R^\Lambda(n)$ is given by

$$\deg(e(\nu)) = 0, \quad \deg(x_i e(\nu)) = 2\mathsf{d}_{\nu_i}, \quad \deg(\psi_i e(\nu)) = -\mathsf{d}_{\nu_i} a_{\nu_i,\nu_{i+1}},$$

with $(\mathsf{d}_0, \mathsf{d}_1, \ldots, \mathsf{d}_{\ell-1}, \mathsf{d}_\ell) = (2, 1, \ldots, 1, 2)$. We call $R^\Lambda(n)$ the cyclotomic quiver Hecke algebra of type $C_\ell^{(1)}$, and this algebra was introduced by Mikhail Khovanov and Aaron Lauda [35]. Note that the (affine) quiver Hecke algebra $R(n)$ obtained by omitting the relation (6) was also introduced by Raphael Rouquier [48], independent of [35]. Thus, the cyclotomic quiver Hecke algebra is also known as the cyclotomic Khovanov–Lauda–Rouquier algebra.

Given a positive root $\beta \in Q_+$ with $|\beta| = n$, we set

$$e(\beta) := \sum_{\nu \in I^\beta} e(\nu) \quad \text{with} \quad I^\beta := \left\{ \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in I^n \;\middle|\; \sum_{i=1}^n \alpha_{\nu_i} = \beta \right\}.$$

This is a central idempotent of $R^\Lambda(n)$. We may distinguish the component of $R^\Lambda(n)$ associated with $e(\beta)$ as follows.

Definition 2.2. We define $R^\Lambda(\beta) := R^\Lambda(n)e(\beta)$.

We may define $R^\Lambda(\beta)$ with the same defining relations of $R^\Lambda(n)$, just by replacing I with I^β.

Remark 2.3. Fix $\Lambda = \sum_{i\in I} m_i\Lambda_i \in P^+_{cl,k}$. It is known, e.g., [48, p. 25] or [10, Lemma 3.2], that $R(n)$ or $R^\Lambda(n)$ (of type $C_\ell^{(1)}$) does not depend on the choice of $Q_{i,j}(u,v)$, up to isomorphism. Let $R_A^\Lambda(n)$ be the cyclotomic KLR algebra of type $A_\ell^{(1)}$ whose definition uses polynomials $Q_{i,i+1}(u,v) = u - v$ for $i \in \mathbb{Z}/(\ell+1)\mathbb{Z}$, and $Q_{i,j}(u,v) = 1$ if $j \not\equiv_{\ell+1} i, i \pm 1$. Suppose that

$$\beta \in \mathbb{Z}_{\geq 0}\alpha_1 \oplus \mathbb{Z}_{\geq 0}\alpha_2 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_{\ell-1}.$$

Then, β may be viewed as an element in the positive cone of the root lattice for the type $A_\ell^{(1)}$. Under this circumstance, we have an isomorphism of algebras $R^\Lambda(\beta) \cong R_A^{\Lambda_A}(\beta)$, where $\Lambda_A = \Lambda - m_0\Lambda_0 - m_\ell\Lambda_\ell$. In the rest of the paper, we write $R_A^\Lambda(\beta)$ instead of $R_A^{\Lambda_A}(\beta)$ by abuse of notation.

Let $\sigma : I \to I$ be the involution given by $\sigma(i) = \ell - i$. Given a dominant integral weight $\Lambda = \sum_{i\in I} m_i\Lambda_i \in P^+_{cl,k}$ and a positive root $\beta = \sum_{i\in I} n_i\alpha_i \in Q_+$, we define

$$\sigma\Lambda := \sum_{i\in I} m_i\Lambda_{\sigma(i)} \quad \text{and} \quad \sigma\beta := \sum_{i\in I} n_i\alpha_{\sigma(i)}. \tag{2.2}$$

Using Remark 2.3, we may assume that $R^\Lambda(\beta)$ and $R^{\sigma\Lambda}(\sigma\beta)$ share the same family of polynomials $Q_{i,j}(u,v) \in \Bbbk[u,v]$.

Proposition 2.4 ([7, Lemma 3.1])**.** *There is an algebra isomorphism*

$$R^\Lambda(\beta) \cong R^{\sigma\Lambda}(\sigma\beta).$$

There is a symmetric bilinear form $(-,-)$ on the weight lattice P such that

$$(\Lambda_i, \alpha_j) = \mathsf{d}_j\delta_{ij}, \quad (\alpha_i, \alpha_j) = \mathsf{d}_i a_{ij}.$$

with $(\mathsf{d}_0, \mathsf{d}_1, \ldots, \mathsf{d}_{\ell-1}, \mathsf{d}_\ell) = (2, 1, \ldots, 1, 2)$. The *defect* of $R^\Lambda(\beta)$ is given by

$$\operatorname{def}_\Lambda(\beta) := (\Lambda, \beta) - (\beta, \beta)/2.$$

We sometimes omit Λ from the subscript and write $\operatorname{def}(\beta)$ instead of $\operatorname{def}_\Lambda(\beta)$. In level one, we experienced the validity of Erdmann–Nakano type theorems, see [13, 22]. Hence, it is of interest to list defect values here. In the representation-finite cases, the value is 1 except for the following three cases.

- (f1): $\operatorname{def}(\beta) = m_a - 1$ if $1 \le a \le \ell - 1$, and $\operatorname{def}(\beta) = 2m_a - 2$ if $a = 0, \ell$.
- (f2) or (f3): $\operatorname{def}(\beta) = 2$ for $m_0 = m_1 = 1$ or $m_{\ell-1} = m_\ell = 1$, and $\operatorname{def}(\beta) = 2m_i - 1$ for $i = 0$ or ℓ.

In the tame cases, the value is 2 only for 5 cases, and the other 16 cases may have different values as listed below.

- (t3) or (t4): $\operatorname{def}(\beta) = 2m_i \ge 4$ for $i = 0$ or ℓ.
- (t5) or (t6): $\operatorname{def}(\beta) = 2m_i \ge 2$ for $i = 0$ or ℓ.
- (t7) or (t8): $\operatorname{def}(\beta) = 3$.
- (t9): $\operatorname{def}(\beta) = m_i \ge 2$ for $i = a$ or b.
- (t10) or (t11): $\operatorname{def}(\beta) = 3$ if $i \ne \ell$ or 0, and $\operatorname{def}(\beta) = 4$ if $i = \ell$ or 0.
- (t13) or (t14): $\operatorname{def}(\beta) = 2$ if $i \ne \ell$ or 0, and $\operatorname{def}(\beta) = 3$ if $i = \ell$ or 0.
- (t16) or (t17): $\operatorname{def}(\beta) = 3$.
- (t19): $\operatorname{def}(\beta) = 4$.
- (t20) or (t21): $\operatorname{def}(\beta) = 4$.

Let $n \ge 1$ be a natural number and $\lambda = (\lambda_1, \lambda_2, \dots)$ a sequence of non-negative integers. We call λ a *partition* of n if $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$. A k-*multipartition* of n is an ordered k-tuple of partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(k)}| = n$. We denote by $\mathcal{P}_{k,n}$ the set of all k-multipartitions of n.

A Young diagram is considered as a realization of a partition. Here, the Young diagram of a k-multipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$ can be visualized as a column vector whose entries are $\lambda^{(i)}$'s in increasing order from top to bottom. We say that a node of $\lambda \in \mathcal{P}_{k,n}$ is *removable* (resp., *addable*) if one obtains a new k-multipartition after removing (resp., adding) the node from (resp., to) λ.

Let $g_\ell : \mathbb{Z} \to \mathbb{Z}/2\ell\mathbb{Z}$ be the natural projection and we define $f_\ell : \mathbb{Z}/2\ell\mathbb{Z} \to I$ by

$$f_\ell(a + 2\ell\mathbb{Z}) := \begin{cases} a & \text{if } 0 \le a \le \ell, \\ 2\ell - a & \text{if } \ell + 1 \le a \le 2\ell - 1. \end{cases}$$

For any $m \in \mathbb{Z}$, we set $\overline{m} := (f_\ell \circ g_\ell)(m) \in I$. In other words, the values periodically repeat in the order of $0\ 1\ 2\ \dots\ \ell-1\ \ell\ \ell-1, \dots, 2\ 1$.

Fix $\Lambda = \Lambda_{i_1} + \Lambda_{i_2} + \cdots + \Lambda_{i_k} \in P^+_{cl,k}$ and $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)}) \in \mathcal{P}_{k,n}$. Let p be a node in the a^{th} row and b^{th} column of $\lambda^{(s)}$. Then, the *residue* of p is defined by

$$\operatorname{res} p := \overline{b - a + i_s} \quad \in I,$$

and p is said to be an *i-node* if $\operatorname{res} p = i$. As $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$ can be visualized as a column vector of Young diagrams, we say that $\lambda^{(s)}$ is below $\lambda^{(t)}$ if $s > t$. We set $\#\text{addable}_{\operatorname{res} p}(\lambda)$ as the number of addable $(\operatorname{res} p)$-nodes of λ below p, and set $\#\text{removable}_{\operatorname{res} p}(\lambda)$ as the number of removable $(\operatorname{res} p)$-nodes of λ below p. If p is a removable i-node of λ, we define

$$d_p(\lambda) := \mathsf{d}_i \cdot (\#\text{addable}_{\operatorname{res} p}(\lambda) - \#\text{removable}_{\operatorname{res} p}(\lambda))$$

with $(\mathsf{d}_0, \mathsf{d}_1, \dots, \mathsf{d}_{\ell-1}, \mathsf{d}_\ell) = (2, 1, \dots, 1, 2)$ as mentioned before.

A standard tableau $T = (T^{(1)}, T^{(2)}, \dots, T^{(k)})$ of shape $\lambda \in \mathcal{P}_{k,n}$ is given by bijectively inserting the integers $1, 2, \dots, n$ into the nodes of the Young diagram of λ, such that each $T^{(i)}$ is a standard tableau of $\lambda^{(i)}$, i.e., the entries in $T^{(i)}$ are strictly increasing along the rows from left to right and down the columns from top to bottom. We denote by $\mathrm{Std}(\lambda)$ the set of all standard tableaux of λ. The *residue sequence* of T is defined as $\mathbf{i}_T := (i_1, i_2, \ldots, i_n) \in I^n$, such that $i_r = \operatorname{res} p$ if the integer r is filled in the node p of λ. We then define the *degree* of T (see [14, (1.4)]) inductively by

$$\deg(T) := \begin{cases} \deg(T\downarrow_n) + d_p(\lambda) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases} \tag{2.3}$$

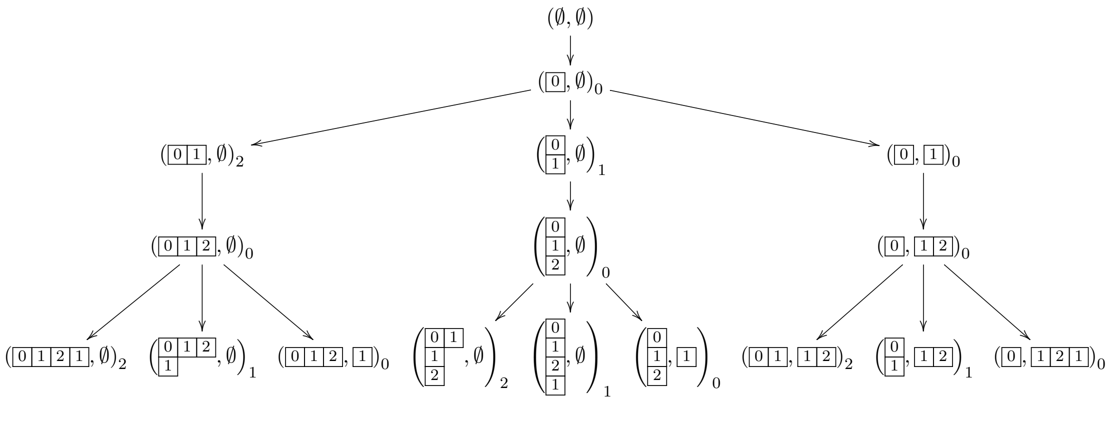
where $T\downarrow_n$ is the tableau obtained by removing p from T and the integer $n > 0$ is filled in the node p of λ.

Using values $\deg(T)$, we may define the action of Chevalley generators on the $\mathbb{Q}[v, v^{-1}]$-span of all k-multipartitions to make it into a module over the quantum group $U_v(\mathfrak{g})$. We call this $U_v(\mathfrak{g})$-module the level k deformed Fock space. We denote the empty k-multipartiton by v_Λ, which generates $V(\Lambda)$ as a $U_v(\mathfrak{g})$-submodule. For the precise definition of the action when $k = 1$, see [13] or [22]. The level k deformed Fock space we use here is the k-fold tensor product of level one deformed Fock spaces. The next theorem follows from the computation in the level k deformed Fock space.

Theorem 2.5 ([14, Theorem 2.5])**.** *For any positive root $\beta \in Q_+$ with $|\beta| = n$ and $\nu, \nu' \in I^\beta$, the graded dimension of $e(\nu)R^\Lambda(\beta)e(\nu')$ is*

$$\dim_q e(\nu)R^\Lambda(\beta)e(\nu') = \sum_{\substack{\mathbf{i}_S=\nu,\, \mathbf{i}_T=\nu', \\ S,T\in\mathrm{Std}(\lambda),\, \lambda\in\mathcal{P}_{k,n}}} q^{\deg(S)+\deg(T)}.$$

Example 2.6. Let $\Lambda = \Lambda_0 + \Lambda_1$ and $\ell = 2$. We consider $R^\Lambda(\delta)$ with $\delta = \alpha_0 + 2\alpha_1 + \alpha_2$. Set $e := e(0121)$. Then, $\dim_q eR^\Lambda(\delta)e = 1 + 2q^2 + 3q^4 + 2q^6 + q^8$ due to the following pattern:



where the subscript number in each vertex gives the corresponding $d_p(\lambda)$.

In the following, we are going to introduce the divided power induction functor $f_i^{(r)}$ (see [20, Section 4.6]) from the category of $R^\Lambda(\beta)$-modules to the category of $R^\Lambda(\beta + r\alpha_i)$-modules, for $r \in \mathbb{Z}_{\geq 0}$. Let $R(\beta)$ be the (affine) KLR algebra, namely, the algebra defined by dropping the cyclotomic condition $x_1^{\langle \alpha_{\nu_1}^\vee, \Lambda\rangle} e(\nu) = 0$ from the defining relations of $R^\Lambda(\beta)$. Then, the definition of $f_i^{(r)}$ starts with the result in [35, Section 2.2] that the polynomial representation $P(i^{(r)}) = \Bbbk[x_1, \dots, x_r]$ over $R(r\alpha_i)$, whose degree is given by

$$\deg(x_1^{m_1} \dots x_r^{m_r}) = \mathsf{d}_i\left(2m_1 + \dots + 2m_r - \frac{r(r-1)}{2}\right),$$

satisfies

$$R(r\alpha_i) \cong P(i^{(r)})\langle \mathsf{d}_i(1-r)\rangle \oplus P(i^{(r)})\langle \mathsf{d}_i(3-r)\rangle \oplus \dots \oplus P(i^{(r)})\langle \mathsf{d}_i(r-1)\rangle,$$

where $R(r\alpha_i)$ is the regular representation.

Example 2.7. $R(2\alpha_i)$ is the $\Bbbk$-algebra generated by x_1, x_2, ψ of degree

$$\deg x_1 = \deg x_2 = 2\mathsf{d}_i, \quad \deg \psi = -2\mathsf{d}_i,$$

which are subject to

$$x_1x_2 = x_2x_1, \quad \psi x_2 - x_1\psi = 1 = x_2\psi - \psi x_1, \quad \psi^2 = 0.$$

Then, $R(2\alpha_i) = \Bbbk[x_1, x_2] \oplus \Bbbk[x_1, x_2]\psi$. Define $e_1 = x_2\psi$ and $e_2 = -\psi x_1$. Then $1 = e_1 + e_2$, $e_se_t = \delta_{st}e_s$, for $s = 1, 2$. Since $\psi = \psi e_1 \in R(2\alpha_i)e_1$, we have

$$P(i^{(2)})\langle -\mathsf{d}_i\rangle \cong \Bbbk[x_1, x_2]\psi = R(2\alpha_i)e_1, \;\; P(i^{(2)})\langle \mathsf{d}_i\rangle \cong \Bbbk[x_1, x_2] = R(2\alpha_i)e_2.$$

Using the $R(r\alpha_i)$-module $P(i^{(r)})$, we define the divided power induction functor $f_i^{(r)}$ as follows.

Definition 2.8. Let $\theta_i^{(r)}(M) := \mathrm{Ind}_{R(\beta)\otimes R(r\alpha_i)}^{R(\beta+r\alpha_i)}(M \otimes P(i^{(r)}))$ for an $R(\beta)$-module M. Based on [20, Lemma 4.4], we define

$$f_i^{(r)} := \mathrm{pr} \circ \theta_i^{(r)} \circ \mathrm{Infl}\Big\langle r^2 - r(\Lambda - \beta, \alpha_i)\Big\rangle,$$

where pr is the tensor functor defined by the $(R^\Lambda(\beta+r\alpha), R(\beta+r\alpha))$-bimodule $R^\Lambda(\beta+r\alpha)$, and Infl is the inflation functor from the category of $R^\Lambda(\beta)$-modules to the category of $R(\beta)$-modules with respect to the quotient algebra homomorphism $R(\beta) \to R^\Lambda(\beta)$.

We need the following lemma proved in [20, Lemma 4.8].

Lemma 2.9. *The divided power induction functor $f_i^{(r)}$ is an exact functor and it sends projective modules to projective modules.*

Indeed, if $\beta = \sum_{j=1}^s n_j\alpha_{i_j}$ for some $n_j \in \mathbb{Z}_{\geq 0}$ and $i_j \in I$, the element

$$f_{i_s}^{(n_s)}, \dots, f_{i_2}^{(n_2)} f_{i_1}^{(n_1)} v_\Lambda$$

in the level k deformed Fock space of type $C_\ell^{(1)}$ uniquely determines the projective module which is one of the direct summands of $R^\Lambda(\beta)e(\nu)$ where $\nu = (i_1^{n_1}, i_2^{n_2}, \dots, i_s^{n_s})$, and all the other direct summands are shifts of this projective module. This fact together with Theorem 2.5 allows us to compute the graded dimension of the endomorphism algebra of a certain well-chosen direct sum of indecomposable projective $R^\Lambda(\beta)$-modules, and to apply lemmas on graded dimensions in the next subsection to prove wildness of $R^\Lambda(\beta)$.

Remark 2.10. The divided restriction functor $e_i^{(r)}$ is also an exact functor and it sends projective modules to projective modules.

2.4. **Some tame and wild algebras.** We review a few tame and wild algebras in this subsection. Besides, it is well-known that $\Bbbk[x]/(x^n)$ for any $n \geq 2$ is a representation-finite local algebra. The wild algebras below will give us a reduction method for proving wildness, because, if e is an idempotent of a finite-dimensional algebra A and a factor algebra of eAe is wild, then A is wild.

Proposition 2.11. *Let $A = \Bbbk Q/J$ be a local algebra with*

$$Q: \ x \circlearrowleft \circ \circlearrowright y \ .$$

(1) If $J = \langle x^2, y^2, xy - yx \rangle$, then A is tame.
(2) If $J = \langle x^2 - y^2, xy, yx \rangle$, then A is wild.
(3) If $J = \langle x^3, y^2, x^2y, xy - yx \rangle$, then A is wild.
(4) If $J = \langle x^m - y^n, xy, yx \rangle$ for some $m, n \geq 2$ and $m + n \geq 5$, then A is tame.

Proof. See [47] for (1)–(3) and see [27, Theorem III.1 (a)] for (4). □

Lemma 2.12. *If the graded dimension of a graded local algebra A satisfies*

$$\dim_q A - 1 - mq \in q^2\mathbb{Z}_{\geq 0}[q] \quad \text{or} \quad \dim_q A - 1 - mq^2 \in q^3\mathbb{Z}_{\geq 0}[q],$$

for $3 \leq m \in \mathbb{Z}_{\geq 0}$, then A is wild.

Proof. Let J be the span of elements of degree greater than or equal to 2 or 3, respectively. Then, J is a two-sided ideal of A, and we have

$$\dim_q A/J = 1 + mq \quad \text{or} \quad \dim_q A/J = 1 + mq^2,$$

respectively. In either case, A/J is the radical square zero local algebra whose Gabriel quiver has at least 3 loops. Hence, A/J is wild by [27, I.10.10 (a)] or [47, (1.1)], and so is A. □

Lemma 2.13. *If the graded dimension of a graded local algebra A satisfies*

$$\dim_q A - 1 - q - mq^2 \in q^3\mathbb{Z}_{\geq 0}[q],$$

for $3 \leq m \in \mathbb{Z}_{\geq 0}$, then A is wild.

Proof. There exists an $x \in A$ spanning the degree 1 part of A. If $x^2 = 0$, then the degree 2 part of A has a basis $\{y_1, y_2, \ldots, y_{m-1}, y_m\}$. If $x^2 \neq 0$, we have a basis $\{x^2, y_1, y_2, \ldots, y_{m-1}\}$ in the degree 2 part of A. In both cases, the Gabriel quiver of A has at least $m \geq 3$ loops. Hence, A is wild. □

Lemma 2.14. *If the graded dimension of a symmetric graded local algebra A satisfies*

$$\dim_q A - 1 - m_1 q - m_2 q^2 \in q^3\mathbb{Z}_{\geq 0}[q],$$

for $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ with $m_1 + m_2 \geq 5$, then A is wild.

Proof. Note that $\operatorname{Rad}^3 A$ is contained in the span of elements of degree greater than or equal to 3. It follows that

$$\dim\Big(\operatorname{Rad} A/\operatorname{Rad}^2 A\Big) + \dim\Big(\operatorname{Rad}^2 A/\operatorname{Rad}^3 A\Big) \geq m_1 + m_2 \geq 5.$$

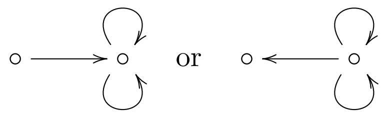
If $\dim(\operatorname{Rad} A/\operatorname{Rad}^2 A) \geq 3$, then the Gabriel quiver of A has at least 3 loops, and A is wild. Otherwise, we have $\dim(\operatorname{Rad}^2 A/\operatorname{Rad}^3 A) \geq 3$, and A is again wild by [27, Theorem III.4]. □

Lemma 2.15. *Let e_1, e_2 be two different primitive idempotents of A. If*

$$\dim_q e_i A e_j - \delta_{ij} - m_{ij} q^2 \in q^3 \mathbb{Z}_{\geq 0}[q]$$

for $m_{ij} \in \mathbb{Z}_{\geq 0}$ such that $m_{11} + m_{22} \geq 3$ and $m_{12} + m_{21} \geq 2$, then A is wild.

Proof. By [7, Lemma 1.3], the Gabriel quiver of $(e_1 + e_2)A(e_1 + e_2)$ has



as a subquiver. Then, A is wild by [31, Theorem 1]. □

Lemma 2.16. *Let $A = \Bbbk[x]/(x^2)$ and $B = \Bbbk Q/J$ be the algebra given by*

$$Q : \circ \underset{\nu}{\overset{\mu}{\rightleftarrows}} \circ \quad \textit{and} \quad J : \langle \mu\nu\mu, \nu\mu\nu \rangle.$$

Then, the tensor product algebra $A \otimes B$ is wild.

Proof. By tensoring A with B, each vertex gets one loop. The tensor product $A \otimes B$ has the minimal wild algebra numbered 32 in [31, Table W] as a factor algebra. □

The next lemma by Kang and Kashiwara [33, Lemma 4.2] is stated for the cyclotomic affine quiver Hecke algebra $R(n)$, but the proof works for $R^\Lambda(\beta)$ (by applying $M = R^\Lambda(\beta)$ there).

Lemma 2.17. *If $\nu \in I^\beta$ satisfies $\nu_i = \nu_{i+1}$ and $fe(\nu) = 0$, for $f \in \Bbbk[x_1, \ldots, x_n]$, then $(\partial_i f)e(\nu) = 0$ and $(s_i f)e(\nu) = 0$, where $\partial_i f = \frac{s_i f - f}{x_i - x_{i+1}}$.*

Proof. First we recall the following equation from [33, (3.7)]

$$(\psi_i f - (s_i f)\psi_i)e(\nu) = (\partial_i f)e(\nu). \tag{2.4}$$

Then, we have

$$\begin{aligned}
0 &= (x_i - x_{i+1})\psi_i f e(\nu)\psi_i \\
&= (x_i - x_{i+1})\psi_i f \psi_i e(\nu) \overset{(2.4)}{=} (x_i - x_{i+1})\big((s_i f)\psi_i + \partial_i f\big)\psi_i e(\nu) \\
&= (x_i - x_{i+1})(\partial_i f)\psi_i e(\nu) \quad (\text{since } \psi_i^2 e(\nu) = 0) \\
&= (s_i f - f)\psi_i e(\nu) \overset{(2.4)}{=} (\psi_i f - \partial_i f - f\psi_i)e(\nu) \\
&= (\partial_i f)e(\nu) \quad (\text{since } fe(\nu) = 0).
\end{aligned}$$

Moreover, we also obtain $(s_i f)e(\nu) = fe(\nu) + (x_i - x_{i+1})(\partial_i f)e(\nu) = 0$. □

The following tensor product lemma is useful. We prove the lemma only for $C_\ell^{(1)}$ here by using the graded dimension formula, but the lemma holds for general Lie type by a different argument [43]. See the appendix.

Lemma 2.18. *Suppose that we have two intervals I_1 and I_2 in $I = \{0, 1, \ldots, \ell\}$ which satisfy $a_{ij} = 0$ for $(i, j) \in I_1 \times I_2$, and $\beta = \beta_1 + \beta_2$ with*

$$\beta_1 \in \sum_{i \in I_1} \mathbb{Z}_{\geq 0}\alpha_i \quad \textit{and} \quad \beta_2 \in \sum_{i \in I_2} \mathbb{Z}_{\geq 0}\alpha_i.$$

We denote by $\nu_1 * \nu_2$ the concatenation of $\nu_1 \in I^{\beta_1}$ and $\nu_2 \in I^{\beta_2}$, and we define

$$e := \sum_{\substack{\nu_1 \in I^{\beta_1},\\ \nu_2 \in I^{\beta_2}}} e(\nu_1 * \nu_2).$$

Then, there is an isomorphism of graded algebras

$$eR^{\Lambda}(\beta)e \cong R^{\Lambda'}(\beta_1) \otimes R^{\Lambda''}(\beta_2)$$

such that $\Lambda' = \sum_{i \in I_1} \langle \alpha_i^\vee, \Lambda \rangle \Lambda_i$ and $\Lambda'' = \sum_{i \in I_2} \langle \alpha_i^\vee, \Lambda \rangle \Lambda_i$. Moreover, $R^{\Lambda}(\beta)$ is graded Morita equivalent to $R^{\Lambda'}(\beta_1) \otimes R^{\Lambda''}(\beta_2)$.

Proof. We define an algebra homomorphism $\mathscr{F} : R^{\Lambda'}(\beta_1) \otimes R^{\Lambda''}(\beta_2) \to eR^{\Lambda}(\beta)e$ by the following assignment:

$$\begin{aligned} 1 \otimes 1 &\longmapsto e, & e(\nu_1) \otimes e(\nu_2) &\longmapsto e(\nu_1 * \nu_2), \\ \psi_i \otimes 1 &\longmapsto \psi_i, & 1 \otimes \psi_i &\longmapsto \psi_{|\beta_1|+i}, \\ x_i \otimes 1 &\longmapsto x_i, & 1 \otimes x_i &\longmapsto x_{|\beta_1|+i}. \end{aligned}$$

Indeed, it is clear that the images of $e(\nu_1) \otimes 1$, $x_i \otimes 1$ and $\psi_i \otimes 1$ commute with the images of $1 \otimes e(\nu_2)$, $1 \otimes x_j$ and $1 \otimes \psi_j$. Since e is the unit of $eR^{\Lambda}(\beta)e$, the unit maps to the unit and

$$e = \sum_{\nu_1 \in I^{\beta_1}} \left(\sum_{\nu_2 \in I^{\beta_2}} e(\nu_1 * \nu_2) \right) = \sum_{\nu_1 \in I^{\beta_1}} \left(\sum_{\nu_2 \in I^{\beta_2}} \mathscr{F}(e(\nu_1) \otimes e(\nu_2)) \right)$$

such that $\mathscr{F}(1 \otimes 1) = \sum_{\nu_1 \in I^{\beta_1}} \mathscr{F}(e(\nu_1) \otimes 1)$ is satisfied. Similarly, $\mathscr{F}(1 \otimes 1) = \sum_{\nu_2 \in I^{\beta_2}} \mathscr{F}(1 \otimes e(\nu_2))$ is satisfied. Then, the orthogonality relations among $\mathscr{F}(e(\nu_1) \otimes 1)$ and among $\mathscr{F}(1 \otimes e(\nu_2))$ hold by the same rewriting of the unit 1.

It is also easy to see that other commutation relations among the generators of $R^{\Lambda'}(\beta_1)$ and the generators of $R^{\Lambda''}(\beta_2)$ hold on their images.

Now, let $m := |\beta_1|$, $\nu_1 = (i_1, i_2, \ldots, i_m)$ and ν_2 starts with $i \in I^{\beta_2}$. Then,

$$\begin{aligned} x_{m+1}^{\langle \alpha_i^\vee, \Lambda'' \rangle} \psi_m^2 e(i_1, i_2, \ldots, i_m, i, \ldots) &= \psi_m x_m^{\langle \alpha_i^\vee, \Lambda'' \rangle} e(i_1, i_2, \ldots, i_{m-1}, i, i_m, \ldots) \psi_m \\ &= \psi_m x_m^{\langle \alpha_i^\vee, \Lambda'' \rangle} \psi_{m-1}^2 e(i_1, \ldots, i_{m-1}, i, i_m, \ldots) \psi_m \\ &= \ldots\ldots \\ &= \psi_m \ldots \psi_1 x_1^{\langle \alpha_i^\vee, \Lambda'' \rangle} e(i, i_1, \ldots, i_m, \ldots) \psi_1 \ldots \psi_m = 0. \end{aligned}$$

Here, the last equality uses $\langle \alpha_i^\vee, \Lambda'' \rangle = \langle \alpha_i^\vee, \Lambda \rangle$. Hence, we have

$$\mathscr{F}\left(1 \otimes x_1^{\langle \alpha_i^\vee, \Lambda'' \rangle} e(\nu_2)\right) = \sum_{\nu_1 \in I^{\beta_1}} x_{m+1}^{\langle \alpha_i^\vee, \Lambda'' \rangle} e(\nu_1 * \nu_2) = 0,$$

and $\mathscr{F}$ induces an algebra homomorphism $R^{\Lambda'}(\beta_1) \otimes R^{\Lambda''}(\beta_2) \to eR^{\Lambda}(\beta)e$. We then observe that $e\psi_w e \neq 0$ implies $w = w_1 w_2$ with $(w_1, w_2) \in \mathfrak{S}_{|\beta_1|} \times \mathfrak{S}_{|\beta_2|}$. Hence, the algebra homomorphism $\mathscr{F}$ is surjective.

To show the injectivity of $\mathscr{F}$, we look at the graded dimensions. Let $K(\nu,\lambda)$ be the sum of monomials $q^{\deg(T)}$ over standard tableaux T of λ and $\mathbf{i}_T=\nu$. Then, we have

$$\dim_q R^{\Lambda'}(\beta_1)=\sum_{\lambda\vdash|\beta_1|}\left(\sum_{\nu_1,\nu_1'\in I^{\beta_1}}K(\nu_1,\lambda)K(\nu_1',\lambda)\right),$$

$$\dim_q R^{\Lambda''}(\beta_2)=\sum_{\lambda\vdash|\beta_2|}\left(\sum_{\nu_2,\nu_2'\in I^{\beta_2}}K(\nu_2,\lambda)K(\nu_2',\lambda)\right),$$

$$\dim_q eR^{\Lambda}(\beta)e=\sum_{\lambda\vdash|\beta|}\left(\sum_{\substack{\nu_1,\nu_1'\in I^{\beta_1}\\ \nu_2,\nu_2'\in I^{\beta_2}}}K(\nu_1*\nu_2,\lambda)K(\nu_1'*\nu_2',\lambda)\right).$$

Since $K(\nu_1*\nu_2,\lambda)\neq 0$ only if the multipartition λ with respect to Λ is a union of multipartitions λ_1 with respect to Λ' and λ_2 with respect to Λ'', we have

$$\dim_q eR^{\Lambda}(\beta)e=\sum_{\substack{\lambda_1\vdash|\beta_1|\\ \lambda_2\vdash|\beta_2|}}\left(\sum_{\substack{\nu_1,\nu_1'\in I^{\beta_1}\\ \nu_2,\nu_2'\in I^{\beta_2}}}K(\nu_1,\lambda_1)K(\nu_2,\lambda_2)K(\nu_1',\lambda_1)K(\nu_2',\lambda_2)\right),$$

which shows $\dim_q eR^{\Lambda}(\beta)e=\dim_q R^{\Lambda'}(\beta_1)\dim_q R^{\Lambda''}(\beta_2)$.

Finally, we prove that $R^{\Lambda}(\beta)$ and $R^{\Lambda'}(\beta_1)\otimes R^{\Lambda''}(\beta_2)$ are graded Morita equivalent. To see this, it suffices to show that the indecomposable projective $R^{\Lambda}(\beta)$-modules that appear as direct summands of $R^{\Lambda}(\beta)e(\nu)$, for any $\nu\in I^{\beta}$, appear as direct summands of $R^{\Lambda}(\beta)e$. Let $n_1:=|\beta_1|$, $n_2:=|\beta_2|$ and $n:=n_1+n_2$. Each $\nu\in I^{\beta}$ defines a black-white sequence of length n with n_1 black entries and n_2 white entries. Let $w\in\mathfrak{S}_n$ be the distinguished right coset representative of $(\mathfrak{S}_{n_1}\times\mathfrak{S}_{n_2})\backslash\mathfrak{S}_n$ which changes the black-white sequence by place permutation to the black-white sequence whose first n_1 entries are black and the remaining n_2 entries are white. We choose a reduced expression of w and define ψ_w. Then, there exist $\nu_1\in I^{\beta_1}$ and $\nu_2\in I^{\beta_2}$ such that we have an $R^{\Lambda}(\beta)$-module homomorphism $R^{\Lambda}(\beta)e(\nu)\to R^{\Lambda}(\beta)e(\nu_1*\nu_2)$ defined by the right multiplication with ψ_w.

Using the same reduced expression but in the reversed order, we have another $R^{\Lambda}(\beta)$-module homomorphism $R^{\Lambda}(\beta)e(\nu_1*\nu_2)\to R^{\Lambda}(\beta)e(\nu)$ by the right multiplication with $\psi_{w^{-1}}$. We compute the composition: they are given by right multiplication with

$$e(\nu_1*\nu_2)\psi_{w^{-1}}\psi_w e(\nu_1*\nu_2)\quad\text{or}\quad e(\nu)\psi_w\psi_{w^{-1}}e(\nu).$$

Write $\psi_w=\psi_{i_1}\psi_{i_2}\dots\psi_{i_r}$. Then,

$$\begin{aligned}e(\nu)\psi_w\psi_{w^{-1}}&=e(\nu)\psi_{i_1}\dots\psi_{i_r}^2\dots\psi_{i_1}\\&=\psi_{i_1}\dots\psi_{i_{r-1}}e(s_{i_{r-1}}\dots s_{i_1}\nu)\psi_{i_r}^2\psi_{i_{r-1}}\dots\psi_{i_1}.\end{aligned}$$

By the minimality of the right coset representative w, the entries at i_r and i_r+1 are neither (white, white) nor (black, black). It follows that $e(s_{i_{r-1}}\dots s_{i_1}\nu)\psi_{i_r}^2=e(s_{i_{r-1}}\dots s_{i_1}\nu)$. We continue the same argument. Then,

$$e(\nu)\psi_w\psi_{w^{-1}}=e(\nu)\psi_{i_1}\dots\psi_{i_{r-1}}^2\dots\psi_{i_1}=\cdots=e(\nu)\psi_{i_1}^2=e(\nu),$$

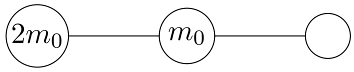
and $e(\nu_1 * \nu_2)\psi_{w^{-1}}\psi_w = e(\nu_1 * \nu_2)$. Hence, we have $R^\Lambda(\beta)e(\nu) \cong R^\Lambda(\beta)e(\nu_1 * \nu_2)$, and this suffices to see that $R^\Lambda(\beta)$ is graded Morita equivalent to $R^{\Lambda'}(\beta_1) \otimes R^{\Lambda''}(\beta_2)$. □

2.5. **Brauer graph algebra.** It is well-known in the literature that Brauer tree algebras are representation-finite, and other Brauer graph algebras, i.e., the remaining algebras whose Brauer graph is either not a tree or with multiple exceptional vertices, are tame. There is an in-depth introduction to Brauer graph algebras, see [49]. Besides, some of the latest progress on the derived equivalence of Brauer graph algebras can be found in [6] and [44]. We then will not review the definition of the Brauer graph and its associated algebra. We use the same conventions in this paper as we have given in [15]. Although any tame cyclotomic KLR algebra in type $A_\ell^{(1)}$ can be realized as a Brauer graph algebra up to Morita equivalence, we point out that it is not always the case in type $C_\ell^{(1)}$, as we mentioned in the introduction.

We remark that, [22, Lemma 3.1] refers to [11] for the tame algebra $R^{\Lambda_1}(\delta)$ with $\ell = 2$, because the assumption that $\operatorname{char} \Bbbk \neq 2$ in [11] is put only for guaranteeing Morita invariant property of cellularity, and the bound quiver algebra mentioned there is tame in $\operatorname{char} \Bbbk = 2$ as well. Hence, as long as we are content with representation type, the characteristic of the field $\Bbbk$ does not matter, but if we want to determine the Morita equivalent classes of a cellular algebra, we must note that the basic algebra of a cellular algebra is not necessarily cellular unless $\operatorname{char} \Bbbk \neq 2$ or the algebra itself is basic.

We give two examples of Brauer graph algebras in the following, which appear as tame cyclotomic KLR algebras in type $C_\ell^{(1)}$.

Lemma 2.19. *Suppose* $\Lambda = m_0\Lambda_0 + m_1\Lambda_1 + \cdots + m_\ell\Lambda_\ell \in P^+_{cl,k}$. *Then,* $R^\Lambda(\alpha_0 + \alpha_1)$ *is tame if* $m_0 \geq 2$ *and* $m_1 = 1$, *namely* (t3) *in Main Theorem A. More precisely, it is Morita equivalent to the Brauer graph algebra whose Brauer graph is displayed as*



Proof. Let $A := R^\Lambda(\alpha_0 + \alpha_1)$. We define $e_1 := e(01)$ and $e_2 := e(10)$. Then,

$$\dim_q e_1Ae_1 = 1 + \sum_{i=1}^{m_0} q^{2(2i-1)} + \sum_{i=1}^{m_0-1} 2q^{4i} + q^{4m_0},$$

$$\dim_q e_2Ae_2 = 1 + \sum_{i=1}^{m_0} q^{2i}, \quad \dim_q e_1Ae_2 = \dim_q e_2Ae_1 = \sum_{i=1}^{m_0} q^{2(2i-1)}.$$

We show that e_iAe_j has a basis as follows.

$$\begin{aligned} e_1Ae_1 &= \Bbbk\text{-span}\Big\{x_1^a x_2^b e_1 \,\Big|\, 0 \leq a \leq m_0 - 1,\ 0 \leq b \leq 2\Big\}, \\ e_2Ae_2 &= \Bbbk\text{-span}\{x_2^a e_2 \,|\, 0 \leq a \leq m_0\}, \\ e_1Ae_2 &= \Bbbk\text{-span}\{\psi_1 x_2^a e_2 \,|\, 0 \leq a \leq m_0 - 1\}, \\ e_2Ae_1 &= \Bbbk\text{-span}\{\psi_1 x_1^a e_1 \,|\, 0 \leq a \leq m_0 - 1\}. \end{aligned}$$

The required basis for e_2Ae_2 follows from $x_1e_2 = 0$ and the graded dimension above. Moreover, $\psi_1^2 e_1 = (x_1 - x_2^2)e_1$ implies that $0 = \psi_1 x_1 e_2 \psi_1 = x_2\psi_1 e_2 \psi_1 = x_2\psi_1^2 e_1 = x_2(x_1 - x_2^2)e_1$, and hence $x_2^3 e_1 = x_1x_2e_1$. This together with $x_1^{m_0}e_1 = 0$ and the graded dimensions imply the required bases for e_1Ae_1, e_1Ae_2 and e_2Ae_1. For e_2Ae_1, apply the anti-involution which fixes the generators $e_1, e_2, x_1, x_2, \psi_1$ elementwise.

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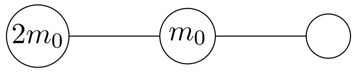
Set $\alpha := x_2 e_1$, $\mu := \psi_1 e_2$ and $\nu := \psi_1 e_1$. We have

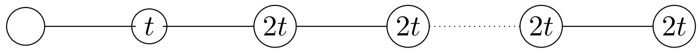
$$\alpha\mu = x_2\psi_1 e_2 = \psi_1 x_1 e_2 = 0, \quad \nu\alpha = \psi_1 x_2 e_1 = x_1 \psi_1 e_1 = 0.$$

Moreover, $\mu\nu = \psi_1^2 e_1 = (x_1 - x_2^2)e_1 = x_1 e_1 - \alpha^2$ such that $(\mu\nu)^{m_0} = -\alpha^{2m_0}$. By comparing dimensions, A is isomorphic to the Brauer graph algebra whose Brauer graph is



proving the assertion. ☐

Lemma 2.20. *Suppose* $\Lambda = \Lambda_a + t\Lambda_\ell$ *with* $t \geq 1$ *and* $\beta = \alpha_a + \alpha_{a+1} + \cdots + \alpha_\ell$, *for some* $1 \leq a \leq \ell - 2$. *This is* (t6) *in Main Theorem A and the basic algebra of* $R^\Lambda(\beta)$ *is isomorphic to the Brauer graph algebra whose Brauer graph is displayed as*



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where the number of vertices is $\ell - a + 2$.

Proof. Let $b := \ell - a + 1$ and $e := e_1 + e_2 + \cdots + e_b$, where $e_i = e(\nu_i)$ for $1 \leq i \leq b$, and

$$\begin{aligned}
\nu_1 &= (a, a+1, a+2, \ldots, \ell-3, \ell-2, \ell-1, \ell), \\
\nu_2 &= s_{b-1}\nu_1 = (a, a+1, a+2, \ldots, \ell-3, \ell-2, \ell, \ell-1), \\
\nu_3 &= s_{b-1}s_{b-2}\nu_2 = (a, a+1, a+2, \ldots, \ell-3, \ell, \ell-1, \ell-2), \\
&\cdots \\
\nu_{b-1} &= s_{b-1}s_{b-2}\ldots s_3 s_2 \nu_{b-2} = (a, \ell, \ell-1, \ell-2, \ldots, a+2, a+1), \\
\nu_b &= s_{b-1}s_{b-2}\ldots s_2 s_1 \nu_{b-1} = (\ell, \ell-1, \ell-2, \ldots, a+2, a+1, a).
\end{aligned}$$

Write $A := eR^\Lambda(\beta)e$. We may compute the graded dimensions as follows.

$$\begin{aligned}
\dim_q e_1 A e_1 &= 1 + \sum_{i=1}^{t} q^{4i}, \\
\dim_q e_2 A e_2 &= 1 + \sum_{i=1}^{2t} q^{2i} + \sum_{i=t}^{t-1} q^{4i}, \\
\dim_q e_i A e_i &= 1 + \sum_{i=1}^{2t-1} 2q^{2i} + q^{4t}, \quad \text{for } 3 \leq i \leq b, \\
\dim_q e_i A e_j &= \begin{cases} \sum_{1\leq i\leq t} q^{4i-2} & \text{if } (i,j) = (1,2), (2,1), \\ \sum_{1\leq i\leq 2t} q^{2i-1} & \text{if } |i-j| = 1, i, j \geq 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

We then find that the basis of $e_i A e_j$ is given as

$$\begin{aligned}
e_1 A e_1 &= \Bbbk\text{-span}\{x_b^m e_1 \mid 0 \leq m \leq t\}, \\
e_2 A e_2 &= \Bbbk\text{-span}\{x_{b-1}^s x_b^m e_2 \mid 0 \leq s \leq t-1, 0 \leq m \leq 2\}, \\
e_1 A e_2 &= \Bbbk\text{-span}\{x_b^a \psi_{b-1} e_2 \mid 0 \leq a \leq t-1\}, \\
e_2 A e_1 &= \Bbbk\text{-span}\{\psi_{b-1} x_b^a e_1 \mid 0 \leq a \leq t-1\},
\end{aligned}$$

and for any $i \geq 2$,

$$e_{i+1}Ae_{i+1} = \Bbbk\text{-span}\{x_{b-1}^m e_{i+1}, x_{b-1}^m x_b e_{i+1} \mid 0 \leq m \leq 2t-1\},$$
$$e_iAe_{i+1} = \Bbbk\text{-span}\{x_{b-1}^a x_b^m \psi_{b-i}\psi_{b-i+1}\dots\psi_{b-1}e_{i+1} \mid 0 \leq a \leq t-1, 0 \leq m \leq 1\},$$
$$e_{i+1}Ae_i = \Bbbk\text{-span}\{\psi_{b-1}\psi_{b-2}\dots\psi_{b-i+1}\psi_{b-i}x_{b-1}^a x_b^m e_i \mid 0 \leq a \leq t-1, 0 \leq m \leq 1\}.$$

- $x_1e_1 = 0$ and $\psi_ie_1 = 0$ for $1 \leq i \leq b-2$ imply that $x_je_1 = 0$ for $2 \leq j \leq b-1$. Then, we have the required basis for e_1Ae_1 by the graded dimension. Similarly, we have
$$x_ie_j = 0 \quad \text{for} \quad 1 \leq i \leq b-j, \quad \text{and} \quad x_1^t e_b = 0. \tag{2.5}$$
Moreover, for any $1 \leq j \leq b$, we have
$$\begin{aligned} x_{b-j+1}^t e_j &= x_{b-j+1}^t \psi_{b-j}^2 e_j = \psi_{b-j}x_{b-j}^t e(s_{b-j}\nu_j)\psi_{b-j} \\ &= \dots \\ &= \psi_{b-j}\dots\psi_2\psi_1 x_1^t e(s_1s_2\dots s_{b-j}\nu_j)\psi_1\psi_2\dots\psi_{b-j} = 0. \end{aligned} \tag{2.6}$$
In particular, $x_{b-1}^t e_2 = 0$. On the other hand, $x_b\psi_{b-1}^2 e_2 = \psi_{b-1}x_{b-1}e_1\psi_{b-1} = 0$. This implies
$$x_b^3 e_2 = x_{b-1}x_b e_2 \tag{2.7}$$
and hence, the required basis for e_2Ae_2 is obtained by the graded dimension.
- For $j \geq 3$, $\psi_he_j = 0$ with $b-j+1 \leq h \leq b-2$ implies $(x_{b-j+2}^2 - x_{b-j+1})e_j = \psi_{b-j+1}^2 e_j = 0$ and $(x_{h+1} - x_h)e_j = \psi_h^2 e_j = 0$ for $b-j+2 \leq h \leq b-2$. Therefore,
$$x_{b-j+2}^{2t} e_j = 0 \tag{2.8}$$
by (2.6), and
$$x_h e_j = x_{b-j+2}e_j \quad \text{for} \quad b-j+3 \leq h \leq b-1. \tag{2.9}$$
- For $j \geq 3$, we have
$$\begin{aligned} x_b\psi_{b-1}^2 e_j &= \psi_{b-1}x_{b-1}e(s_{b-1}\nu_j)\psi_{b-1} \\ &= \psi_{b-1}x_{b-1}\psi_{b-2}^2 e(s_{b-1}\nu_j)\psi_{b-1} \\ &= \dots \\ &= \psi_{b-1}\psi_{b-2}\dots\psi_{b-j+1}x_{b-j+1}e_{j-1}\psi_{b-j+1}\dots\psi_{b-2}\psi_{b-1} \\ &\overset{(2.5)}{=} 0. \end{aligned}$$
This implies that
$$x_b^2 e_j = x_b x_{b-1} e_j \quad \text{for} \quad 3 \leq j \leq b, \tag{2.10}$$
and it gives the required basis of e_jAe_j for $3 \leq j \leq b$. Furthermore, the required basis of e_iAe_j with $|i-j| = 1$ follows from (2.5)–(2.9) and the graded dimensions.

We now are able to find the basic algebra of $R^\Lambda(\beta)$. For any $1 \leq i \leq b-1$, we set

$$\mu_i := \psi_{b-i}\psi_{b-i+1}\dots\psi_{b-1}e_{i+1} \in e_iAe_{i+1}, \quad \nu_i := \psi_{b-1}\psi_{b-2}\dots\psi_{b-i+1}\psi_{b-i}e_i \in e_{i+1}Ae_i,$$

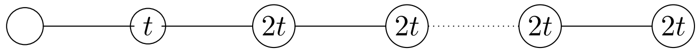
and $\alpha := x_be_b \in e_bAe_b$. Then, $\mu_i\mu_{i+1} = 0 = \nu_{i+1}\nu_i$ for $1 \leq i \leq b-2$, and

$$\begin{aligned} \mu_{b-1}\alpha &= \psi_1\psi_2\dots\psi_{b-1}x_be_b &&= x_1\psi_1\psi_2\dots\psi_{b-1}e_b &&\overset{(2.5)}{=} 0, \\ \alpha\nu_{b-1} &= x_b\psi_{b-1}\psi_{b-2}\dots\psi_1e_{b-1} &&= \psi_{b-1}\psi_{b-2}\dots\psi_1x_1e_{b-1} &&\overset{(2.5)}{=} 0. \end{aligned}$$

We compute $\mu_i\nu_i$ and $\nu_i\mu_i$ as follows.

- $\nu_1\mu_1 = \psi_{b-1}^2 e_2 = (x_b^2 - x_{b-1})e_2$ and
$$\mu_2\nu_2 = \psi_{b-2}\psi_{b-1}^2\psi_{b-2}e_2 = \psi_{b-2}(x_{b-1} - x_b)\psi_{b-2}e_2 \overset{(2.5)}{=} -x_b\psi_{b-2}^2 e_2 = -x_b e_2.$$
This together with (2.7) and (2.8) imply $(\nu_1\mu_1)^t = -(\mu_2\nu_2)^{2t}$.
- Similar computation shows that $\mu_i\nu_i = -x_b e_i$ for $3 \leq i \leq b-1$, and $\nu_j\mu_j = (x_{b-1} - x_b)e_{j+1}$ for $2 \leq j \leq b-1$. This together with (2.8) and (2.10) imply that
$$(\nu_i\mu_i)^{2t} = -(\mu_{i+1}\nu_{i+1})^{2t} \quad \text{for } 2 \leq i \leq b-2, \quad \text{and} \quad (\nu_{b-1}\mu_{b-1})^{2t} = -\alpha^{2t}.$$

We conclude that A is isomorphic to the Brauer graph algebra whose Brauer graph is



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where the number of vertices is $b+1$. By the crystal computation, we see that the number of simple modules of $R^\Lambda(\beta)$ is exactly b. Therefore, A is the basic algebra of $R^\Lambda(\beta)$. □

2.6. **Tilting mutation and derived equivalence.** In this subsection only, we denote by $\mathsf{mod}\, A$ the category of finitely generated right A-modules and by $\mathsf{proj}\, A$ the full subcategory of $\mathsf{mod}\, A$ consisting of projective A-modules. This is harmless when we apply the silting theory to a cyclotomic quiver Hecke algebra, because the algebra admits an anti-involution which fixes generators and relations, and the anti-involution swaps left modules and right modules.

Let $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ be the homotopy category of bounded complexes of finitely generated projective A-modules. We denote by $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ the derived category of $\mathsf{mod}\, A$, which is the localization of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with respect to quasi-isomorphisms. Both $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ are triangulated categories. Two algebras A and B are said to be *Morita equivalent* if there is a category equivalence $\mathsf{mod}\, A \cong \mathsf{mod}\, B$, while A and B are said to be *derived equivalent* if there is a triangle equivalence between the derived categories $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B)$. If A is a local algebra, then the derived equivalence implies Morita equivalence [53, Theorem 2.3]. The remarkable derived equivalences of algebras are induced by classical tilting modules, and this area of study has developed into a very extensive research direction now. We refer readers to the *Handbook of Tilting Theory* [5] to find more details. In particular, it is proven in [45, Theorem 6.4] by Rickard that A is derived equivalent to B if and only if there exists a *tilting complex* T in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ satisfying $B \cong \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(T)$. Further, $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is triangle equivalent to $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$ if and only if A and B are derived equivalent. Thus, it suffices to study tilting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ in order to understand the derived equivalence of A.

Let us review the silting theory, a generalization of tilting theory. Silting is also known as *half-tilting*. A core concept in silting theory is *silting mutation* introduced by Aihara and Iyama in [4]. In ideal cases, we can classify Morita equivalence classes of algebras in the derived equivalence class of A by computing a finite number of tilting complexes by mutation and their endomorphism algebras, as we will see below. We refer to [4] for more definitions of silting theory.

Let $\mathsf{silt}\, A$ be the set of isomorphism classes of basic silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. We construct a directed graph $\mathcal{H}(\mathsf{silt}\, A)$ by drawing an arrow from T to S if S is an irreducible left silting mutation of T. On the other hand, we may regard $\mathsf{silt}\, A$ as a poset concerning a partial order: $T \geqslant S$ if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(T, S[i]) = 0$ for any $i > 0$. Then, the directed graph $\mathcal{H}(\mathsf{silt}\, A)$ is exactly the Hasse quiver of the poset $\mathsf{silt}\, A$. In other words, the Hasse quiver of $\mathsf{silt}\, A$ realizes the left/right silting mutations of silting complexes.

Since mutation produces strictly decreasing silting complexes with respect to the partial order, $\mathcal{H}(\mathsf{silt}\,A)$ is an infinite quiver in general. However, the set of endomorphism algebras of silting complexes in $\mathsf{silt}\,A$ may not be infinite, due to the existence of a certain cyclic phenomenon. Such a cyclic phenomenon has already appeared in the literature, e.g., [8, 16, 52]. To explain this, we start with the following proposition.

Proposition 2.21 ([16, Lemma 2.8])**.** *Let A and B be two algebras with a triangle equivalence $\mathscr{T} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B)$. Then, the following statements hold.*

(1) $\mathscr{T}$ sends silting/tilting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ to that in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B)$.
(2) $\mathscr{T}$ preserves the partial order on the set of silting complexes.
(3) If T is a silting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$, then $\mathscr{T}(\mu_X^-(T)) \cong \mu_{\mathscr{T}(X)}^-(\mathscr{T}(T))$, where $\mu_X^-(T)$ is the irreducible left silting mutation for some direct summand X of T.

Let $T = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ be a tilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ and let B be the endomorphism algebra of T. We denote by $Q_1, Q_2, \ldots, Q_n$ the indecomposable projective B-modules. Then, the triangle equivalence $\mathscr{T} : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A) \to \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B)$ is induced by mapping X_i to Q_i for $i = 1, 2, \ldots, n$. We consider the following irreducible left silting mutation:

$$\begin{array}{ccccc} T & \longrightarrow & \mu_{X_i}^-(T) & & \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A) \\ \downarrow{\scriptstyle \mathscr{T}} & & \downarrow{\scriptstyle \mathscr{T}} & & \\ B & \longrightarrow & \mu_{Q_i}^-(B) & & \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B). \end{array}$$

Note that $\mu_{X_i}^-(T)$ and $\mu_{Q_i}^-(B)$ are again silting but they are not necessarily tilting.

As $\mathscr{T}$ sends $\mathsf{add}(T/X_i)$-approximation to $\mathsf{add}(B/Q_i)$-approximation, we have the following statement.

Corollary 2.22. *We have $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)}\,\mu_{X_i}^-(T) \cong \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B)}\,\mu_{Q_i}^-(B)$.*

We define $2\text{-}\,\mathsf{silt}\,A := \{T \mid A \geq T \geq A[1]\} \subset \mathsf{silt}\,A$, and elements in $2\text{-}\,\mathsf{silt}\,A$ are called *2-term silting complexes*. Then, $2\text{-}\,\mathsf{silt}\,A$ is again a poset, so that its Hasse quiver $\mathcal{H}(2\text{-}\,\mathsf{silt}\,A)$ is a subquiver of $\mathcal{H}(\mathsf{silt}\,A)$. It is also worth mentioning that there is a poset isomorphism between $2\text{-}\,\mathsf{silt}\,A$ and the set of support τ-tilting A-modules in the sense of τ-tilting theory, see [1] for more details.

Symmetric algebras admit a nice feature in silting theory. Let A be a symmetric algebra. It is proved in [2] that any silting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ is a tilting complex. Therefore, $\mathsf{silt}\,A$ coincides with $\mathsf{tilt}\,A$, the set of isomorphism classes of tilting complexes. We obtain the following theorem for symmetric algebras.

Theorem 2.23. *Let $A_1, A_2, \ldots, A_s$ be finite-dimensional symmetric algebras which are derived equivalent to each other and identify $\mathcal{T} = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A_i)$ for all $1 \leq i \leq s$. Suppose the following conditions hold.*

(1) The set $2\text{-}\,\mathsf{silt}\,A_i$ is finite[6], for $1 \leq i \leq s$.
(2) For each indecomposable projective direct summand X of the left regular module A_i, for $1 \leq i \leq s$, we have $\mathrm{End}_{\mathcal{T}}(\mu_X^-(A_i)) \cong A_j$, for some $1 \leq j \leq s$.

[6]This condition is equivalent to that the algebras A_i are τ-tilting finite or brick-finite, see [1, 24].

Then, any finite-dimensional algebra B which has derived equivalence

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B) \cong \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A_1)\Big(\cong \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A_2) \cong \ldots \cong \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A_s)\Big)$$

is Morita equivalent to A_i, for some $1 \leq i \leq s$.

Proof. We need the concept of silting-discreteness in silting theory: an algebra A is said to be *silting-discrete* if there is a silting object T such that $\{S \mid T \geq S \geq T[k]\} \subset \mathsf{silt}\, A$ is a finite set, for any $k \in \mathbb{N}$. A nice property (see [2]) of a silting-discrete algebra A is that each silting complex in $\mathsf{silt}\, A$ can be obtained by iterated irreducible left silting mutation from a shift of the stalk complex A. It is then shown in [3, Theorem 16] that A is silting-discrete if and only if there is a silting object $T \in \mathsf{silt}\, A$ such that $\{S \mid U \geq S \geq U[1]\}$ is finite, for any iterated irreducible left silting mutation U of T.

Note that silting-discreteness is equivalent to tilting-discreteness since A_1 is a symmetric algebra. Let X be an indecomposable projective summand of A. We set

$$\mu_Y^- \circ \mu_X^-(A) := \mu_Y^-\Big(\mathrm{End}_{\mathcal{T}}\, \mu_X^-(A)\Big),$$

where Y is an indecomposable projective summand of $\mathrm{End}_{\mathcal{T}}\, \mu_X^-(A)$.

Suppose that U is an iterated irreducible left silting mutation of $A_1 \in \mathsf{silt}\, A_1$. Using Corollary 2.22 repeatedly, we obtain

$$U \cong \mu_{X_k}^- \circ \cdots \circ \mu_{X_2}^- \circ \mu_{X_1}^-(A_1),$$

for some $k \in \mathbb{N}$ and some indecomposable projective summands X_i's of $\mathrm{End}_{\mathcal{T}}(U_{i-1})$, where $U_i := \mu_{X_i}^- \circ \cdots \circ \mu_{X_1}^-(A_1)$ for $2 \leq i \leq k$. Then, assumption (2) says that $\mathrm{End}_{\mathcal{T}}(U_1) \cong A_j$ for some $1 \leq j \leq s$. We assume that $\mathrm{End}_{\mathcal{T}}(U_{i-1}) \cong A_h$, for some $1 \leq h \leq s$, holds. Then, Rickard's Morita theorem implies that there is an auto-equivalence $\mathscr{T} : \mathcal{T} \cong \mathcal{T}$ providing $\mathscr{T}(U_{i-1}) = A_h$. See [37, Chapter 3]. Hence, we have

$$\mathrm{End}_{\mathcal{T}}(U_i) = \mathrm{End}_{\mathcal{T}}(\mu_{X_i}(U_{i-1})) \cong \mathrm{End}_{\mathcal{T}}\Big(\mu_{\mathscr{T}(X_i)}(A_h)\Big).$$

In particular, $\mathscr{T}(X_i)$ is an indecomposable projective direct summand of A_h. We deduce by assumption (2) that $\mathrm{End}_{\mathcal{T}}(U_i) \cong A_j$ for some $1 \leq j \leq s$. It finally gives that $\mathrm{End}_{\mathcal{T}}(U) \cong A_j$ for some $1 \leq j \leq s$. On the other hand, using Rickard's Morita theorem again, the set $\{S \mid U \geq S \geq U[1]\}$ is in bijection with the set $\{S \mid A_j \geq S \geq A_j[1]\}$. By assumption (1), we conclude that A_1 is tilting-discrete.

Let B be the algebra which is derived equivalent to A_1. By Rickard's Morita theorem, there is a tilting complex $T \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A_1)$ such that $B \cong \mathrm{End}_{\mathcal{T}}(T)$. Since A_1 is tilting-discrete, T is obtained by iterated irreducible left silting mutation from a shift of the stalk complex A_1. Then, by the above argument, $\mathrm{End}_{\mathcal{T}}(T) \cong A_j$, for some $1 \leq j \leq s$. □

2.7. **The derived equivalence class of** (t7)**.** There is a tame Case (t7) of cyclotomic KLR algebras in affine type C, which cannot be realized as a Brauer graph algebra. Then, we may use Theorem 2.23 to find all Morita equivalence classes of algebras that are derived equivalent to (t7). We consider the following quiver:

$$Q:\ \alpha \circlearrowright \circ \underset{\nu}{\overset{\mu}{\rightleftarrows}} \circ \circlearrowleft \beta\ ,$$

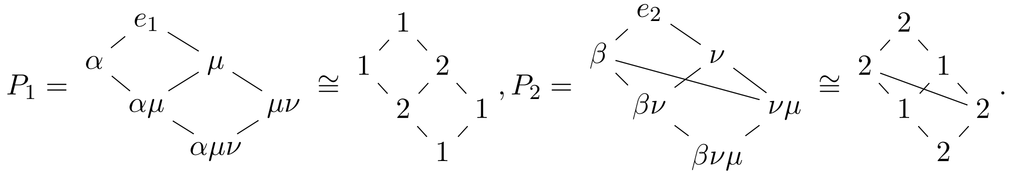
and define

- $A := \Bbbk Q/\{\alpha^2 = 0, \beta^2 = \nu\mu, \alpha\mu = \mu\beta, \beta\nu = \nu\alpha\}$.
- $B := \Bbbk Q/\{\alpha^2 = \mu\nu, \beta^2 = \nu\mu, \alpha\mu = \mu\beta, \beta\nu = \nu\alpha, \mu\nu\mu = \nu\mu\nu = 0\}$.

Here, A is the tame algebra (t7) (See Lemma 7.2) and B[7] is a factor algebra of the tame algebra numbered (21) in [31, Table T]. In this subsection, we refer to the arXiv version [9] of this paper for most of the proofs.

Lemma 2.24. *The algebras A and B are cellular.*

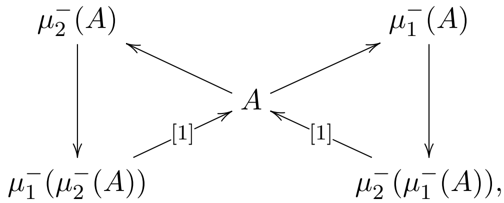
Since the cyclotomic quiver Hecke algebra has an anti-involution which fixes generators and relations, the category of left A-modules and the category of right A-modules are equivalent. Thus, it is harmless to work with right A-modules instead of left A-modules as we mentioned in Subsection 2.6, and we compute with right modules in this subsection. Let P_i be the indecomposable projective A-module at vertex $i \in \{1, 2\}$. We may read the non-zero paths starting from e_i and connect them using an undirected line. It gives the structure of P_i as follows.



It gives

Hom	1	2
1	$e_1, \alpha, \mu\nu, \alpha\mu\nu$	$\nu, \beta\nu$
2	$\mu, \alpha\mu$	$e_2, \beta, \nu\mu, \beta\nu\mu$

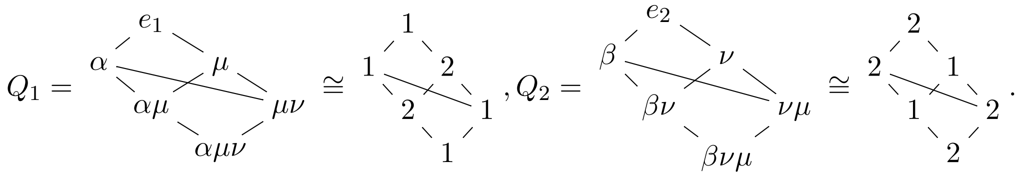
By direct calculation, the Hasse quiver $\mathcal{H}(2\text{-}\mathsf{silt}\, A)$ is given as



where $\mu_i^-(-) := \mu_{P_i}^-(-)$, $X \xrightarrow{[1]} Y$ means $X \to Y[1]$.

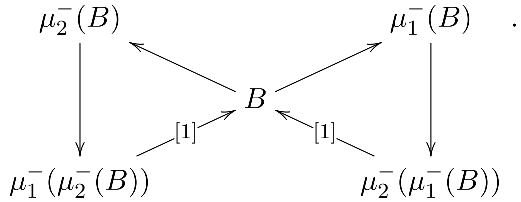
Proposition 2.25. *We have* $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}\, \mu_1^-(A) \cong B$.

Let Q_i be the indecomposable projective B-module at vertex $i \in \{1, 2\}$. Then,



[7]We have not checked whether B appears as the basic algebra of some $R^\Lambda(\beta)$.

The Hasse quiver $\mathcal{H}(2\text{-}\mathsf{silt}\, B)$ is displayed as



In particular, we have

$$\mu_1^-(B) = \begin{bmatrix} Q_1 \xrightarrow{\nu} Q_2 \\ \oplus \\ 0 \longrightarrow Q_2 \end{bmatrix} \quad \text{and} \quad \mu_2^-(B) = \begin{bmatrix} 0 \longrightarrow Q_1 \\ \oplus \\ Q_2 \xrightarrow{\mu} Q_1 \end{bmatrix}.$$

It gives $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)}\, \mu_1^-(B) \cong B$ and $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)}\, \mu_2^-(B) \cong A$.

Proposition 2.26. *If a basic algebra C is derived equivalent to A, then C is isomorphic to A or B.*

Proof. By direct calculation, we have found that both $2\text{-}\mathsf{silt}\, A$ and $2\text{-}\mathsf{silt}\, B$ are finite. We also obtained in the above that

- $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}\, \mu_1^-(A) \cong B$ and $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}\, \mu_2^-(A) \cong A$.
- $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)}\, \mu_1^-(B) \cong B$ and $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)}\, \mu_2^-(B) \cong A$.

Then, the algebra C is Morita equivalent to A or B by Theorem 2.23. □

3. A connected quiver in affine type C

Similar to the construction in [15], we may construct a connected quiver whose vertex set is $\max^+(\Lambda)$. Let us start with the description of $\max^+(\Lambda)$, which was introduced in [36]. Given a dominant weight $\Lambda \in P^+_{cl,k}$, we define

$$P^+_{cl,k}(\Lambda) := \left\{ \Lambda' \in P^+_{cl,k} \,\middle|\, \Lambda \sim \Lambda' \right\},$$

where the equivalence $\Lambda \sim \Lambda'$ was defined in Subsection 2.1. In Proposition 3.6 below, we recall the bijection between $P^+_{cl,k}(\Lambda)$ and $\max^+(\Lambda)$.

Definition 3.1. For any $\Lambda = \sum_{i=0}^{\ell} m_i \Lambda_i \in P^+_{cl,k}$, we set

$$\mathrm{ev}(\Lambda) := m_1 + m_3 + \cdots + m_{2\lfloor(\ell-1)/2\rfloor+1}.$$

Proposition 3.2 ([36, Theorem 2.14])**.** $P^+_{cl,k}(\Lambda) = \{\Lambda' \in P^+_{cl,k} \mid \mathrm{ev}(\Lambda) - \mathrm{ev}(\Lambda') \in 2\mathbb{Z}\}$.

The distinguished representatives $\mathrm{DR}(P^+_{cl,k}) = P^+_{cl,k}/\sim$ of the equivalence classes of $P^+_{cl,k}$ under $\sim$ are given in [36, Table 2.2]. It follows that we have either $P^+_{cl,k}(\Lambda) = P^+_{cl,k}(k\Lambda_0)$ or $P^+_{cl,k}(\Lambda) = P^+_{cl,k}((k-1)\Lambda_0 + \Lambda_1)$, for any $\Lambda \in P^+_{cl,k}$.

Example 3.3. Set $k = 2$, $\ell = 4$. Then,

$$P^+_{cl,2}(2\Lambda_0) = \{2\Lambda_0, 2\Lambda_1, 2\Lambda_2, 2\Lambda_3, 2\Lambda_4, \Lambda_0 + \Lambda_2, \Lambda_1 + \Lambda_3, \Lambda_2 + \Lambda_4, \Lambda_0 + \Lambda_4\}$$

and

$$P^+_{cl,2}(\Lambda_0 + \Lambda_1) = \{\Lambda_0 + \Lambda_1, \Lambda_1 + \Lambda_2, \Lambda_2 + \Lambda_3, \Lambda_3 + \Lambda_4, \Lambda_0 + \Lambda_3, \Lambda_1 + \Lambda_4\}.$$

 https://doi.org/10.5802/art.34

For any $X = (x_0, x_1, \ldots, x_\ell) \in \mathbb{Z}_{\geq 0}^{\ell+1}$, we define

$$\min X := \min\{x_i \mid 0 \leq i \leq \ell\} \quad \text{and} \quad \max X := \max\{x_i \mid 0 \leq i \leq \ell\}.$$

Lemma 3.4. *Suppose that $Y = (y_0, y_1, \ldots, y_\ell) \in \mathbb{Z}^{\ell+1}$ satisfies*

$$y_0 + y_1 + \cdots + y_\ell = 0 \quad \textit{and} \quad y_1 + 2y_2 + \cdots + \ell y_\ell \in 2\mathbb{Z}.$$

There exists a unique solution $X = (x_0, x_1, \ldots, x_\ell) \in \mathbb{Z}^{\ell+1}$ of $\mathsf{A}X^t = Y^t$, such that $\min\{x_0, x_1, \ldots, x_\ell\} \geq 0$ and $\min\{x_0 - 1, x_1 - 2, \ldots, x_{\ell-1} - 2, x_\ell - 1\} < 0$.

Proof. We define $\widehat{X} = (\widehat{x}_0, \widehat{x}_1, \ldots, \widehat{x}_\ell)$ by

$$\begin{gathered} \widehat{x}_0 = 0, \quad \widehat{x}_1 = -y_0, \quad \widehat{x}_2 = -2y_0 - y_1, \ldots, \\ \widehat{x}_{\ell-1} = -(\ell-1)y_0 - (\ell-2)y_1 - \cdots - 2y_{\ell-3} - y_{\ell-2}, \\ 2\widehat{x}_\ell = -\ell y_0 - (\ell-1)y_1 - \cdots - 2y_{\ell-2} - y_{\ell-1} = y_1 + 2y_2 + \cdots + \ell y_\ell. \end{gathered}$$

It is obvious that $\widehat{X} \in \mathbb{Z}^{\ell+1}$. By our assumption, one may easily check that $\mathsf{A}\widehat{X}^t = Y^t$. Thus, the set of integral solutions of $\mathsf{A}X^t = Y^t$ is $\widehat{X} + \mathbb{Z}(1, 2, \ldots, 2, 1)$. We may adjust $m \in \mathbb{Z}$ in $\widehat{X} + m(1, 2, \ldots, 2, 1)$ to obtain the desired solution. It is also clear that such a solution is unique. □

Definition 3.5. For any $\Lambda \in P$, the hub of Λ is defined to be

$$\mathrm{hub}(\Lambda) := (\langle \alpha_0^\vee, \Lambda \rangle,\ \langle \alpha_1^\vee, \Lambda \rangle, \ldots,\ \langle \alpha_\ell^\vee, \Lambda \rangle).$$

In particular, if $\Lambda = \sum_{i=0}^{\ell} m_i \Lambda_i \in P^+_{cl,k}$, then $\mathrm{hub}(\Lambda) = (m_0, m_1, \ldots, m_\ell)$.

Fix $\Lambda = \sum_{i=0}^{\ell} m_i \Lambda_i \in P^+_{cl,k}$ and $\Lambda' = \sum_{i=0}^{\ell} n_i \Lambda_i \in P^+_{cl,k}(\Lambda)$. We define

$$Y^\Lambda_{\Lambda'} = (y_0, y_1, \ldots, y_\ell) := \mathrm{hub}(\Lambda) - \mathrm{hub}(\Lambda').$$

Then,

$$y_0 + y_1 + \cdots + y_\ell = \sum_{i=0}^{\ell} m_i - \sum_{i=0}^{\ell} n_i = k - k = 0,$$

and $\mathrm{ev}(\Lambda) - \mathrm{ev}(\Lambda') \in 2\mathbb{Z}$ implies

$$y_1 + 2y_2 + \cdots + \ell y_\ell \in \mathrm{ev}(\Lambda) - \mathrm{ev}(\Lambda') + 2\mathbb{Z} \subseteq 2\mathbb{Z}.$$

Hence, we may apply Lemma 3.4. Using the unique solution $X^\Lambda_{\Lambda'} := (x_0, x_1, \ldots, x_\ell)$ in Lemma 3.4, we define

$$\beta^\Lambda_{\Lambda'} := \sum_{i=0}^{\ell} x_i \alpha_i \in Q_+.$$

If there is no confusion of Λ, we will simply write $X_{\Lambda'}$, $Y_{\Lambda'}$ and $\beta_{\Lambda'}$ for $X^\Lambda_{\Lambda'}$, $Y^\Lambda_{\Lambda'}$ and $\beta^\Lambda_{\Lambda'}$, respectively. Now, we are able to explain the bijection between $P^+_{cl,k}(\Lambda)$ and $\max^+(\Lambda)$.

Proposition 3.6. *Let $\Lambda \in P^+_{cl,k}$. Then, the correspondence $\Lambda' \in P^+_{cl,k}(\Lambda) \mapsto \Lambda - \beta^\Lambda_{\Lambda'} \in \Lambda - Q_+$ gives a bijection between $P^+_{cl,k}(\Lambda)$ and $\max^+(\Lambda)$.*

Proof. Since $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_\ell \oplus \mathbb{Z}\delta$, we may write

$$\Lambda - \beta^\Lambda_{\Lambda'} = \sum_{i=0}^{\ell} n_i \Lambda_i + n\delta,$$

 https://doi.org/10.5802/art.34

for some $n_0, n_1, \ldots, n_\ell, n \in \mathbb{Z}$. We have $\langle \alpha_i^\vee, \Lambda \rangle - n_i = \langle \alpha_i^\vee, \beta_{\Lambda'}^\Lambda \rangle$. On the other hand,

$$\langle \alpha_i^\vee, \Lambda \rangle - \langle \alpha_i^\vee, \Lambda' \rangle = \sum_{j=0}^{\ell} \langle \alpha_i^\vee, \alpha_j \rangle x_j = \left\langle \alpha_i^\vee, \beta_{\Lambda'}^\Lambda \right\rangle \tag{3.1}$$

by the definition of $\beta_{\Lambda'}^\Lambda$. Hence, $n_i = \langle \alpha_i^\vee, \Lambda' \rangle$ for $0 \leq i \leq \ell$, and they are nonnegative integers due to $\Lambda' \in P_{cl,k}^+(\Lambda)$. Therefore, $\langle \alpha_i^\vee, \Lambda - \beta_{\Lambda'}^\Lambda \rangle \geq 0$ for $0 \leq i \leq \ell$, and

$$\Lambda - \beta_{\Lambda'}^\Lambda \in P^+ \cap (\Lambda - Q_+) \subseteq P(\Lambda).$$

By the minimality of the solution $X_{\Lambda'}^\Lambda \in \mathbb{Z}^{\ell+1}$, we also have $\Lambda - \beta_{\Lambda'}^\Lambda + \delta \notin \Lambda - Q_+$. We have proved that the correspondence defines a map from $P_{cl,k}^+(\Lambda)$ to $\max^+(\Lambda)$.

Suppose $\Lambda - \sum_{j=0}^{\ell} x_j \alpha_j \in \max^+(\Lambda)$. In particular, x_j's are nonnegative integers for $0 \leq j \leq \ell$. We may write

$$\Lambda - \sum_{j=0}^{\ell} x_j \alpha_j = \sum_{i=0}^{\ell} m_i \Lambda_i + n\delta,$$

for some $m_0, m_1, \ldots, m_\ell, n \in \mathbb{Z}$ as before. We set $\Lambda' = \sum_{i=0}^{\ell} m_i \Lambda_i$. Then,

$$m_i = \langle \alpha_i^\vee, \Lambda' \rangle = \langle \alpha_i^\vee, \Lambda' + n\delta \rangle = \langle \alpha_i^\vee, \Lambda \rangle - \sum_{j=0}^{\ell} \langle \alpha_i^\vee, \alpha_j \rangle x_j.$$

This implies that $X = (x_0, x_1, \ldots, x_\ell) \in \mathbb{Z}_{\geq 0}^{\ell+1}$ is a solution of $\mathsf{A}X^t = Y^t$ for $Y = \mathrm{hub}(\Lambda) - \mathrm{hub}(\Lambda')$. Since $\Lambda' + n\delta \in \max^+(\Lambda)$ is a dominant integral weight, we have $m_i \geq 0$ for $0 \leq i \leq \ell$. Moreover, $(1, 1, \ldots, 1)\mathsf{A} = (0, 0, \ldots, 0)$ implies

$$\langle c, \Lambda' \rangle = \sum_{i=0}^{\ell} m_i = \langle c, \Lambda \rangle - \sum_{i,j=0}^{\ell} \langle \alpha_i^\vee, \alpha_j \rangle x_j = \langle c, \Lambda \rangle - (1, 1, \ldots, 1)\mathsf{A}X^t = k.$$

Hence, Λ' belongs to $P_{cl,k}^+$. By the maximality of $\Lambda - \sum_{j=0}^{\ell} x_j \alpha_j$, X is the unique solution of $\mathsf{A}X^t = Y^t$ in the sense of Lemma 3.4. We conclude that $\sum_{j=0}^{\ell} x_j \alpha_j = \beta_{\Lambda'}^\Lambda$. Therefore, the map $P_{cl,k}^+(\Lambda) \to \max^+(\Lambda)$ is surjective.

If we have the same solution $X \in \mathbb{Z}_{\geq 0}^{\ell+1}$ for

$$Y' = \mathrm{hub}(\Lambda) - \mathrm{hub}(\Lambda') \quad \text{and} \quad Y'' = \mathrm{hub}(\Lambda) - \mathrm{hub}(\Lambda''),$$

then $Y' = X\mathsf{A}^t = Y''$. Thus, the map $P_{cl,k}^+(\Lambda) \to \max^+(\Lambda)$ is injective. □

We have the following corollary immediately, and we leave the proof to readers.

Corollary 3.7. *Suppose* $\Lambda = \overline{\Lambda} + \widetilde{\Lambda}$ *with* $\Lambda \in P_{cl,k}^+$, $\overline{\Lambda} \in P_{cl,k'}^+$ *and* $\widetilde{\Lambda} \in P_{cl,k-k'}^+$. *Then,*

$$P_{cl,k'}^+(\overline{\Lambda}) + \widetilde{\Lambda} \subset P_{cl,k}^+(\Lambda) \quad \textit{and} \quad \beta_{\Lambda'}^{\overline{\Lambda}} = \beta_{\Lambda'+\tilde{\Lambda}}^{\Lambda}$$

for any $\Lambda' \in P_{cl,k'}^+(\overline{\Lambda})$.

Our task is to make $\max^+(\Lambda)$ into a connected quiver in such a way that if there is an arrow $\Lambda' \to \Lambda''$ which corresponds to $\Lambda - \sum_{i=0}^{\ell} x_i' \alpha_i$ and $\Lambda - \sum_{i=0}^{\ell} x_i'' \alpha_i$, there is a sequence of simple coroots $\alpha_{i_1}^\vee, \alpha_{i_1}^\vee, \ldots, \alpha_{i_s}^\vee$ such that

$$\left\langle \alpha_{i_t}^\vee, \Lambda - \sum_{i=0}^{\ell} x_i' \alpha_i - \alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_{t-1}} \right\rangle \geq 1,$$

and $\sum_{i=0}^{\ell} x_i'\alpha_i + \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_s} = \sum_{i=0}^{\ell} x_i''\alpha_i$, for $1 \le t \le s$.

3.1. **A connected graph of** $\max^+(\Lambda)$**.** Fix $\Lambda \in P^+_{cl,k}$. Suppose $\Lambda' = \Lambda_i + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ for some $i \in I$ and $\widetilde{\Lambda} \in P^+_{cl,k-1}$, we define

$$\Lambda'_{i^+} := \Lambda_{i+2} + \widetilde{\Lambda} \quad \text{if} \quad 0 \le i \le \ell-2,$$
$$\Lambda'_{i^-} := \Lambda_{i-2} + \widetilde{\Lambda} \quad \text{if} \quad 2 \le i \le \ell.$$

Suppose $\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ for some $i, j \in I$ and $\widetilde{\Lambda} \in P^+_{cl,k-2}$, we define

$$\Lambda'_{i^+,j^+} = \Lambda'_{j^+,i^+} := \Lambda_{i+1} + \Lambda_{j+1} + \widetilde{\Lambda}$$

if $0 \le i \le j \le \ell-1$, and

$$\Lambda'_{i^-,j^-} = \Lambda'_{j^-,i^-} := \Lambda_{i-1} + \Lambda_{j-1} + \widetilde{\Lambda}$$

if $1 \le i \le j \le \ell$.

Suppose $\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ for some $i, j \in I$ and $\widetilde{\Lambda} \in P^+_{cl,k-2}$, we define

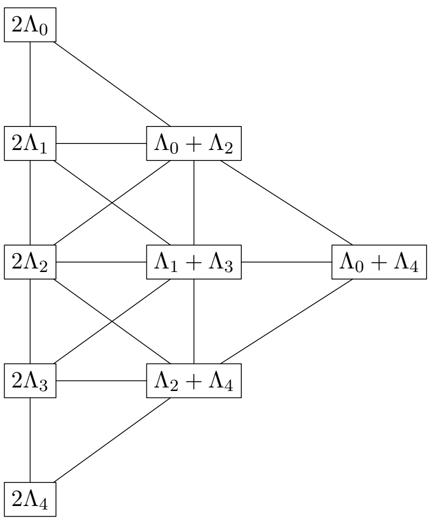
$$\Lambda'_{i^-,j^+} = \Lambda'_{j^+,i^-} := \Lambda_{i-1} + \Lambda_{j+1} + \widetilde{\Lambda}$$

if $i \ne 0$, $j \ne \ell$, $i-1 \ne j$.

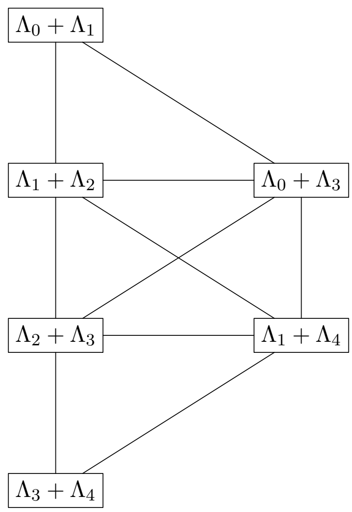
Note that $\Lambda'_{i^+,(i+1)^+} = \Lambda'_{i^+}$ for $0 \le i \le \ell-2$ and $\Lambda'_{i^-,(i+1)^-} = \Lambda'_{(i+1)^-}$ for $1 \le i \le \ell-1$. It is obvious that $\Lambda'_{i^\pm}$, $\Lambda'_{i^\pm,j^\pm}$, $\Lambda'_{i^\pm,j^\mp} \in P^+_{cl,k}(\Lambda)$.

Definition 3.8. Fix $\Lambda \in P^+_{cl,k}$. Let $C(\Lambda)$ be the undirected graph with vertex set $P^+_{cl,k}(\Lambda)$, such that an edge between Λ' and Λ'' exists if $\Lambda'' = \Lambda'_{i^\pm}$ or $\Lambda'_{i^\pm,j^\pm}$ or Λ'_{i^-,j^+}.

Example 3.9. Set $k=2$, $\ell=4$. The graphs $C(2\Lambda_2)$ and $C(\Lambda_1+\Lambda_2)$ are displayed as



and



respectively.

Lemma 3.10. *For any $\Lambda', \Lambda'' \in P^+_{cl,k}(\Lambda)$, there exists an undirected path from Λ' to Λ'' in $C(\Lambda)$. In particular, $C(\Lambda)$ is a finite connected graph.*

Proof. It suffices to consider $\Lambda \in \mathrm{DR}(P^+_{cl,k}) = \{k\Lambda_0, (k-1)\Lambda_0 + \Lambda_1\}$. If $k = 1$, then the assertion is obviously true by level one case, as we will mention in Subsection 3.3 . Suppose $k \geq 2$. We show that there is an undirected path from Λ to Λ', for any $\Lambda' \in P^+_{cl,k}(\Lambda)$.

Set $\Lambda' = \sum_{i\in I} m_i\Lambda_i \in P^+_{cl,k}(\Lambda)$. If $m_0 = k$, then $\Lambda' = \Lambda$ and the assertion is trivial.

If $m_0 = k-1$, then $\Lambda' = (k-1)\Lambda_0 + \Lambda_i$ for some $i \neq 0$. For $i \equiv_2 0$ (i.e., $\Lambda = k\Lambda_0$), we have an undirected path

$$\boxed{k\Lambda_0} \text{———} \boxed{(k-1)\Lambda_0 + \Lambda_2} \text{———} \cdots \text{———} \boxed{(k-1)\Lambda_0 + \Lambda_i}.$$

For $i \equiv_2 1$ (i.e., $\Lambda = (k-1)\Lambda_0 + \Lambda_1$), we have an undirected path

$$\boxed{(k-1)\Lambda_0 + \Lambda_1} \text{———} \boxed{(k-1)\Lambda_0 + \Lambda_3} \text{———} \cdots \text{———} \boxed{(k-1)\Lambda_0 + \Lambda_i}.$$

Suppose $m_0 \leq k-2$. Then, $\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda}$ for some $i \leq j \in I$. If $i \equiv_2 0$ or $j \equiv_2 0$, then there is an undirected path from Λ_0 to Λ_i or Λ_j; this yields an undirected path from $\Lambda_0 + \Lambda_j + \widetilde{\Lambda}$ or $\Lambda_0 + \Lambda_i + \widetilde{\Lambda}$ to Λ'. By the induction hypothesis on $k - m_0$, we have an undirected path from Λ to $\Lambda_0 + \Lambda_j + \widetilde{\Lambda}$ and $\Lambda_0 + \Lambda_i + \widetilde{\Lambda}$, so that there is an undirected path from Λ to Λ'. If $i \equiv_2 j \equiv_2 1$, then $j - i \equiv_2 0$ and there is an undirected path

$$\boxed{2\Lambda_i} \text{———} \boxed{\Lambda_i + \Lambda_{i+2}} \text{———} \cdots \text{———} \boxed{\Lambda_i + \Lambda_j}.$$

Hence, we have an undirected path from $2\Lambda_0$ to $\Lambda_i + \Lambda_j$; this yields an undirected path from $2\Lambda_0 + \widetilde{\Lambda}$ to Λ'. By the induction hypothesis on $k - m_0$, we have an undirected path from Λ to Λ'. $\square$

In order to attach a direction to each edge in $C(\Lambda)$, we compare $X_{\Lambda'}$ and $X_{\Lambda''}$ if there is an edge between Λ' and Λ'', i.e., $\Lambda'' = \Lambda'_{i^\pm}$ or Λ'_{i^-,j^+} or $\Lambda'_{i^\pm,j^\pm}$. To simplify the notation, we will also denote $\delta = (1, 2, 2, \ldots, 2, 1) \in \mathbb{Z}^{\ell+1}$ if there is no confusion in the context.

For $0 \leq i \leq \ell - 2$ and $2 \leq j \leq \ell$, we define

$$\Delta_{i^+} = \left(1, 2^i, 1, 0^{\ell-i-1}\right) \in \mathbb{Z}^{\ell+1}, \quad \Delta_{j^-} = \left(0^{j-1}, 1, 2^{\ell-j}, 1\right) \in \mathbb{Z}^{\ell+1}.$$

Then, we have

$$\delta - \Delta_{i^+} = \Delta_{(i+2)^-}. \tag{3.2}$$

Lemma 3.11. *Suppose $\Lambda' = \Lambda_i + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ for some $0 \leq i \leq \ell - 2$ and $\widetilde{\Lambda} \in P^+_{cl,k-1}$. Set $\Lambda'' := \Lambda'_{i^+}$. Then, $\Lambda''_{(i+2)^-} = \Lambda'$ and one of the following holds.*

(1) If $\min(X_{\Lambda'} + \Delta_{i^+} - \delta) < 0$, then $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i^+}$ and $\min(X_{\Lambda''} + \Delta_{(i+2)^-} - \delta) \geq 0$,

(2) If $\min(X_{\Lambda'} + \Delta_{i^+} - \delta) \geq 0$, then

$$X_{\Lambda''} = X_{\Lambda'} - \Delta_{(i+2)^-} \quad \text{and} \quad \min\left(X_{\Lambda''} + \Delta_{(i+2)^-} - \delta\right) < 0.$$

Proof. We have proved in Lemma 3.4 that $X_{\Lambda'}$ is the unique solution of $\mathsf{A}X^t = Y^t_{\Lambda'}$, satisfying $X_{\Lambda'} \in \mathbb{Z}^{\ell+1}_{\geq 0}$ and $\min(X_{\Lambda'} - \delta) < 0$. We then find

$$\mathsf{A}X^t_{\Lambda''} - \mathsf{A}X^t_{\Lambda'} = Y^t_{\Lambda''} - Y^t_{\Lambda'} = \left(0^i, 1, 0, -1, 0^{\ell-i-2}\right)^t = \mathsf{A}\Delta^t_{i^+}.$$

This gives $\mathsf{A}X^t_{\Lambda''} = \mathsf{A}(X^t_{\Lambda'} + \Delta^t_{i^+})$. It is obvious that $X_{\Lambda'} + \Delta_{i^+} \in \mathbb{Z}^{\ell+1}_{\geq 0}$. If $\min(X_{\Lambda'} + \Delta_{i^+} - \delta) < 0$, then $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i^+}$ by the uniqueness of the solution, and $\min(X_{\Lambda''} + \Delta_{(i+2)^-} - \delta) = \min(X_{\Lambda'}) \geq 0$ by (3.2).

Suppose $\min(X_{\Lambda'} + \Delta_{i^+} - \delta) \geq 0$. Due to $\min(X_{\Lambda'} - \delta) < 0$ and $\Delta_{i^+} - \delta \notin \mathbb{Z}_{\geq 0}^{\ell+1}$, we have $\min(X_{\Lambda'} + \Delta_{i^+} - 2\delta) \leq \min(X_{\Lambda'} - \delta) + \max(\Delta_{i^+} - \delta) < 0$. This implies

$$X_{\Lambda''} = X_{\Lambda'} + \Delta_{i^+} - \delta = X_{\Lambda'} - \Delta_{(i+2)^-}$$

by the uniqueness of the solution, and $\min(X_{\Lambda''} + \Delta_{(i+2)^-} - \delta) = \min(X_{\Lambda'} - \delta) < 0$. □

For any $0 \leq i \leq j \leq \ell - 1$ and $1 \leq s \leq t \leq \ell$, we define two vectors in $\mathbb{Z}^{\ell+1}$ as

$$\Delta_{i^+,j^+} = \Delta_{j^+,i^+} = \Big(1, 2^i, 1^{j-i}, 0^{\ell-j}\Big), \quad \Delta_{s^-,t^-} = \Delta_{t^-,s^-} = \Big(0^s, 1^{t-s}, 2^{\ell-t}, 1\Big).$$

It turns out that $\delta - \Delta_{i^+,j^+} = \Delta_{(i+1)^-,(j+1)^-}$.

Lemma 3.12. *Suppose $\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ for some $0 \leq i \leq j \leq \ell - 1$ and $\widetilde{\Lambda} \in P^+_{cl,k-2}$. Set $\Lambda'' := \Lambda'_{i^+,j^+}$. Then, $\Lambda''_{(i+1)^-,(j+1)^-} = \Lambda'$ and one of the following holds.*

(1) If $\min(X_{\Lambda'} + \Delta_{i^+,j^+} - \delta) < 0$, then $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i^+,j^+}$ and

$$\min\Big(X_{\Lambda''} + \Delta_{(i+1)^-,(j+1)^-} - \delta\Big) \geq 0.$$

(2) If $\min(X_{\Lambda'} + \Delta_{i^+,j^+} - \delta) \geq 0$, then $X_{\Lambda''} = X_{\Lambda'} - \Delta_{(i+1)^-,(j+1)^-}$ and $\min(X_{\Lambda''} + \Delta_{(i+1)^-,(j+1)^-} - \delta) < 0$.

Proof. Since $Y_{\Lambda''} - Y_{\Lambda'} = (0^i, 1, -1, 0^{\ell-i-1}) + (0^j, 1, -1, 0^{\ell-j-1})$ and

$$\mathsf{A}\Big(0^{i+1}, 1^{\ell-i-1}, 1/2\Big)^t = \Big(0^i, -1, 1, 0^{\ell-i-1}\Big)^t,$$

we obtain

$$\begin{aligned} X_{\Lambda''} - X_{\Lambda'} &\in -\Big(0^{i+1}, 1^{\ell-i-1}, 1/2\Big) - \Big(0^{j+1}, 1^{\ell-j-1}, 1/2\Big) + \mathbb{Z}\delta \\ &= -\Delta_{(i+1)^-,(j+1)^-} + \mathbb{Z}\delta = \Delta_{i^+,j^+} + \mathbb{Z}\delta. \end{aligned}$$

Then, the proof is similar to that of Lemma 3.11. □

For any $0 \leq i, j \leq \ell$ with $i \neq 0, j \neq \ell$ with $i - 1 \neq j$, we define two vectors in $\mathbb{Z}^{\ell+1}$ as

$$\Delta_{i^-,j^+} = \Delta_{j^+,i^-} := \begin{cases} \Big(0^i, 1^{j-i+1}, 0^{\ell-j}\Big) & \text{if } i \leq j, \\ \Big(1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1\Big) & \text{if } i \geq j+2. \end{cases}$$

It gives that $\delta - \Delta_{i^-,j^+} = \Delta_{(j+1)^-,(i-1)^+}$.

Lemma 3.13. *Suppose $\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ for some $0 \leq i, j \leq \ell$ satisfying $i \neq 0$, $j \neq \ell$, $i - 1 \neq j$ and $\widetilde{\Lambda} \in P^+_{cl,k-2}$. Set $\Lambda'' = \Lambda'_{i^-,j^+}$. Then, $\Lambda''_{(j+1)^-,(i-1)^+} = \Lambda'$ and one of the following holds.*

(1) If $\min(X_{\Lambda'} + \Delta_{i^-,j^+} - \delta) < 0$, then $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i^-,j^+}$ and

$$\min\Big(X_{\Lambda''} + \Delta_{(j+1)^-,(i-1)^+} - \delta\Big) \geq 0.$$

(2) If $\min(X_{\Lambda'} + \Delta_{i^-,j^+} - \delta) \geq 0$, then $X_{\Lambda''} = X_{\Lambda'} - \Delta_{(j+1)^-,(i-1)^+}$ and

$$\min\Big(X_{\Lambda''} + \Delta_{(j+1)^-,(i-1)^+} - \delta\Big) < 0.$$

Proof. Similar to the proof of Lemma 3.12, we obtain

$$X_{\Lambda''} - X_{\Lambda'} \in \Big(0^i, 1^{\ell-i}, 1/2\Big) - \Big(0^{j+1}, 1^{\ell-j-1}, 1/2\Big) + \mathbb{Z}\delta = \Delta_{i^-,j^+} + \mathbb{Z}\delta.$$

We omit the details. □

One may also find the relation between $X_{\Lambda'}$ and $X_{\Lambda''}$ if $\Lambda'' = \Lambda'_{i^-}$ or Λ'_{i^+,j^-} or Λ'_{i^-,j^-}. We list the corresponding lemmas below and leave the proofs to readers.

Lemma 3.14. *Suppose* $\Lambda' = \Lambda_i + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ *for some* $2 \le i \le \ell$ *and* $\widetilde{\Lambda} \in P^+_{cl,k-1}$. *Then,*

(1) $X_{\Lambda'_{i^-}} = X_{\Lambda'} + \Delta_{i^-}$, *if* $\min(X_{\Lambda'} + \Delta_{i^-} - \delta) < 0$.

(2) $X_{\Lambda'_{i^-}} = X_{\Lambda'} - \Delta_{i-2}$, *if* $\min(X_{\Lambda'} + \Delta_{i^-} - \delta) \ge 0$.

Lemma 3.15. *Suppose* $\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ *with* $1 \le i \le j \le \ell$, $\widetilde{\Lambda} \in P^+_{cl,k-2}$. *Then,*

(1) $X_{\Lambda'_{i^-,j^-}} = X_{\Lambda'} + \Delta_{i^-,j^-}$, *if* $\min(X_{\Lambda'} + \Delta_{i^-,j^-} - \delta) < 0$.

(2) $X_{\Lambda'_{i^-,j^-}} = X_{\Lambda'} - \Delta_{(i-1)^+,(j-1)^+}$, *if* $\min(X_{\Lambda'} + \Delta_{i^-,j^-} - \delta) \ge 0$.

The following lemma is a restatement of Lemma 3.13, if we observe that $\Lambda_{i^+,j^-} = \Lambda_{j^-,i^+}$ and $\Delta_{i^-,j^+} = \Delta_{j^+,i^-}$.

Lemma 3.16. *Suppose* $\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ *for* $0 \le i, j \le \ell$ *satisfying* $i \ne \ell$, $j \ne 0$, $j - 1 \ne i$ *and* $\widetilde{\Lambda} \in P^+_{cl,k-2}$. *Then,*

(1) $X_{\Lambda'_{i^+,j^-}} = X_{\Lambda'} + \Delta_{i^+,j^-}$, *if* $\min(X_{\Lambda'} + \Delta_{i^+,j^-} - \delta) < 0$.

(2) $X_{\Lambda'_{i^+,j^-}} = X_{\Lambda'} - \Delta_{(i+1)^-,(j-1)^+}$, *if* $\min(X_{\Lambda'} + \Delta_{i^+,j^-} - \delta) \ge 0$.

For any $\Lambda' \in P^+_{cl,k}(\Lambda)$, we set $|X_{\Lambda'}| := |\beta_{\Lambda'}|$, i.e., $|X_{\Lambda'}| = \sum_{i \in I} x_i$ if $X_{\Lambda'} = (x_0, x_1, \dots, x_\ell)$. According to the above lemmas, we have either $|X_{\Lambda'}| > |X_{\Lambda''}|$ or $|X_{\Lambda'}| < |X_{\Lambda''}|$ if there is an edge between Λ' and Λ''. This leads to the following definition.

3.2. **A connected quiver of** $\max^+(\Lambda)$. Fix $\Lambda \in P^+_{cl,k}$.

Definition 3.17. We define $\vec{C}(\Lambda)$ to be the quiver having $C(\Lambda)$ as its underlying graph, and the orientation of an edge $\Lambda' \longrightarrow \Lambda'' \in C(\Lambda)$ is given as $\Lambda' \longrightarrow \Lambda''$ if $|X_{\Lambda''}| > |X_{\Lambda'}|$, or equivalently, $\beta_{\Lambda''} - \beta_{\Lambda'} \in Q_+$.

It is clear that the choice of the orientation of $\Lambda' — \Lambda''$ is always possible and unique. We may explain the details of drawing arrows in $\vec{C}(\Lambda)$ as follows.

Fix $\Lambda' \in P^+_{cl,k}(\Lambda)$. We draw an arrow $\Lambda' \xrightarrow{\Delta} \Lambda''$ if $\min(X_{\Lambda'} + \Delta - \delta) < 0$, and then $X_{\Lambda''} = X_{\Lambda'} + \Delta$. According to the lemmas we have given in the previous subsection, there are only 5 choices for Δ, as listed below.

(1) For $0 \le i \le \ell - 2$ with $\langle \alpha_i^\vee, \Lambda' \rangle \ge 1$, we set $\Lambda'' := \Lambda'_{i^+}$ and

$$\Delta := \Delta_{i^+} = \left(1, 2^i, 1, 0^{\ell-i-1}\right).$$

(2) For $2 \le i \le \ell$ with $\langle \alpha_i^\vee, \Lambda' \rangle \ge 1$, we set $\Lambda'' := \Lambda'_{i^-}$ and

$$\Delta := \Delta_{i^-} = \left(0^{i-1}, 1, 2^{\ell-i}, 1\right).$$

(3) For $0 \le i \le j \le \ell - 1$ with $i + 1 \ne j$, $\langle \alpha_i^\vee, \Lambda' \rangle \ge 1$, $\langle \alpha_j^\vee, \Lambda' \rangle \ge 1$, we set $\Lambda'' := \Lambda'_{i^+,j^+} = \Lambda'_{j^+,i^+}$ and

$$\Delta := \Delta_{i^+,j^+} = \Delta_{j^+,i^+} = \left(1, 2^i, 1^{j-i}, 0^{\ell-j}\right).$$

If $i + 1 = j$, then $\Lambda'_{i^+,(i+1)^+} = \Lambda'_{i^+}$ and this coincides with Case (1).

 https://doi.org/10.5802/art.34

(4) For $1 \le i \le j \le \ell$ with $i+1 \ne j$, $\langle \alpha_i^\vee, \Lambda' \rangle \ge 1$, $\langle \alpha_j^\vee, \Lambda' \rangle \ge 1$, we set $\Lambda'' := \Lambda'_{i^-,j^-} = \Lambda'_{j^-,i^-}$ and

$$\Delta := \Delta_{i^-,j^-} = \Delta_{j^-,i^-} = \left(0^i, 1^{j-i}, 2^{\ell-j}, 1\right).$$

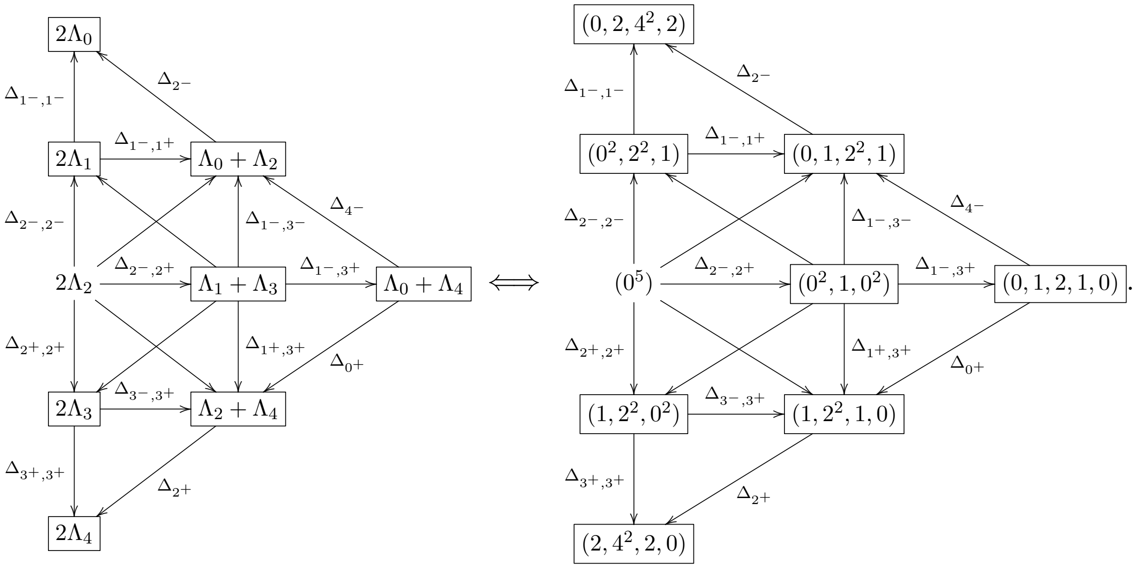
If $i+1=j$, then $\Lambda'_{(j-1)^-,j^-} = \Lambda'_{j^-}$ and this coincides with Case (2).

(5) For $0 \le i,j \le \ell$ with $i \ne 0, j \ne \ell, i-1 \ne j$, $\langle \alpha_i^\vee, \Lambda' \rangle \ge 1$, $\langle \alpha_j^\vee, \Lambda' \rangle \ge 1$, we set $\Lambda'' := \Lambda'_{i^-,j^+} = \Lambda'_{j^+,i^-}$ and

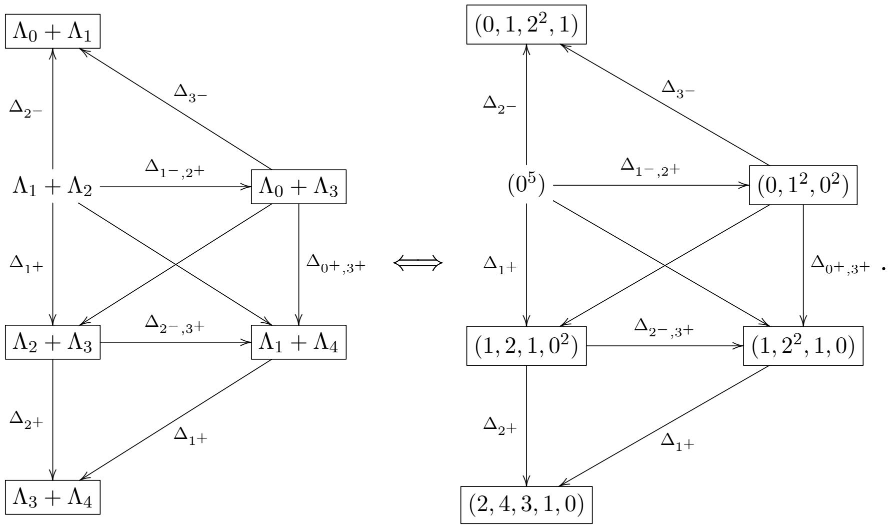
$$\Delta := \Delta_{i^-,j^+} = \Delta_{j^+,i^-} = \begin{cases} \left(0^i, 1^{j-i+1}, 0^{\ell-j}\right) & \text{if } i \le j, \\ \left(1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1\right) & \text{if } i \ge j+2. \end{cases}$$

We remind the reader that it is still needed to check $\min(X_{\Lambda'} + \Delta - \delta)$ in each case.

Example 3.18. Set $k=2$, $\ell=4$. The quiver $\vec{C}(2\Lambda_2)$ associated with $X_{\Lambda'}$ is displayed as



Besides, the quiver $\vec{C}(\Lambda_1+\Lambda_2)$ associated with $X_{\Lambda'}$ is displayed as



Recall that $\Delta_{\rm re}^+ = \{\beta + m\delta \mid m \geq 0, \beta \in \Delta_{\rm fin}^+ \text{ or } \ \delta - \Delta_{\rm fin}^+\}$ with

$$\Delta_{\rm fin}^+ = \{2\epsilon_i \mid 1 \leq i \leq \ell\} \sqcup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \ell\}.$$

We call $\overline{\Delta_{\rm re}^+} := \{\beta \in \Delta_{\rm re}^+ \mid \beta \in \Delta_{\rm fin}^+ \text{ or } \ \delta - \Delta_{\rm fin}^+\}$ the first layer of $\Delta_{\rm re}^+$. If an arrow $\Lambda' \xrightarrow{\Delta} \Lambda''$ defined in the above (1)–(5) exists (i.e., $\min(X_{\Lambda'} + \Delta - \delta) < 0$), then Δ corresponds to a certain element in $\overline{\Delta_{\rm re}^+}$. We then observe that all arrows in $\vec{C}(\Lambda)$ are labeled by elements in $\overline{\Delta_{\rm re}^+}$. Let us check it case by case.

(1) $\Delta = \Delta_{i^+} = (1, 2^i, 1, 0^{\ell-i-1}) = \delta - (\epsilon_{i+1} + \epsilon_{i+2})$ for $0 \leq i \leq \ell - 2$. This gives

$$\delta - \{\epsilon_i + \epsilon_{i+1} \,|\, 1 \leq i \leq \ell - 1\} \subseteq \overline{\Delta_{\rm re}^+}.$$

(2) $\Delta = \Delta_{i^-} = (0^{i-1}, 1, 2^{\ell-i}, 1) = \epsilon_{i-1} + \epsilon_i$ for $2 \leq i \leq \ell$. This gives

$$\{\epsilon_i + \epsilon_{i+1} \mid 1 \leq i \leq \ell - 1\} \subseteq \overline{\Delta_{\rm re}^+}.$$

(3) $\Delta = \Delta_{i^+,j^+} = (1, 2^i, 1^{j-i}, 0^{\ell-j})$ for $0 \leq i \leq j \leq \ell - 1$ with $i + 1 \neq j$. This gives

$$\delta - \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq \ell - 1, \ i + 1 \neq j\} \subseteq \overline{\Delta_{\rm re}^+}.$$

(4) $\Delta = \Delta_{i^-,j^-} = (0^i, 1^{j-i}, 2^{\ell-j}, 1)$ for $1 \leq i \leq j \leq \ell$ with $i + 1 \neq j$. This gives

$$\{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq \ell - 1, \ i + 1 \neq j\} \subseteq \overline{\Delta_{\rm re}^+}.$$

(5) For $0 \leq i, j \leq \ell$ with $i \neq 0, j \neq \ell, i - 1 \neq j$,

$$\Delta = \Delta_{i^-,j^+} = \begin{cases} \left(0^i, 1^{j-i+1}, 0^{\ell-j}\right) = \epsilon_i - \epsilon_{j+1} & \text{if } i \leq j, \\ \left(1, 2^j, 1^{i-j-1}, 2^{\ell-i}, 1\right) = \delta - (\epsilon_{j+1} - \epsilon_i) & \text{if } i \geq j + 2. \end{cases}$$

This gives

$$\{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \,|\, 1 \leq i < j \leq \ell - 1\} \subseteq \overline{\Delta_{\rm re}^+}.$$

Remark 3.19. In type $A_\ell^{(1)}$, we have $\Delta_{\rm fin}^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq \ell + 1\}$ and

$$\overline{\Delta_{\rm re}^+} = \{\epsilon_i - \epsilon_j, \delta - (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq \ell + 1\}.$$

Elements in $\overline{\Delta_{\rm re}^+}$ label all arrows in $\vec{C}(\Lambda)$ of type $A_\ell^{(1)}$. More precisely, in [15, Section 3], we draw an arrow

$$\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda} \xrightarrow{\Delta_{i,j}} \Lambda'' = \Lambda_{i-1} + \Lambda_{j+1} + \widetilde{\Lambda} \in \vec{C}(\Lambda)$$

if $i - 1 \not\equiv_{\ell+1} j$ and $\min(X_{\Lambda'} + \Delta_{i,j} - \delta) < 0$. Under this setting, $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell = (1, 1, \ldots, 1)$ and $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i,j}$ with

$$\Delta_{i,j} := \begin{cases} \left(0^i, 1^{j-i+1}, 0^{\ell-j}\right) = \epsilon_i - \epsilon_{j+1} & \text{if } 0 < i \leq j \leq \ell, \\ \left(1^{j+1}, 0^{\ell-j}\right) = \delta - (\epsilon_{j+1} - \epsilon_{\ell+1}) & \text{if } 0 = i \leq j \leq \ell - 1, \\ \left(1^{j+1}, 0^{i-j-1}, 1^{\ell-i+1}\right) = \delta - (\epsilon_{j+1} - \epsilon_i) & \text{if } 0 \leq j < i \leq \ell. \end{cases}$$

Lemma 3.20. *Suppose $\Lambda \in P_{cl,k}^+$ and $\Lambda \neq \Lambda' \in P_{cl,k}^+(\Lambda)$. Then, there is a directed path from Λ to Λ' in $\vec{C}(\Lambda)$.*

Proof. We prove the assertion by induction on $|X_{\Lambda'}|$. More precisely, we may construct a certain Λ'' such that $|X_{\Lambda''}| < |X_{\Lambda'}|$. Using a suitable lemma given in the previous subsection, we obtain a directed path displayed as $\Lambda \longrightarrow \cdots \longrightarrow \Lambda'' \longrightarrow \Lambda'$.

Write $\Lambda' = \sum_{i=0}^{\ell} m_i\Lambda_i$ and $X_{\Lambda'} = (x_0, x_1, \ldots, x_\ell)$. Since $\Lambda' \neq \Lambda$, we have $|X_{\Lambda'}| > 0$. Since $\min(X_{\Lambda'} - \delta) < 0$, we have $\min X_{\Lambda'} \in \{0,1\}$. If moreover, $\min X_{\Lambda'} = 1$, we have $x_i = 1$ for some $1 \le i \le \ell - 1$. We divide the proof into the following 4 cases.

Case 1. Suppose that there are some $0 \le i, j \le \ell$ satisfying $i + 1 < j$, $x_i = x_j = 0$, $x_{i+1} = x_{i+2} = \cdots = x_{j-1} \ge 1$. Then, by (3.1), we have

$$\langle \alpha_i^\vee, \Lambda - \Lambda' \rangle = \langle \alpha_i^\vee, \beta_{\Lambda'} \rangle < 0, \quad \left\langle \alpha_j^\vee, \Lambda - \Lambda' \right\rangle = \left\langle \alpha_j^\vee, \beta_{\Lambda'} \right\rangle < 0.$$

This implies that $m_i, m_j \ge 1$ and $\Lambda' = \Lambda_i + \Lambda_j + \widetilde{\Lambda} \in P^+_{cl,k}(\Lambda)$ for some $\widetilde{\Lambda} \in P^+_{cl,k-2}$. Since $i < j - 1$, Λ'_{j^-,i^+} is well-defined and $\Delta_{j^-,i^+} = (1, 2^i, 1^{j-i-1}, 2^{\ell-j}, 1)$. Since $x_{i+1} = x_{i+2} = \cdots = x_{j-1} \ge 1$, we have $\min(X_{\Lambda'} + \Delta_{j^-,i^+} - \delta) \ge 0$. By Lemma 3.13, we have $\Lambda'_{j^-,i^+} \longrightarrow \Lambda'$ with

$$X_{\Lambda'} = X_{\Lambda'_{j^-,i^+}} - \Delta_{j^-,i^+} + \delta = X_{\Lambda'_{j^-,i^+}} + \Delta_{(i+1)^-,(j-1)^+}.$$

In this case, we have $\Lambda'' := \Lambda'_{j^-,i^+}$.

Case 2. Suppose $x_i = 0$ for some $0 \le i \le \ell - 1$ and $x_t \ge 1$ for all $i + 1 \le t \le \ell$.

$i = \ell - 1$. Then, $\langle \alpha_{\ell-1}^\vee, \beta_{\Lambda'} \rangle \le -2x_\ell \le -2$ and hence, $m_{\ell-1} \ge 2$. We may write $\Lambda' = 2\Lambda_{\ell-1} + \widetilde{\Lambda}$ for some $\widetilde{\Lambda} \in P^+_{cl,k-2}$. Using $\min(X_{\Lambda'} + \Delta_{i^+,i^+} - \delta) \ge 0$, we obtain an arrow from $\Lambda'' := \Lambda'_{(\ell-1)^+,(\ell-1)^+}$ to Λ' by Lemma 3.12.

$i = \ell - 2$. Then, $\langle \alpha_{\ell-2}^\vee, \beta_{\Lambda'} \rangle \le -1$ and $m_{\ell-2} \ge 1$ such that Λ'_{i^+} is well-defined. Using $\min(X_{\Lambda'} + \Delta_{i^+} - \delta) \ge 0$, we obtain an arrow from $\Lambda'' := \Lambda'_{i^+}$ to Λ' by Lemma 3.11.

$i \le \ell - 3$ *and* $x_{\ell-1} > 2x_\ell$. Then, $\langle \alpha_i^\vee, \beta_{\Lambda'} \rangle < 0$ and $\langle \alpha_\ell^\vee, \beta_{\Lambda'} \rangle = 2x_\ell - x_{\ell-1} < 0$. It gives $m_i, m_\ell > 0$ and Λ_{ℓ^-,i^+} is well-defined. We have $\Lambda'' := \Lambda'_{\ell^-,i^+}$ similar to Case 1.

$i \le \ell - 3$, $x_j \le x_{j+1} \le \cdots \le x_{\ell-1} \le 2x_\ell$ *and* $x_{j-1} > x_j$ *for some* $i + 2 \le j \le \ell - 1$. Then, $\langle \alpha_{\ell-1}^\vee, \beta_{\Lambda'} \rangle = (x_{\ell-1} - x_{\ell-2}) - (2x_\ell - x_{\ell-1}) < 0$ if $j = \ell - 1$, and $\langle \alpha_j^\vee, \beta_{\Lambda'} \rangle = (x_j - x_{j-1}) - (x_{j+1} - x_j) < 0$ if $j < \ell - 1$; in both cases, we have $m_j > 0$. We also have $m_i > 0$ due to $\langle \alpha_i^\vee, \beta_{\Lambda'} \rangle < 0$. Thus, Λ'_{j^-,i^+} is well-defined and we may choose $\Lambda'' := \Lambda'_{j^-,i^+}$.

$i \le \ell - 3$ *and* $x_{i+1} \le x_{i+2} \le \cdots \le x_{\ell-1} \le 2x_\ell$.

- If $x_{i+1} \ge 2$, then $\langle \alpha_i^\vee, \beta_{\Lambda'} \rangle \le -2$ and Λ'_{i^+,i^+} is well-defined. We set $\Lambda'' := \Lambda'_{i^+,i^+}$ due to $\min(X_{\Lambda'} + \Delta_{i^+,i^+} - \delta) \ge 0$.
- If $x_{i+1} = x_{i+2} = \cdots = x_j = 1$ and $x_{j+1} \ge 2$ for some $i + 2 \le j \le \ell - 1$, then $\langle \alpha_i^\vee, \beta_{\Lambda'} \rangle < 0$ and $\langle \alpha_j^\vee, \beta_{\Lambda'} \rangle < 0$. It gives $m_i, m_j > 0$, such that $\Lambda'' := \Lambda'_{j^-,i^+}$ is well-defined.
- If $x_{i+1} = x_{i+2} = \cdots = x_\ell = 1$, then $\langle \alpha_{\ell-1}^\vee, \beta_{\Lambda'} \rangle = -1$ and $m_{\ell-1} \ge 1$. It turns out that $\Lambda'' := \Lambda'_{(\ell-1)^-,i^+}$.

Case 3. Suppose $x_i = 0$ for some $1 \le i \le \ell$ and $x_t \ge 1$ for all $0 \le t \le i - 1$. One may check this case using a similar method as in Case 2.

Case 4. Suppose $\min X_{\Lambda'} = 1$ (i.e., $x_i \neq 0$ for all $0 \leq i \leq \ell$). Since $\min(X_{\Lambda'} - \delta) < 0$, there must exist $x_i = 1$ for some $1 \leq i \leq \ell - 1$. We denote by i (resp., j) the minimal (resp., maximal) number in $\{1, 2, \dots, \ell-1\}$ satisfying $x_i = 1$ (resp., $x_j = 1$). If $i = j$, then $\langle \alpha_i^\vee, \beta_{\Lambda'} \rangle \leq -2$ and $m_i \geq 2$. If $i < j$, then $\langle \alpha_i^\vee, \beta_{\Lambda'} \rangle \leq -1$ and $\langle \alpha_j^\vee, \beta_{\Lambda'} \rangle \leq -1$, such that $m_i, m_j \geq 1$. In both cases, $\Lambda'' := \Lambda'_{i^-, j^+}$ is well-defined and $\min(X_{\Lambda'} + \Delta_{i^-, j^+} - \delta) \geq 0$.

We have completed the proof of Lemma 3.20. □

We have a natural embedding of quivers from lower level to higher level as follows. We omit the proof because it is easy to verify the assertion by the definition of arrows.

Corollary 3.21. *Suppose* $\Lambda = \overline{\Lambda} + \widetilde{\Lambda}$ *with* $\Lambda \in P^+_{cl,k}, \overline{\Lambda} \in P^+_{cl,k'}$ *and* $\widetilde{\Lambda} \in P^+_{cl,k-k'}$. *There is a directed path*

$$\Lambda^{(1)} \xrightarrow{\Delta^{(1)}} \Lambda^{(2)} \xrightarrow{\Delta^{(2)}} \dots \xrightarrow{\Delta^{(m-1)}} \Lambda^{(m)} \in \vec{C}(\overline{\Lambda})$$

if and only if there is a directed path

$$\Lambda^{(1)} + \widetilde{\Lambda} \xrightarrow{\Delta^{(1)}} \Lambda^{(2)} + \widetilde{\Lambda} \xrightarrow{\Delta^{(2)}} \dots \xrightarrow{\Delta^{(m-1)}} \Lambda^{(m)} + \widetilde{\Lambda} \in \vec{C}(\Lambda).$$

We are able to show that our quiver $\vec{C}(\Lambda)$ serves the same role as that for type $A^{(1)}_\ell$ in [15].

Theorem 3.22. *Suppose* $\Lambda' \to \Lambda'' \in \vec{C}(\Lambda)$ *and* $s := |X_{\Lambda''}| - |X_{\Lambda'}|$. *There is an element* $\mathbf{i} = (i_1, i_2, \dots, i_s) \in I^s$ *and a sequence* $\beta_{\Lambda'} = \beta_0, \beta_1, \dots, \beta_s = \beta_{\Lambda''} \in Q_+$ *such that* $\beta_t = \beta_{t-1} + \alpha_{i_t}$ *and* $\langle \alpha_{i_t}^\vee, \Lambda - \beta_{t-1} \rangle \geq 1$, *for* $1 \leq t \leq s$.

Proof. We divide the proof into the following 5 cases.

Case 1: $\Lambda'' = \Lambda'_{i^+}$. By Definition 3.17, $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i^+}$ for some $0 \leq i \leq \ell - 2$. This gives $s = 2(i+1)$ and $\beta_{\Lambda''} = \beta_{\Lambda'} + \alpha_0 + 2\alpha_1 + \dots + 2\alpha_i + \alpha_{i+1}$. We set

$$\mathbf{i} = \begin{cases} (0,1) & \text{if } i = 0, \\ (i, i-1, \dots, 2, 1, 0, 1, 2, \dots, i-1, i+1, i) & \text{if } i \neq 0. \end{cases}$$

We obviously obtain $\beta_t = \beta_{t-1} + \alpha_{i_t}$ for $1 \leq t \leq s$. By (3.1), we have $\langle \alpha_{i_t}^\vee, \Lambda - \beta_{\Lambda'} \rangle = \langle \alpha_{i_t}^\vee, \Lambda' \rangle$. We have $\langle \alpha_{i_1}^\vee, \Lambda - \beta_{\Lambda'} \rangle = \langle \alpha_i^\vee, \Lambda' \rangle \geq 1$ since Λ' is of the form $\Lambda_i + \widetilde{\Lambda}'$ in this case. For $2 \leq t \leq s$, we have

$$\begin{aligned} \langle \alpha_{i_t}^\vee, \Lambda - \beta_{t-1} \rangle &= \left\langle \alpha_{i_t}^\vee, \Lambda - \left(\beta_0 + \sum_{j=1}^{t-1} \alpha_{i_j} \right) \right\rangle \\ &= \left\langle \alpha_{i_t}^\vee, \Lambda' - \sum_{j=1}^{t-1} \alpha_{i_j} \right\rangle \geq - \left\langle \alpha_{i_t}^\vee, \sum_{j=1}^{t-1} \alpha_{i_j} \right\rangle, \end{aligned}$$

which implies $\langle \alpha_{i_t}^\vee, \Lambda - \beta_{t-1} \rangle \geq 2$ if $i = 0$, and $\langle \alpha_{i_t}^\vee, \Lambda - \beta_{t-1} \rangle \geq 1$ if $i \neq 0$.

Case 2: $\Lambda'' = \Lambda'_{i^-}$. In this case, $X_{\Lambda''} = X_{\Lambda'} + \Delta_{i^-}$ for some $2 \leq i \leq \ell$. We have $s = 2(\ell - i)$ and $\beta_{\Lambda''} = \beta_{\Lambda'} + \alpha_{i-1} + 2(\alpha_i + \dots + \alpha_{\ell-1}) + \alpha_\ell$. Set

$$\mathbf{i} = \begin{cases} (i, i+1, \dots, \ell-1, \ell, \ell-1, \dots, i+3, i+2, i-1, i) & \text{if } i \neq \ell, \\ (\ell, \ell-1) & \text{if } i = \ell. \end{cases}$$

We then omit the details since they are quite similar to the Case 1.

Case 3: $\Lambda''=\Lambda'_{i^-,j^+}$. Then, $X_{\Lambda''}=X_{\Lambda'}+\Delta_{i^-,j^+}$ for some $0\le i,j\le \ell$ with $i\neq 0$, $j\neq \ell$, $i-1\neq j$. If $i\le j$, then $s=j-i+1$ and $\beta_{\Lambda''}=\beta_{\Lambda'}+\alpha_i+\cdots+\alpha_j$, we set $\mathbf{i}=(i,i+1,\dots,j)$. If $i\ge j+2$, then $s=2\ell+j-i+1$ and $\beta_{\Lambda''}=\beta_{\Lambda'}+\alpha_0+2(\alpha_1+\cdots+\alpha_j)+(\alpha_{j+1}+\cdots+\alpha_{i-1})+2(\alpha_i+\cdots+\alpha_{\ell-1})+\alpha_\ell$, we set

$$\mathbf{i}=\begin{cases}(0,1,\dots,i-1,i,\dots,\ell-1,\ell,\ell-1,\dots,i+1,i) & \text{if } i\neq \ell,\\ (0,1,\dots,\ell) & \text{if } i=\ell.\end{cases}$$

for $j=0$, and $\mathbf{i}=(j,j-1,\dots,1,0,1,\dots,j-1,j+1,j,j+2,\dots,i-1,i,\dots,\ell-1,\ell,\ell-1,\dots,i+1,i)$ for $j\ge 1$. In both cases, we have $\beta_t=\beta_{t-1}+\alpha_{i_t}$ for $1\le t\le s$. Similar to Case 1, we have $\langle\alpha^\vee_{i_1},\Lambda-\beta_{\Lambda'}\rangle=\langle\alpha^\vee_i,\Lambda'\rangle$ or $\langle\alpha^\vee_j,\Lambda'\rangle\ge 1$. For $2\le t\le s$, we have

$$\langle\alpha^\vee_{i_t},\Lambda-\beta_{t-1}\rangle=\left\langle\alpha^\vee_{i_t},\Lambda'-\sum_{r=1}^{t-1}\alpha_{i_r}\right\rangle\ge-\left\langle\alpha^\vee_{i_t},\sum_{r=1}^{t-1}\alpha_{i_r}\right\rangle,$$

it gives $\langle\alpha^\vee_{i_t},\Lambda-\beta_{t-1}\rangle\ge 2$ if $i=\ell, j=0$, and $\langle\alpha^\vee_{i_t},\Lambda-\beta_{t-1}\rangle\ge 1$ otherwise.

Case 4: $\Lambda''=\Lambda'_{i^+,j^+}$. Then, $X_{\Lambda''}=X_{\Lambda'}+\Delta_{i^+,j^+}$ for some $0\le i\le j\le\ell-1$. The case of $j=i+1$ has been proven in Case 1 since $\Delta_{i^+,(i+1)^+}=\Delta_{i^+}$.

Suppose $i=j$. We have $s=2i+1$ and $\beta_{\Lambda''}=\beta_{\Lambda'}+\alpha_0+2(\alpha_1+\cdots+\alpha_i)$, and we set

$$\mathbf{i}=\begin{cases}(0) & \text{if } i=0,\\ (i,i-1,\dots,1,0,1,\dots,i) & \text{if } i\neq 0.\end{cases}$$

It gives $\langle\alpha^\vee_{i_1},\Lambda-\beta_{\Lambda'}\rangle=\langle\alpha^\vee_i,\Lambda'\rangle\ge 2$ by our assumption. For $2\le t\le s$, we obtain $\langle\alpha^\vee_{i_t},\Lambda-\beta_{t-1}\rangle\ge-\langle\alpha^\vee_{i_t},\sum_{r=1}^{t-1}\alpha_{i_r}\rangle=1$ if $t\neq s$, and $\langle\alpha^\vee_{i_t},\Lambda-\beta_{t-1}\rangle=\langle\alpha^\vee_i,\Lambda'\rangle\ge 2$ if $t=s$. In fact, set $t=s\ge 2$, we have

$$\langle\alpha^\vee_{i_s},\Lambda-\beta_{s-1}\rangle=\left\langle\alpha^\vee_i,\Lambda-\left(\beta_0+\sum_{r=1}^{s-1}\alpha_{i_r}\right)\right\rangle=\left\langle\alpha^\vee_i,\Lambda'-\sum_{r=1}^{s-1}\alpha_{i_r}\right\rangle,$$

combining this with $\langle\alpha^\vee_1,\sum_{r=1}^{s-1}\alpha_{i_r}\rangle=a_{11}+a_{10}=0$ if $i=1$ and $\langle\alpha^\vee_i,\sum_{r=1}^{s-1}\alpha_{i_r}\rangle=a_{ii}+2a_{i(i-1)}=0$ if $2\le i\le\ell-1$, we obtain the result.

Suppose $i+2\ge j$. We have a path

$$\Lambda'\longrightarrow\Lambda'_{i^+}\longrightarrow(\Lambda'_{i^+})_{(i+2)^-,j^+}=\Lambda''.$$

Then, the statement holds by composing the results in Case 1 and Case 3.

Case 5: $\Lambda''=\Lambda'_{i^-,j^-}$. Then, $X_{\Lambda''}+\Delta_{i^-,j^-}$ for some $1\le i\le j\le\ell$, and the case of $i=j-1$ has been proven in Case 1 due to $\Delta_{(j-1)^-,j^-}=\Delta_{j^-}$. If $i=j$, then $s=2(\ell-j)+1$ and $\beta_{\Lambda''}=\beta_{\Lambda'}+2(\alpha_j+\cdots+\alpha_{\ell-1})+\alpha_\ell$, we set

$$\mathbf{i}=\begin{cases}(\ell) & \text{if } j=\ell,\\ (j,j+1,\dots,\ell-1,\ell,\ell-1,\dots,j) & \text{if } j<\ell.\end{cases}$$

One may show the statement using a similar analysis with Case 1. If $i\le j-2$, there is a path

$$\Lambda'\longrightarrow\Lambda'_{j^-}\longrightarrow(\Lambda'_{j^-})_{(i)^-,(j-2)^+}=\Lambda''.$$

Then, the statement follows from the results in Case 2 and 3.

We have completed the proof of Theorem 3.22. □

3.3. **Comparison with previous level one results.** We may understand the construction of [13, 22] in our broader setting as follows.

In [13, Proposition 5.1], it was shown that

$$\max^+(\Lambda_0) = \left\{ \Lambda_0 + \varpi_i - \frac{i}{2}\delta \,\middle|\, 0 \le i \le \ell,\ i \in 2\mathbb{Z}_{\ge 0} \right\},$$

where $\varpi_0 := 0$ and

$$\varpi_i := \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i\Big(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{\ell-1} + \frac{i}{2}\alpha_\ell\Big).$$

We remark that this is the solution of $\mathsf{A}X^t = Y^t$ for $Y = \mathrm{hub}(\Lambda_i) - \mathrm{hub}(\Lambda_0)$ in the sense of Lemma 3.4. Substituting this into our setting, we have

$$\beta_{\Lambda_i}^{\Lambda_0} = \frac{i}{2}\delta - \varpi_i.$$

This gives an arrow $\Lambda_i \longrightarrow \Lambda_{i+2}$ in $\vec{C}(\Lambda_0)$ because

$$\Big(\frac{i+2}{2}\delta - \varpi_{i+2}\Big) - \Big(\frac{i}{2}\delta - \varpi_i\Big) = \alpha_{i+1} + 2\alpha_{i+2} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell \in Q_+.$$

Thus, the quiver $\vec{C}(\Lambda_0)$ is displayed as

$$\boxed{\Lambda_0} \longrightarrow \boxed{\Lambda_2} \longrightarrow \boxed{\Lambda_4} \longrightarrow \cdots \longrightarrow \boxed{\Lambda_{2\lfloor \ell/2 \rfloor}}. \tag{3.3}$$

In [22, Proposition 2.8], the authors showed that, for $0 \le s \le \ell$,

$$\max^+(\Lambda_s) = \left\{ \Lambda_s + \xi_{s,\pm i} - \frac{i}{2}\delta \,\middle|\, 0 \le i \le \ell,\ i \in 2\mathbb{Z}_{\ge 0} \right\},$$

where $\xi_{0,i} = \varpi_i$, and

$$\frac{i}{2}\delta - \xi_{s,i} = \frac{i}{2}\alpha_0 + i\sum_{j=1}^{s}\alpha_j + (i-1)\alpha_{s+1} + (i-2)\alpha_{s+2} + \cdots + \alpha_{s+i-1},$$

$$\frac{i}{2}\delta - \xi_{s,-i} = \alpha_{s-i+1} + 2\alpha_{s-i+2} + \cdots + (i-1)\alpha_{s-1} + i\sum_{j=s}^{\ell-1}\alpha_j + \frac{i}{2}\alpha_\ell.$$

This leads to the identities

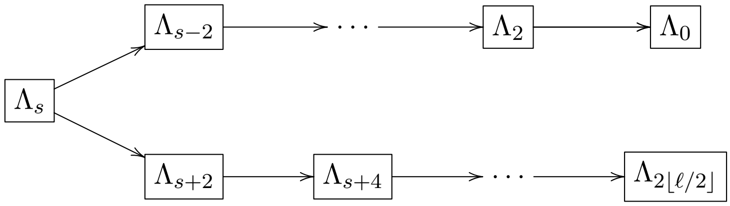
$$\beta_{\Lambda_{s+i}}^{\Lambda_s} = \frac{i}{2}\delta - \xi_{s,i} \quad \text{and} \quad \beta_{\Lambda_{s-i}}^{\Lambda_s} = \frac{i}{2}\delta - \xi_{s,-i}.$$

Moreover, if we multiply A with coefficient vectors of $\beta_{\Lambda_{s+i}}^{\Lambda_s}$ or $\beta_{\Lambda_{s-i}}^{\Lambda_s}$, we always obtain a vector with exactly one 1 and one -1 while all other entries are 0. One may check that

$$\Big(\frac{i+2}{2}\delta - \xi_{s,i+2}\Big) - \Big(\frac{i}{2}\delta - \xi_{s,i}\Big) = \alpha_0 + 2\sum_{j=1}^{s+i}\alpha_j + \alpha_{s+i+1} \in Q_+,$$

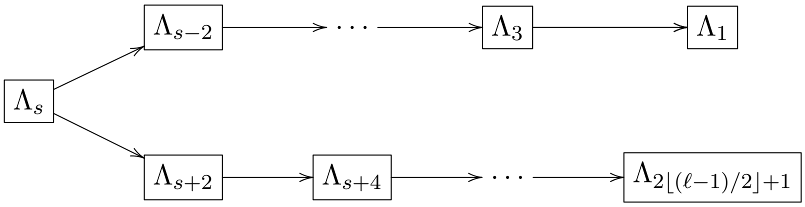
$$\Big(\frac{i+2}{2}\delta - \xi_{s,-i-2}\Big) - \Big(\frac{i}{2}\delta - \xi_{s,-i}\Big) = \alpha_{s-i-1} + 2\sum_{j=s-i}^{\ell-1}\alpha_j + \alpha_\ell \in Q_+.$$

Hence, there are arrows $\Lambda_{s+i} \longrightarrow \Lambda_{s+i+2}$ and $\Lambda_{s-i} \longrightarrow \Lambda_{s-i-2}$ in $\vec{C}(\Lambda_s)$. We conclude that the quiver $\vec{C}(\Lambda_s)$ is displayed as



(3.4)

if s is even, and



(3.5)

if s is odd.

4. Proof strategy for the Main Theorem A

In this section, we review some well-known features in the representation theory of $R^\Lambda(\beta)$ in type $C_\ell^{(1)}$. We recall the results from [13] and [22] for level one cases. We then focus on the case $k \geq 2$ and prove our main theorem given in the introduction: we prove (1) of Main Theorem A in Section 5; we give the proofs for (2)(a) and (2)(b) of Main Theorem A in Section 6 and Section 7 respectively; we prove (2)(c) of Main Theorem A in the remaining sections. We also introduce some reduction lemmas to reduce the problem on $R^\Lambda(\beta)$ to cases with small levels of Λ and small heights of β, similar to the strategy in [15] for type $A_\ell^{(1)}$. These reduction methods play a crucial role in the proof process.

Let us start with the fact that $R^\Lambda(\beta)$ is a symmetric algebra (see [50, Appendix]). It gives that the representation type of $R^\Lambda(\beta)$ is preserved under derived equivalence, see [38, 46]. Then, the problem we consider relies on figuring out when $R^\Lambda(\beta)$ and $R^\Lambda(\beta')$ are derived equivalent. By Chuang and Rouquier's result [21], we know that $R^\Lambda(\beta)$ is derived equivalent to $R^\Lambda(\beta')$ if $\Lambda-\beta$ and $\Lambda-\beta'$ lie in the same W-orbit of $P(\Lambda)$. Furthermore, by (2.1) and Proposition 3.6, the representatives of W-orbits of $P(\Lambda)$ with $\Lambda \in P^+_{cl,k}$ are given by $\{\Lambda-\beta_{\Lambda'}-m\delta \mid \Lambda' \in P^+_{cl,k}(\Lambda), m \in \mathbb{Z}_{\geq 0}\}$, where $P^+_{cl,k}(\Lambda)$ is defined at the beginning of Section 3. All in all, it suffices to consider the representation type of $R^\Lambda(\gamma)$ for $\gamma \in O(\Lambda)$, where

$$O(\Lambda) := \left\{\beta_{\Lambda'}+m\delta \,\middle|\, \Lambda' \in P^+_{cl,k}(\Lambda),\ m \in \mathbb{Z}_{\geq 0}\right\}. \tag{4.1}$$

Remark 4.1. If $\Lambda' = \Lambda$, i.e., $\beta_{\Lambda'} = 0$, then $R^\Lambda(\beta_\Lambda) \cong \Bbbk$ is a simple algebra.

4.1. **Results in level one cases.** We have given the quiver $\vec{C}(\Lambda_s)$ for $0 \leq s \leq \ell$ in the previous section, see (3.3), (3.4), (3.5). Then, the main results of [13, 22] can be summarized as follows.

Theorem 4.2. *Set $\Lambda_s \in P^+_{cl,1}$ with $0 \leq s \leq \ell$ and $\Lambda' \in P^+_{cl,1}(\Lambda_s)$. Then, the cyclotomic KLR algebra $R^{\Lambda_s}(\beta_{\Lambda'}+m\delta)$ is representation-finite if $m=0$ and $\Lambda' \in \{\Lambda_s, \Lambda_{s-2}, \Lambda_{s+2}\}$, tame if $m=1$, $\ell=2$ and $\Lambda'=\Lambda_s$, wild otherwise.*

 https://doi.org/10.5802/art.34

It implies that $R^{\Lambda_s}(\beta_{\Lambda'}+m\delta)$ is wild for all $m\geq 1$ if $\beta_{\Lambda'}\neq 0$, and for all $m\geq 2$ if $\beta_{\Lambda'}=0$. Then, the representation type of $R^{\Lambda_s}(\beta_{\Lambda'})$ and $R^{\Lambda_s}(\delta)$ are determined as in Theorem 4.2.

4.2. **Reduction methods.** In [15, Section 5], level lowering argument and the quiver $\vec{C}(\Lambda)$ are used to show the wildness of $R^{\Lambda}(\beta^{\Lambda}_{\Lambda'}+m\delta)$ in type $A^{(1)}_{\ell}$, for $m\geq 1+\delta_{\Lambda,\Lambda'}$, where $\delta_{\Lambda,\Lambda'}$ is the Kronecker delta. Similarly, we have

Lemma 4.3. *Suppose $\Lambda=\overline{\Lambda}+\widetilde{\Lambda}$ for some $\Lambda\in P^+_{cl,k}$, $\overline{\Lambda}\in P^+_{cl,k'}$ and $\widetilde{\Lambda}\in P^+_{cl,k-k'}$. Then, the representation-infiniteness (resp., wildness) of $R^{\overline{\Lambda}}(\gamma)$ implies the representation-infiniteness (resp., wildness) of $R^{\Lambda}(\gamma)$.*

Proof. This is similar to the proof of [15, Lemma 4.1]. ☐

Lemma 4.4. *Suppose $\Lambda'\longrightarrow\Lambda''$ in $\vec{C}(\Lambda)$. Then, the representation-infiniteness (resp., wildness) of $R^{\Lambda}(\beta_{\Lambda'}+m\delta)$ implies the representation-infiniteness (resp., wildness) of $R^{\Lambda}(\beta_{\Lambda''}+m\delta)$, for any $m\in\mathbb{Z}_{\geq 0}$.*

Proof. This is similar to the proof of [15, Lemma 4.2], by using Theorem 3.22, [28, Proposition 2.3] and [33, Theorem 5.2]. ☐

Corollary 4.5. *If $R^{\Lambda}(\beta_{\Lambda'}+m\delta)$ for $\Lambda'\in\vec{C}(\Lambda)$ and $m\in\mathbb{Z}_{\geq 0}$ is representation-infinite (resp., wild) and there is a directed path from Λ' to Λ'' in $\vec{C}(\Lambda)$, then $R^{\Lambda}(\beta_{\Lambda''}+m\delta)$ is also representation-infinite (resp., wild).*

5. Proof of part (1) of Main Theorem A

We are able to show the following result.

Theorem 5.1. *Suppose $\Lambda\in P^+_{cl,k}$ with $k\geq 2$. Then, $R^{\Lambda}(\beta_{\Lambda'}+m\delta)$ is wild for any $m\geq 1$ and $\Lambda'\in P^+_{cl,k}(\Lambda)$.*

Proof. Set $\Lambda=\Lambda_s+\widetilde{\Lambda}$ with $0\leq s\leq\ell$. If $m\geq 2$, then $R^{\Lambda_s}(m\delta)$ is wild by Theorem 4.2, and so is $R^{\Lambda}(m\delta)$ by Lemma 4.3. Since there exists a directed path from Λ to any $\Lambda\neq\Lambda'\in P^+_{cl,k}(\Lambda)$, we deduce that $R^{\Lambda}(\beta_{\Lambda'}+m\delta)$ is wild for any $m\geq 2$ and $\Lambda'\in P^+_{cl,k}(\Lambda)$, by Corollary 4.5. If $m=1$ and $\ell\geq 3$, then $R^{\Lambda_s}(\delta)$ is wild following Theorem 4.2, which implies that $R^{\Lambda}(\beta_{\Lambda'}+\delta)$ is wild for any $\Lambda'\in P^+_{cl,k}(\Lambda)$.

Suppose $m=1$ and $\ell=2$. Then, $\delta=\alpha_0+2\alpha_1+\alpha_2$. We have to consider the cases $\Lambda\in\{2\Lambda_0,2\Lambda_1,2\Lambda_2,\Lambda_0+\Lambda_1,\Lambda_1+\Lambda_2,\Lambda_0+\Lambda_2\}$.

Case 1. Set $A:=eR^{2\Lambda_0}(\delta)e$ with $e=e(0121)$. Then, $\dim_q A=1+2q^2+2q^4+2q^6+q^8$. We show that A has a basis $\{x_2^a e, x_2^a x_4 e \mid 0\leq a\leq 3\}$. First, we have $x_1^2e=x_1^2e'=0$, where $e':=e(\nu')=e(0112)$. Since $e(s_1\nu)=e(s_1\nu')=e(s_2\nu)=0$, we have $\psi_1e=\psi_2e=\psi_1e'=0$ and hence $\psi_1^2e=\psi_2^2e=\psi_1^2e'=0$. This implies $x_1e=x_2^2e$, $x_3e=x_2^2e$, so that we may replace x_1e and x_3e with x_2^2e, and $x_1e'=x_2^2e'$. Let $f=x_1-x_2^2$ and $\partial_2 f=\frac{s_2f-f}{x_2-x_3}$. Then Lemma 2.17 implies $(\partial_2 f)e'=0$ since $\nu'_2=\nu'_3$ and $fe'=0$. Hence, $x_3e'=-x_2e'$. This implies that

$$x_4\psi_3\psi_2\psi_3e=x_4\psi_3e'\psi_2\psi_3=\psi_3x_3e'\psi_2\psi_3=-x_2\psi_3\psi_2\psi_3e.$$

On the other hand, we have $\psi_3\psi_2\psi_3e=(\psi_3\psi_2\psi_3-\psi_2\psi_3\psi_2)e=(x_2+x_4)e$. Hence,

$$x_4(x_2+x_4)e=-x_2(x_2+x_4)e,$$

and we may replace $x_4^2 e$ with $-(x_2^2 + 2x_2x_4)e$. Moreover, if $e\psi_w e \neq 0$, then we can choose $\psi_w = 1$ or $\psi_w = \psi_2\psi_3\psi_2$. The latter one can not happen since $\psi_2 e = 0$. Therefore, we obtain the required basis following the graded dimension. Further, we have a surjective algebra homomorphism from A to $B := \Bbbk[X,Y]/(X^3, Y^2, X^2Y)$ sending x_2 and $x_2 + x_4$ to X and Y, respectively. Since B is a wild local algebra by Proposition 2.11, A is also wild.

Case 2. Set $A := (e_1 + e_2)R^{2\Lambda_1}(\delta)(e_1 + e_2)$ with $e_1 = e(1210)$ and $e_2 = e(1201)$. We have

$$\dim_q e_1Ae_1 = \dim_q e_2Ae_2 = 1 + 2q^2 + 2q^4 + 2q^6 + q^8,$$
$$\dim_q e_1Ae_2 = \dim_q e_2Ae_1 = q^2 + 2q^4 + q^6.$$

Then, A is wild by Lemma 2.15.

Case 3. Set $A := (e_1 + e_2)R^{\Lambda_0+\Lambda_1}(\delta)(e_1 + e_2)$ with $e_1 = e(0121)$ and $e_2 = e(1201)$. Then,

$$\dim_q e_1Ae_1 = 1 + 2q^2 + 3q^4 + 2q^6 + q^8,$$
$$\dim_q e_2Ae_2 = 1 + q^2 + 2q^4 + q^6 + q^8,$$
$$\dim_q e_1Ae_2 = \dim_q e_2Ae_1 = q^2 + q^4 + q^6.$$

Then, A is wild by Lemma 2.15.

Case 4. Set $A := eR^{\Lambda_0+\Lambda_2}(\delta)e$ with $e = e(2101)$. We obtain

$$\dim_q eAe = 1 + 3q^2 + 4q^4 + 3q^6 + q^8.$$

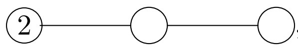
Then, A is wild by Lemma 2.12.

In the above 4 cases, $R^\Lambda(\delta)$ is wild since we find an idempotent truncation of $R^\Lambda(\delta)$ being wild. Using Proposition 2.4, we conclude that all the remaining cases are wild. □

Combining with the bijection between $P^+_{cl,k}(\Lambda)$ and $\max^+(\Lambda)$ as we mentioned in Proposition 3.6, we conclude that $R^\Lambda(\beta)$ is wild if $\Lambda - \beta$ is not a maximal dominant weight. This gives a proof of Main Theorem A (1). Now, in the case of $k \geq 2$, we only need to determine the representation type of $R^\Lambda(\beta_{\Lambda'})$ for $\Lambda' \in P^+_{cl,k}(\Lambda)$. This will be accomplished in the following sections.

6. Proof of the second part–finite representation type

In the Case (f1), $R^\Lambda(\beta) \cong \Bbbk[X]/(X^{m_a})$. For the first case in (f2), we have $e_1 = e(01) = 1$ by $e_2 = e(10) = x_1^{\langle\alpha_1, m_0\Lambda_0\rangle} e(10) = 0$, and $\psi = \psi e_2 = e_1\psi = 0$, $(x_2^2 - x_1)e_1 = \psi^2 e_1 = 0$, so that $R^\Lambda(\beta) \cong \Bbbk[X]/(X^{2m_0})$. For the second case in (f2), we have $x_1 = 0$ and that $P_1 = \langle e_1, \psi e_1, x_2e_1, \psi^2 e_1\rangle$, $P_2 = \langle e_2, \psi e_2, \psi^2 e_2\rangle$ are indecomposable projective $R^\Lambda(\beta)$-modules. Then, we see that $R^\Lambda(\beta)$ is a Brauer tree algebra whose Brauer tree is given as



,

which is of finite representation type. By symmetry, we have the results for the Case (f3). The Case (f4) is treated in [15, Proposition 6.8] and it is also a Brauer tree algebra. If $R^\Lambda(\beta)$ is derived equivalent to this algebra, we recall that $R^\Lambda(\beta)$ is a cellular algebra when $\operatorname{char}\Bbbk \neq 2$ by [29, Theorem A] because we choose a special value for the parameter t here and Morita invariance of the cellularity holds when $\operatorname{char}\Bbbk \neq 2$. Thus, the Brauer tree is the straight line with $b - a + 2$ vertices without an exceptional vertex. Hence, $R^\Lambda(\beta)$ is Morita equivalent to this algebra when $\operatorname{char}\Bbbk \neq 2$ or $R^\Lambda(\beta)$ is a basic algebra.

The remaining two cases follow from [13, Lemma 3.3 (1)] and [22, Proposition 4.1, Theorem 4.4]: in the Case (f5), $R^{\Lambda}(\beta) \cong R^{\Lambda_a}(\beta_{\Lambda_{a+2}})$. It is the Brauer tree algebra whose Brauer tree is the straight line with $a+2$ vertices without an exceptional vertex, and in the Case (f6), $R^{\Lambda}(\beta) \cong R^{\Lambda_b}(\beta_{\Lambda_{b-2}})$, which is the Brauer tree algebra whose Brauer tree is the straight line with $\ell - b + 2$ vertices without exceptional vertex.

7. Proof of the second part–tame representation type

In this section, we will omit most calculations to make this paper shorter, and one may refer to the arXiv version [9] for more details. Before starting the proof for the tame cases, we consider $A = R^{t\Lambda_{\ell-1}+\Lambda_\ell}(\alpha_{\ell-1}+\alpha_\ell)$, for $t \geq 2$. Define

$$e_1 = e(\ell-1, \ell), \quad e_2 = e(\ell, \ell-1).$$

The graded dimensions are given as follows.

$$\dim_q e_1Ae_1 = 1 + q^2 + 2\sum_{i=2}^{t-1} q^{2i} + q^{2t} + q^{2t+2},$$

$$\dim_q e_2Ae_2 = \sum_{i=0}^{t+1} q^{2i},$$

$$\dim_q e_1Ae_2 = \dim_q e_2Ae_1 = \sum_{i=1}^{t} q^{2i}.$$

In particular, $\dim A = 5t+2$. Then, A is generated by e_1, e_2, ψ, x_1, x_2 such that

$$e_1Ae_1 = \left\langle x_1^a x_2^b e_1 \,\middle|\, 0 \leq a \leq t-1,\ 0 \leq b \leq 1 \right\rangle, \qquad e_2Ae_2 = \left\langle x_2^b e_2 \,\middle|\, 0 \leq b \leq t+1 \right\rangle$$

$$e_1Ae_2 = \left\langle \psi x_2^b e_2 \,\middle|\, 0 \leq b \leq t-1 \right\rangle, \qquad e_2Ae_1 = \langle \psi x_1^a e_1 \mid 0 \leq a \leq t-1 \rangle.$$

If we set

$$\alpha = x_1e_1, \quad \mu = e_1\psi e_2, \quad \nu = e_2\psi e_1, \quad \beta = x_2e_2.$$

Then

$$\alpha^t = x_1^t e_1 = 0, \quad \beta^{t+2} = x_2^{t+2}e_2 = 0, \quad \beta^2 - \nu\mu = x_2^2e_2 - \psi^2 e_2 = 0,$$
$$\alpha\mu - \mu\beta = e_1(x_1\psi - \psi x_2)e_2 = 0, \quad \beta\nu - \nu\alpha = e_2(x_2\psi - \psi x_1)e_1 = 0.$$

Moreover, $\{\alpha, \beta, \mu, \nu\}$ generate A as an algebra.

Lemma 7.1. *Let A' be the algebra with two vertices $1, 2$, a loop α at vertex 1, a loop β at vertex 2, an arrow μ from vertex 1 to vertex 2, an arrow ν from vertex 2 to vertex 1, such that they satisfy the following relations*

$$\alpha^t = 0,\ \ \beta^{t+2} = 0,\ \ \beta^2 = \nu\mu,\ \ \alpha\mu = \mu\beta,\ \ \beta\nu = \nu\alpha.$$

If $t \geq 3$, then A' is isomorphic to A. Moreover, A is wild.

Recall the wild algebra (31) from [31, Table W], which has the same quiver as A and is bounded by

$$\beta\nu = \nu\alpha,\ \ \beta^2 = \nu\mu = \mu\beta = \alpha\mu = \alpha^3 = \nu\alpha^2 = 0.$$

It is clear that if $t \geq 3$ then the following relations hold in this algebra.

$$\alpha^t = 0,\ \ \beta^{t+2} = 0,\ \ \beta^2 = \nu\mu,\ \ \alpha\mu = \mu\beta,\ \ \beta\nu = \nu\alpha.$$

Hence, A has the wild algebra as a factor algebra, so that A is wild if $t \geq 3$.

Lemma 7.2. *Let A' be the algebra with two vertices $1, 2$, a loop α on vertex 1, a loop β on vertex 2, an arrow μ from vertex 1 to vertex 2, an arrow ν from vertex 2 to vertex 1, such that they are bounded by the relations*

$$\alpha^2 = 0, \quad \beta^2 = \nu\mu, \quad \alpha\mu = \mu\beta, \quad \beta\nu = \nu\alpha.$$

If $t = 2$, then A' is isomorphic to A. Moreover, A is tame.

We observe that $A/\operatorname{Rad}^2 A$ is a representation-infinite algebra, since its separated quiver (see [17]) is not a disjoint union of Dynkin quivers. Since A is a symmetric algebra, indecomposable A-modules are either indecomposable projective A-modules or indecomposable $(A/\operatorname{Soc} A)$-modules. Hence, tameness of $A/\operatorname{Soc} A$ implies tameness of A. We then conclude that A is tame since $A/\operatorname{Soc} A = A/\operatorname{Rad}^3 A$ degenerates to a factor algebra of the algebra (18) in [31, Table T].

7.1. **Proof of the tame cases.** We are ready to prove part (b) in Main Theorem A (2). The cases (t1)–(t9) will appear in $R^\Lambda(\beta_{\Lambda'})$, for the first neighbor Λ', that is, those Λ' for which there is an arrow $\Lambda \to \Lambda'$. As we see below, they are Brauer graph algebra except for (t7) and (t8). All the other cases will appear in $R^\Lambda(\beta_{\Lambda''})$, for the second neighbor Λ'', namely those Λ'' for which there is a directed path $\Lambda \to \Lambda' \to \Lambda''$.

In the cases (t9), (t15)–(t19), we have the isomorphism of algebras $R^\Lambda(\beta) \cong R^\Lambda_A(\beta)$. Hence, the results follow from [15]. For the bound quiver presentation of the cases (t9), (t15)–(t19), see [15, 8.2]. Furthermore, it suffices to consider (t2), (t3), (t5), (t7), (t10), (t12), (t13), (t20) in the remaining cases by symmetry. Cases except for (t2) and (t20) are almost complete already.

- (t3) This follows from Lemma 2.19.
- (t5) We have $R^\Lambda(\beta) \cong R^{m_0\Lambda_0+\Lambda_a}(\alpha_0+\cdots+\alpha_a)$, for $1 \le a \le \ell-1$. If $a = 1$ and $m_0 \ge 2$, it follows from Lemma 2.19. If $2 \le a \le \ell - 1$, then it follows from Lemma 2.20.
- (t7) This follows from Lemma 7.2.
- (t10) By Lemma 2.18, $R^\Lambda(\beta)$ is Morita equivalent to

$$R^{2\Lambda_0}(\alpha_0) \otimes R^{2\Lambda_i}(\alpha_i) \cong \Bbbk[X,Y]/\Big(X^2, Y^2\Big),$$

  which is tame by Proposition 2.11.
- (t12) Since $\ell \ge 4$, we may apply Lemma 2.18. Hence, $m_1 = m_{\ell-1} = 0$ implies that $R^\Lambda(\beta)$ is Morita equivalent to

$$R^{\Lambda_0}(\alpha_0+\alpha_1) \otimes R^{\Lambda_\ell}(\alpha_{\ell-1}+\alpha_\ell) \cong \Bbbk[X,Y]/\Big(X^2, Y^2\Big).$$

  Here, we use the proof of (f2) for each of $R^{\Lambda_0}(\alpha_0+\alpha_1)$ and $R^{\Lambda_\ell}(\alpha_{\ell-1}+\alpha_\ell)$ to obtain $\Bbbk[X,Y]/(X^2,Y^2)$.
- (t13) We apply Lemma 2.18 again. Then $m_1 = 0$ implies that $R^\Lambda(\beta)$ is Morita equivalent to $R^{\Lambda_0}(\alpha_0+\alpha_1) \otimes R^{2\Lambda_i}(\alpha_i)$. Then, we use the proof of (f2) again to conclude that $R^\Lambda(\beta)$ is Morita equivalent to $\Bbbk[X,Y]/(X^2,Y^2)$.

In the next two subsections, we prove the remaining cases (t2) and (t20).

7.2. **The Case** (t2)**.** Set $A := R^{2\Lambda_{\ell-1}}(2\alpha_{\ell-1}+\alpha_\ell)$ with

$$e_1 = e(\ell-1, \ell, \ell-1), \quad e_2 = e(\ell-1, \ell-1, \ell), \quad e_2' = x_2\psi_1 e_2.$$

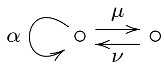
We then have the following graded dimensions.

$$\begin{aligned}\dim_q e_1Ae_1 &= 1+2q^2+q^4,\\ \dim_q e_2Ae_2 &= \left(q+q^{-1}\right)^2\left(1+q^4\right),\\ \dim_q e_1Ae_2 &= \dim_q e_2Ae_1 = \left(q+q^{-1}\right)\left(q+q^3\right).\end{aligned}$$

Let $P_1 := Ae_1$ and $P_2 := Ae_2'\langle 1\rangle$. By looking at the graded dimensions, we know that $Ae_2 = P_2\langle 1\rangle \oplus P_2\langle -1\rangle$ and

$$\dim_q \operatorname{End}(P_1) = 1+2q^2+q^4, \quad \dim_q \operatorname{End}(P_2) = 1+q^4,$$
$$\dim_q \operatorname{Hom}(P_1,P_2) = \dim_q \operatorname{Hom}(P_2,P_1) = q+q^3.$$

By crystal computation [39], We can calculate the number of simple modules, which is two. Hence, the Gabriel quiver is



and the relations are $\nu\mu = \alpha^2 = 0$ and $\alpha\mu\nu = \mu\nu\alpha$.

We see that it is a special biserial algebra[8] Being a symmetric algebra, it is a Brauer graph algebra, whose Brauer graph is as claimed.

7.3. **The Case** (t20)**.** We show that the algebra (t20), namely $A := R^{2\Lambda_0}(2\alpha_0+2\alpha_1)$ in $\operatorname{char}\Bbbk \neq 2$, is tame. First of all, crystal computation shows that the number of simple modules is two. Its basic algebra is $B = \operatorname{End}(P_1 \oplus P_2)^{\mathrm{op}}$ where

$$P_1 = f_1^{(2)} f_0^{(2)} v_\Lambda, \quad P_2 = f_0 f_1^{(2)} f_0 v_\Lambda.$$

Let $e_1 = e(0011)$ and $e_2 = e(0110)$ and $e_3 = e(0101)$. Graded dimension formula computes

$$\dim_q e_1Be_1 = 1+q^2+2q^4+q^6+q^8, \quad \dim_q e_2Be_2 = 1+2q^4+q^8,$$
$$\dim_q e_1Be_2 = \dim_q e_2Be_1 = q^2+q^6.$$

We set $f_1 = x_2\psi_1x_4\psi_3e_1$ and $f_2 = x_3\psi_2e_2$. Then, $P_1 = Af_1\langle 3\rangle$ and $P_2 = Af_2\langle 1\rangle$. Thus, the graded dimensions of f_iAf_j, for $i,j = 1,2$, are as follows.

$$\begin{aligned}\dim_q f_1Af_2 &= \dim_q \operatorname{Hom}_A(Af_1,Af_2) &&= \dim_q \operatorname{Hom}_A\left(P_1\langle -3\rangle, P_2\langle -1\rangle\right)\\ &= \dim_q \operatorname{Hom}_A(P_1,P_2)\langle 2\rangle &&= q^4+q^8,\\ \dim_q f_2Af_1 &= \dim_q \operatorname{Hom}_A(Af_2,Af_1) &&= \dim_q \operatorname{Hom}_A\left(P_2\langle -1\rangle, P_1\langle -3\rangle\right)\\ &= \dim_q \operatorname{Hom}_A(P_2,P_1)\langle -2\rangle &&= 1+q^4,\\ \dim_q f_1Af_1 &= \dim_q \operatorname{Hom}_A(Af_1,Af_1) &&= \dim_q \operatorname{Hom}_A\left(P_1\langle -3\rangle, P_1\langle -3\rangle\right)\\ &= \dim_q \operatorname{Hom}_A(P_1,P_1) &&= 1+q^2+2q^4+q^6+q^8,\\ \dim_q f_2Af_2 &= \dim_q \operatorname{Hom}_A(Af_2,Af_2) &&= \dim_q \operatorname{Hom}_A\left(P_2\langle -1\rangle, P_2\langle -1\rangle\right)\\ &= \dim_q \operatorname{Hom}_A(P_2,P_2) &&= 1+2q^4+q^8.\end{aligned}$$

Let $f = f_1 + f_2$. Then B is isomorphic to fAf as ungraded algebras, and we are going to prove the tameness of A by obtaining the bound quiver presentation of fAf. The computation is lengthy and not straightforward. We start with formulas we will use in the computation. See the arXiv version [9] for the details.

[8]See [27] for the definition of special biserial algebra. It is known that symmetric special biserial algebras are Brauer graph algebras and vice versa. See [49].

Lemma 7.3. *The following formulas hold.*

(1) $(x_1+x_2)e_1=0$, $(x_2+x_3)e_2=0$, $(x_1+x_3)e_3=0$, $x_1e_2=x_2^2e_2=x_3^2e_2$.
(2) $x_3^4e_1=0$, $x_4^2e_2=0$, $(x_3^3+x_3^2x_4+x_3x_4^2+x_4^3)e_1=0$, $(x_3x_4^3+x_3^2x_4^2+x_3^3x_4)e_1=0$.
(3) $f_1\psi_1=0$, $f_2\psi_2=0$, $f_1\psi_3=0$.
(4) $(x_3+x_4)f_1=f_1(x_3+x_4)$, $x_3x_4f_1=f_1x_3x_4$, $x_1f_2=f_2x_1$ *and* $x_4f_2=f_2x_4$.
(5) $x_1f_1=0$, $f_1x_3f_1=0$, $f_1x_3^2f_1=-x_3x_4f_1$, $f_1x_3^3f_1=-(x_3+x_4)x_3x_4f_1$.
(6) $f_2x_3f_2=0$.

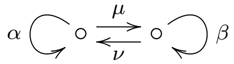
Proposition 7.4. *The bases of* f_iAf_j $(i,j=1,2)$ *are given as follows.*

$$\begin{aligned}
f_1Af_1&=\operatorname{span}\Big\{f_1,\alpha=(x_3+x_4)f_1,\ \alpha'=x_3x_4f_1,\ \alpha^2,\ \alpha\alpha',\ \alpha^2\alpha'\Big\},\\
f_2Af_2&=\operatorname{span}\{f_2,\beta=x_1f_2,\ \beta'=x_4f_2,\ \beta\beta'=\beta'\beta\},\\
f_1Af_2&=\operatorname{span}\{\mu=f_1\psi_2\psi_3f_2,\ f_1\psi_2\psi_3x_1f_2=\mu\beta\},\\
f_2Af_1&=\operatorname{span}\{\nu=f_2\psi_3\psi_2\psi_1f_1,\ f_2x_1\psi_3\psi_2\psi_1f_1=\beta\nu\}.
\end{aligned}$$

Moreover, $\alpha^3=2\alpha\alpha'$ *and* ${\alpha'}^2=\alpha^2\alpha'$ *hold.*

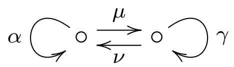
We can find relations among the generators $\alpha,\alpha',\beta,\beta',\mu,\nu$ in order to obtain the bound quiver presentation of $R^{2\Lambda_0}(2\alpha_0+2\alpha_1)$. We leave the computation to the reader.

Proposition 7.5. *Suppose that* $\operatorname{char}\Bbbk\neq 2$. *Then* $R^{2\Lambda_0}(2\alpha_0+2\alpha_1)$ *is Morita equivalent to the following bound quiver algebra.*



$$\alpha\mu=\nu\alpha=0,\quad \beta^2=0,\quad \alpha^4=(\mu\nu)^2=-2\mu\beta\nu,$$
$$\beta\nu\mu=\nu\mu\beta,\quad \nu\mu\nu+2\beta\nu=0,\quad \mu\nu\mu+2\mu\beta=0.$$

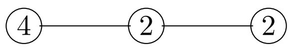
In the above bound quiver presentation, we set $\gamma=\nu\mu+2\beta$ and replace β with $(\gamma-\nu\mu)/2$. Then the bound quiver presentation becomes



$$\alpha\mu=\nu\alpha=0,\quad \gamma\nu=\mu\gamma=0,\quad \alpha^4=(\mu\nu)^2,\quad \gamma^2=-(\nu\mu)^2.$$

We see that the algebra is special biserial. Hence, we have the following corollary.

Corollary 7.6. *If* $\operatorname{char}\Bbbk\neq 2$ *then* $R^{2\Lambda_0}(2\alpha_0+2\alpha_1)$ *is Morita equivalent to the Brauer graph algebra whose Brauer graph is*



8. Representation type in level two cases

The rest of our proof relies on the results when the level is two. In this section, we are aiming to determine the representation type of $R^\Lambda(\beta_{\Lambda'})$ for $\Lambda'\in P^+_{cl,2}(\Lambda)$. There are only two cases to consider: $2\Lambda_a$, for $0\le a\le\ell$, and $\Lambda_a+\Lambda_b$, for $0\le a<b\le\ell$.

Before proceeding to the study of these two cases, we prove the existence of symmetry on the quiver. Let Z be a set of level two dominant integral weights which is stable under $\sigma:\Lambda_a+\Lambda_b\mapsto\Lambda_{\ell-b}+\Lambda_{\ell-a}$ such as $Z=\{2\Lambda_a\mid 0\le a\le\ell\}$ or $Z=\{\Lambda_a+\Lambda_b\mid a\neq b\}$. The lemma below implies that, if $R^\Lambda(\beta_{\Lambda'})$, for some $\Lambda'=\Lambda_i+\Lambda_j$, has a unique common representation type, for all $\Lambda=\Lambda_a+\Lambda_b\in Z$, then we may conclude that $R^\Lambda(\beta_{\Lambda'})$ and $R^\Lambda(\beta_{\sigma\Lambda'})$ have the same representation type for $\Lambda\in Z$.

Lemma 8.1. *Let $0 \le a \le b \le \ell$ and $0 \le i \le j \le \ell$. Then we have an isomorphism of algebras*

$$R^{\Lambda_{\ell-b}+\Lambda_{\ell-a}}\left(\beta_{\Lambda_{\ell-j}+\Lambda_{\ell-i}}\right) \cong R^{\Lambda_a+\Lambda_b}\left(\beta_{\Lambda_i+\Lambda_j}\right).$$

Proof. Let P be the permutation matrix which swaps i and $\ell - i$, for $0 \le i \le \ell$. Then $P\mathsf{A}P = \mathsf{A}$. Hence, if X is the solution of $\mathsf{A}X^t = Y^t$ in the sense of Lemma 3.4, then XP is the solution of $\mathsf{A}PX^t = PY^t$. It implies $\sigma\beta_{\Lambda_{\ell-j}+\Lambda_{\ell-i}} = \beta_{\Lambda_i+\Lambda_j}$. The result follows from Proposition 2.4. □

8.1. **The case $2\Lambda_a$ $(0 \le a \le \ell)$.** Our aim in this subsection is to prove the next theorem.

Theorem 8.2. *Suppose that $\Lambda = 2\Lambda_a$, for $0 \le a \le \ell$.*

(1) If we have an arrow $\Lambda \to \Lambda'$, the representation type of $R^\Lambda(\beta_{\Lambda'})$ is given as follows.
  - *(i′) If $\Lambda' = 2\Lambda_{a-1}$, for $1 \le a \le \ell$, then $R^\Lambda(\beta_{\Lambda'})$ is wild if $1 \le a \le \ell-2$, tame if $a = \ell-1$, finite if $a = \ell$.*
  - *(i″) If $\Lambda' = 2\Lambda_{a+1}$, for $0 \le a \le \ell-1$, then $R^\Lambda(\beta_{\Lambda'})$ is wild if $2 \le a \le \ell-1$, tame if $a = 1$, finite if $a = 0$.*
  - *(ii) If $\Lambda' = \Lambda_{a-1} + \Lambda_{a+1}$, for $1 \le a \le \ell-1$, then $R^\Lambda(\beta_{\Lambda'})$ is finite.*
  - *(iii′) If $\Lambda' = \Lambda_{a-2} + \Lambda_a$, for $2 \le a \le \ell$, then $R^\Lambda(\beta_{\Lambda'})$ is wild if $2 \le a \le \ell-1$, finite if $a = \ell$.*
  - *(iii″) If $\Lambda' = \Lambda_a + \Lambda_{a+2}$, for $0 \le a \le \ell-2$, then $R^\Lambda(\beta_{\Lambda'})$ is wild if $1 \le a \le \ell-2$, finite if $a = 0$.*

(2) If $\Lambda' = \Lambda_{a-2} + \Lambda_{a+2}$, for $2 \le a \le \ell-2$, then $R^\Lambda(\beta_{\Lambda'})$ is tame if $\operatorname{char} \Bbbk \ne 2$*, wild if* $\operatorname{char} \Bbbk = 2$.

(3) (i′) If $\Lambda = 2\Lambda_0$ and $\Lambda' = 2\Lambda_2$, then $R^\Lambda(\beta_{\Lambda'})$ is tame if $\operatorname{char} \Bbbk \ne 2$*, wild otherwise.*
  - *(i″) If $\Lambda = 2\Lambda_\ell$ and $\Lambda' = 2\Lambda_{\ell-2}$, then $R^\Lambda(\beta_{\Lambda'})$ is tame if* $\operatorname{char} \Bbbk \ne 2$*, wild otherwise.*

(4) Other $R^\Lambda(\beta_{\Lambda'})$ are all wild.

Moreover, if $R^\Lambda(\beta_{\Lambda'})$ is finite or tame, then it is an algebra listed in Main Theorem A.

We first give the connected quiver $\vec{C}(2\Lambda_a)$ (Figure 8.1). Once a is fixed, it is easy to verify whether an arrow (or a vertex) exists or not by Definition 3.17.

In the quiver (Figure 8.1), the superscript in the upper right corner of each vertex indicates the representation type of $R^{2\Lambda_a}(\beta_{\Lambda'})$, i.e., the corresponding cyclotomic KLR algebra. In particular, the dashed boxes in the quiver show the cases we have to analyze one by one, and the boxes imply that the corresponding algebra is wild by Lemma 4.4. Here, F means representation-finite, T means tame and W means wild. Finally, all the other remaining vertices of the quiver are wild by Corollary 4.5.

Theorem 8.2 (2) is (t15) if $\operatorname{char} \Bbbk \ne 2$. If $\operatorname{char} \Bbbk = 2$, it is wild by [15, Theorem 4.6], which refers to [7, Theorem B]. There, applying Dynkin automorphism to $2\Lambda_0$ and $\lambda_2^0 = \alpha_\ell + 2\alpha_0 + \alpha_1$, we obtain that $R_A^{2\Lambda_a}(\alpha_{a-1} + 2\alpha_a + \alpha_{a+1})$, for $2 \le a \le \ell-2$, is wild when $\operatorname{char} \Bbbk = 2$.

Proposition 8.3. *Let $\Lambda' = \Lambda_{a-3} + \Lambda_{a+3}$, for $3 \le a \le \ell-3$. Then $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. We have $\beta_{\Lambda'} = \alpha_{a-2} + 2\alpha_{a-1} + 3\alpha_a + 2\alpha_{a+1} + \alpha_{a+2}$. Applying Dynkin automorphism to $2\Lambda_0$ and $\lambda_3^0 = \alpha_{\ell-1} + 2\alpha_\ell + 3\alpha_0 + 2\alpha_1 + \alpha_2$ as above, we see that $R^\Lambda(\beta_{\Lambda'})$ is wild by [15, Theorem 4.6]. □

Proposition 8.3 has the following corollary by Lemma 4.4.

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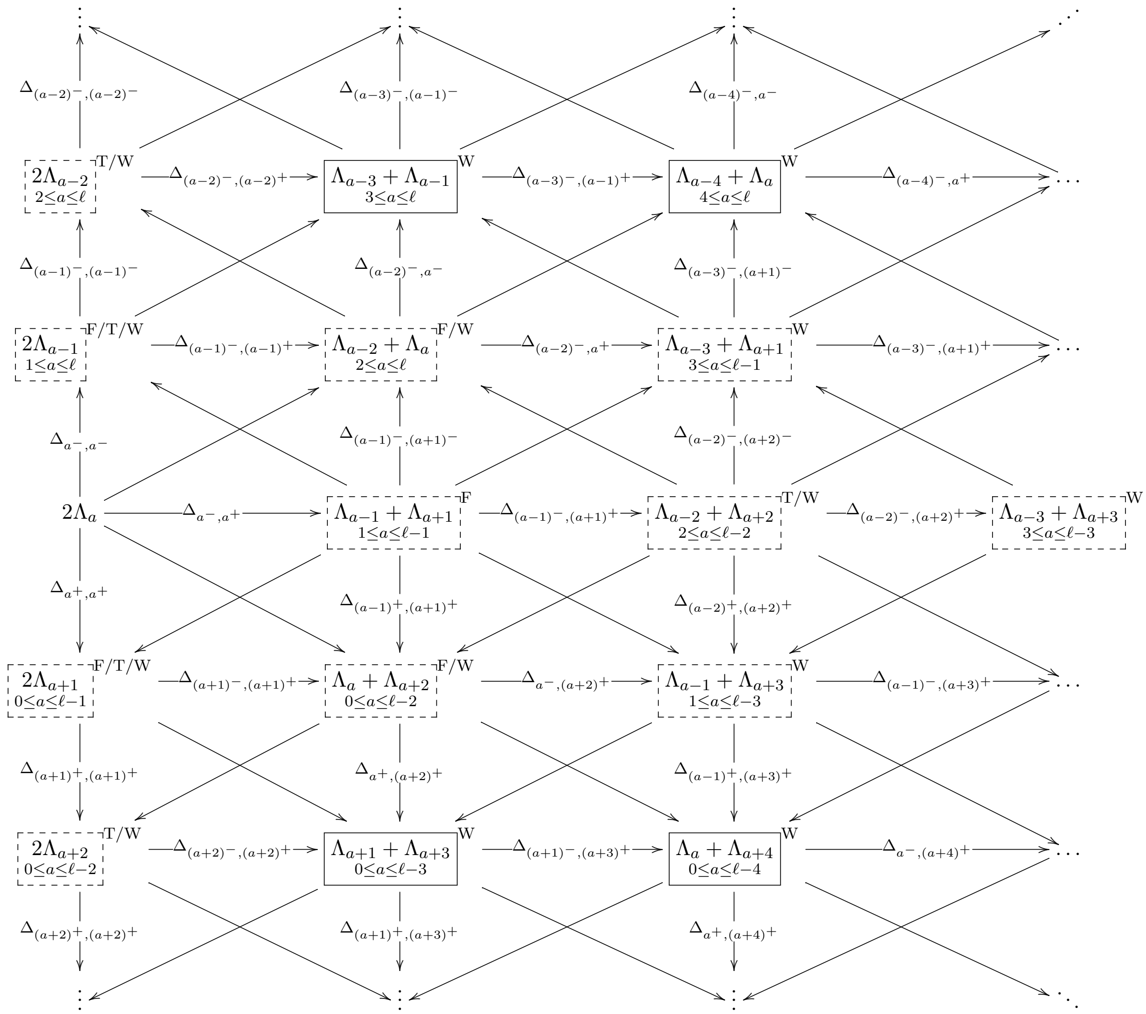


FIGURE 8.1. *The connected quiver* $\vec{C}(2\Lambda_a)$.

Corollary 8.4. *If* Λ' *is one of* $\Lambda_{a-1}+\Lambda_{a+3}$, $\Lambda_{a-3}+\Lambda_{a+1}$, $\Lambda_{a-3}+\Lambda_{a-1}$, $\Lambda_{a+1}+\Lambda_{a+3}$, *for* $3 \le a \le \ell-3$, *then* $R^{\Lambda}(\beta_{\Lambda'})$ *is wild.*

Next, we prove Theorem 8.2 (1). We start with (i′). Then we obtain (i″) by symmetry. Since $\beta_{\Lambda'} = 2\alpha_a + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$, we have the following.

(1) If $a = \ell$, then $\beta_{\Lambda'} = \alpha_\ell$ and it is finite by (f1).
(2) If $a = \ell - 1$, then $\beta_{\Lambda'} = 2\alpha_{\ell-1} + \alpha_\ell$ and it is tame by (t2).

Proposition 8.5. *Let* $\Lambda = 2\Lambda_a$ *and* $\Lambda' = 2\Lambda_{a-1}$, *for* $1 \le a \le \ell-2$. *Then* $R^{\Lambda}(\beta_{\Lambda'})$ *is wild.*

Proof. The readers may refer to the arXiv version [9] for the proof. □

The Case (ii) has $\beta_{\Lambda'} = \alpha_a$, so that it is finite by (f1). We consider (iii′). Then (iii″) is obtained by symmetry. Then

$$\beta_{\Lambda'} = \alpha_{a-1} + 2\alpha_a + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

If $a = \ell$, $\beta_{\Lambda'} = \alpha_{\ell-1} + \alpha_\ell$ and it is finite by (f3).

Proposition 8.6. *Let $\Lambda = 2\Lambda_a$ and $\Lambda' = \Lambda_{a-2} + \Lambda_a$, for $2 \le a \le \ell - 1$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. If $2 \le a \le \ell - 2$, then $R^\Lambda(\beta_{\Lambda'})$ is wild by Proposition 8.5 and Corollary 4.5 since there is an arrow from $2\Lambda_{a-1}$ to $\Lambda_{a-2} + \Lambda_a$.

If $a = \ell - 1$, then $\beta_{\Lambda'} = \alpha_{\ell-2} + 2\alpha_{\ell-1} + \alpha_\ell$ and set $e = e(\ell-1, \ell, \ell-1, \ell-2)$. We have

$$\dim_q eR^\Lambda(\beta_{\Lambda'})e = 1 + 3q^2 + 3q^4 + q^6.$$

Using Lemma 2.12, we deduce that $R^\Lambda(\beta_{\Lambda'})$ is wild. □

Theorem 8.2 (3) in the case $\operatorname{char} \Bbbk \neq 2$ is (t20) and (t21). When $\operatorname{char} \Bbbk = 2$, we use the computation in the proof of Proposition 7.4 to show the wildness as follows.

Lemma 8.7. *Let $\Lambda = 2\Lambda_0$ and $\Lambda' = 2\Lambda_2$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild if* $\operatorname{char} \Bbbk = 2$.

Proof. $\beta_{\Lambda'} = 2\alpha_0 + 2\alpha_1$. Let $f_1 = x_2\psi_1 x_4 \psi_3 e(0011)$. Then Proposition 7.4 implies that

$$f_1 A f_1 \cong \Bbbk[X, Y]/\Big(X^3 - 2XY, XY^2, Y^2 - X^2Y, Y^3\Big)$$

and it admits $\Bbbk[X, Y]/(X^3, Y^2, X^2Y)$ as a quotient algebra when $\operatorname{char} \Bbbk = 2$. It follows that $R^{2\Lambda_0}(2\alpha_0 + 2\alpha_1)$ in $\operatorname{char} \Bbbk = 2$ is wild, by Proposition 2.11. □

To prove the part (4) of Theorem 8.2, namely to prove that all the other $R^\Lambda(\beta_{\Lambda'})$ in level two are wild, it suffices to prove the wildness for:

(1) $\Lambda' = 2\Lambda_{a-2}$, for $2 \le a \le \ell$,
(2) $\Lambda' = 2\Lambda_{a+2}$, for $0 \le a \le \ell - 2$,
(3) $\Lambda_{a-3} + \Lambda_{a+1}$, for $a = \ell - 2$ and $a = \ell - 1$.
(4) $\Lambda_{a+3} + \Lambda_{a-1}$, for $a = 1$ and $a = 2$,
(5) $\Lambda_{a+1} + \Lambda_{a+3}$, for $0 \le a \le 2$,
(6) $\Lambda_{a-3} + \Lambda_{a-1}$, for $\ell - 2 \le a \le \ell$.

Proposition 8.8. *The algebra $R^{2\Lambda_a}(\beta_{\Lambda'})$ is wild, if $\Lambda' = 2\Lambda_{a-2}$, for $2 \le a \le \ell - 1$.*

Proof. It follows from Proposition 8.6 and Lemma 4.4. □

By symmetry, $R^{2\Lambda_a}(\beta_{\Lambda'})$ is wild, if $\Lambda' = 2\Lambda_{a+2}$, for $1 \le a \le \ell - 2$.

The cases (3) and (4) are covered by Lemma 8.9 below. Then, the lemma covers the cases (5) and (6), except for the case $a = 0$ in (5) and the case $a = \ell$ in (6), respectively. These two exceptions are covered by Lemma 8.10.

Lemma 8.9. *The algebra $R^{2\Lambda_a}(\beta_{\Lambda'})$ is wild, if $\Lambda' = \Lambda_{a-3} + \Lambda_{a+1}$, for $3 \le a \le \ell - 1$, or $\Lambda' = \Lambda_{a+3} + \Lambda_{a-1}$, for $1 \le a \le \ell - 3$.*

Proof. Suppose that $\Lambda' = \Lambda_{a-3} + \Lambda_{a+1}$ for $3 \le a \le \ell - 1$. Then by Proposition 8.6 $R^{2\Lambda_a}(\beta_{\Lambda''})$ is wild for $\Lambda'' = \Lambda_{a-2} + \Lambda_a$. This implies $R^{2\Lambda_a}(\beta_{\Lambda'})$ is wild since we have an arrow from Λ'' to Λ'. The other case holds by symmetry. □

When $a = 0$, there is an arrow $\Lambda_1 + \Lambda_3 \to 2\Lambda_3$. When $a = \ell$, there is an arrow $\Lambda_{\ell-3} + \Lambda_{\ell-1} \to 2\Lambda_{\ell-3}$. Thus, the wildness of $R^{2\Lambda_0}(\beta_{2\Lambda_3})$ and $R^{2\Lambda_\ell}(\beta_{2\Lambda_{\ell-3}})$ follow from that of $R^{2\Lambda_0}(\beta_{\Lambda_1+\Lambda_3})$ and $R^{2\Lambda_\ell}(\beta_{\Lambda_{\ell-3}+\Lambda_{\ell-1}})$.

Lemma 8.10. *Let $\Lambda = 2\Lambda_0$ and $\Lambda' = \Lambda_1 + \Lambda_3$. Then $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. We have $\beta_{\Lambda'} = 2\alpha_0 + 2\alpha_1 + \alpha_2$. Let $e_1 = e(01201)$ and $e_2 = e(01210)$. Then

$$\begin{aligned}
\dim_q e_1 R^{\Lambda}(\beta_{\Lambda'}) e_1 &= 1 + 2q^2 + 3q^4 + 3q^6 + 2q^8 + q^{10} \\
\dim_q e_2 R^{\Lambda}(\beta_{\Lambda'}) e_2 &= 1 + q^2 + 2q^4 + 2q^6 + q^8 + q^{10} \\
\dim_q e_1 R^{\Lambda}(\beta_{\Lambda'}) e_2 &= \dim_q e_2 R^{\Lambda}(\beta_{\Lambda'}) e_1 = q^2 + q^4 + q^6 + q^8.
\end{aligned}$$

By Lemma 2.15, $R^{\Lambda}(\beta_{\Lambda'})$ is wild. □

8.2. **The case $\Lambda_a + \Lambda_b$ $(0 \le a < b \le \ell)$.** Our aim in this subsection is to prove the next theorem.

Theorem 8.11. *Suppose that $\Lambda = \Lambda_a + \Lambda_b$, for $0 \le a < b \le \ell$.*

(1) If we have an arrow $\Lambda \to \Lambda'$, the representation type of $R^{\Lambda}(\beta_{\Lambda'})$ is given as follows.
- *(iv′) If $\Lambda' = \Lambda_{a-1} + \Lambda_{b-1}$, for $1 \le a < b \le \ell$, then $R^{\Lambda}(\beta_{\Lambda'})$ is wild if $1 \le a < b \le \ell - 1$, tame if $1 \le a \le \ell - 2$, $b = \ell$, finite if $a = \ell - 1$, $b = \ell$.*
- *(iv″) If $\Lambda' = \Lambda_{a+1} + \Lambda_{b+1}$, for $0 \le a < b \le \ell - 1$, then $R^{\Lambda}(\beta_{\Lambda'})$ is wild if $1 \le a < b \le \ell - 1$, tame if $a = 0$, $1 \le b \le \ell - 1$, finite if $a = 0$, $b = 1$.*
- *(v) If $\Lambda' = \Lambda_{a-1} + \Lambda_{b+1}$, for $1 \le a < b \le \ell - 1$, then $R^{\Lambda}(\beta_{\Lambda'})$ is finite.*
- *(vi) If $\Lambda' = \Lambda_{a+1} + \Lambda_{b-1}$, for $0 \le a < b \le \ell$ and $a \le b - 2$, then $R^{\Lambda}(\beta_{\Lambda'})$ is wild.*
- *(vii′) If $\Lambda' = \Lambda_a + \Lambda_{b-2}$, for $0 \le a < b \le \ell, a \le b - 2$, then $R^{\Lambda}(\beta_{\Lambda'}) \cong R^{\Lambda_b}(\beta_{\Lambda_{b-2}})$ is finite.*
- *(vii″) If $\Lambda' = \Lambda_{a+2} + \Lambda_b$, for $0 \le a < b \le \ell, a \le b - 2$, then $R^{\Lambda}(\beta_{\Lambda'}) \cong R^{\Lambda_a}(\beta_{\Lambda_{a+2}})$ is finite.*
- *(viii′) If $\Lambda' = \Lambda_a + \Lambda_{b+2}$, for $0 \le a < b \le \ell - 2$, then $R^{\Lambda}(\beta_{\Lambda'})$ is wild.*
- *(viii″) If $\Lambda' = \Lambda_{a-2} + \Lambda_b$, for $2 \le a < b \le \ell$, then $R^{\Lambda}(\beta_{\Lambda'})$ is wild.*

(2) If $\Lambda' = \Lambda_{a+2} + \Lambda_{b-2}$ for $0 \le a \le b - 4 \le \ell$, then $R^{\Lambda}(\beta_{\Lambda'})$ is tame if $a = 0$ and $b = \ell$. Otherwise, it is wild.

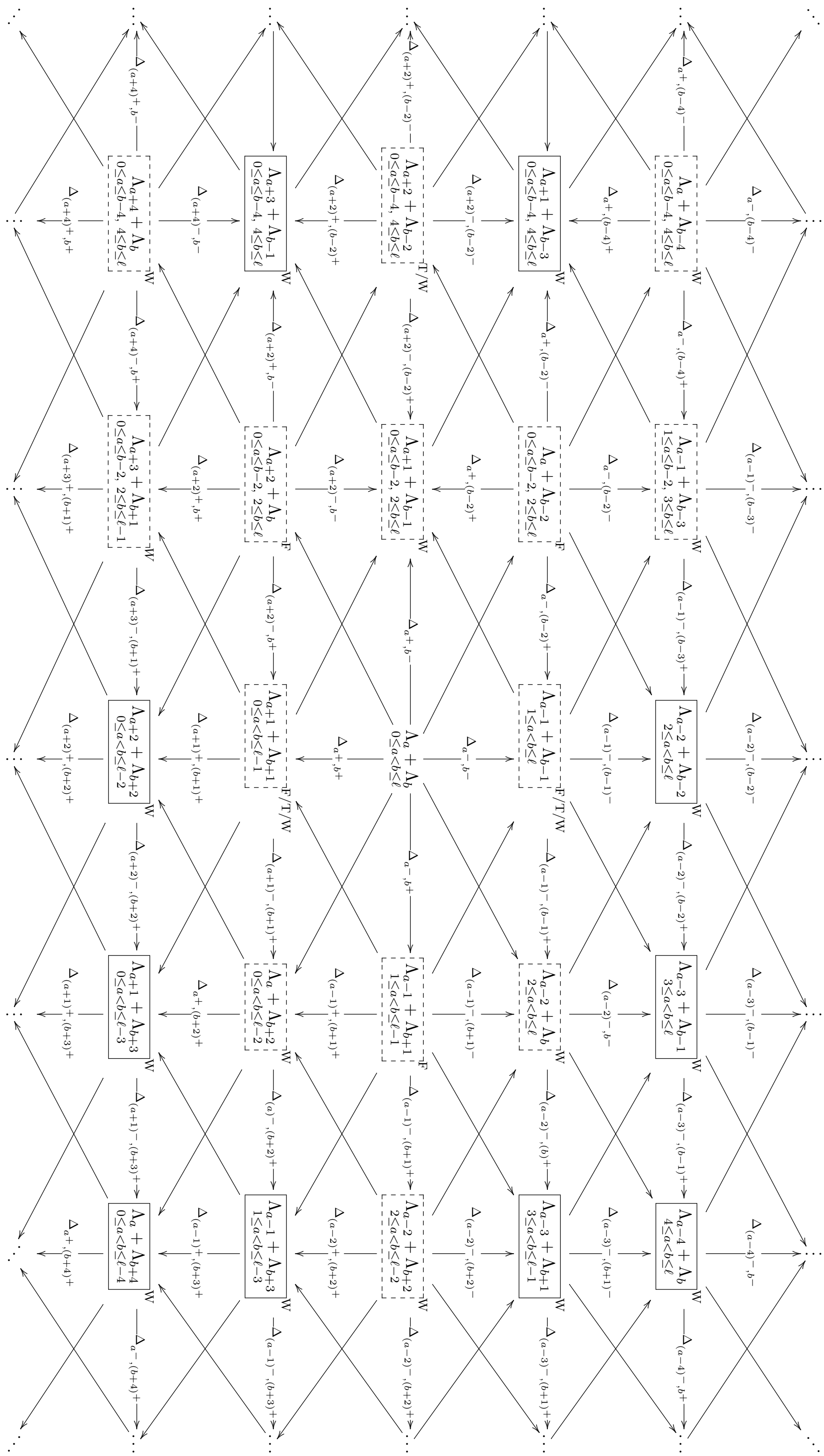
(3) All the other $R^{\Lambda}(\beta_{\Lambda'})$ in level two are wild.

Moreover, if $R^{\Lambda}(\beta_{\Lambda'})$ is finite or tame, then it is an algebra listed in Main Theorem A.

Set $\Lambda = \Lambda_a + \Lambda_b$ with $0 \le a < b \le \ell$. We observe that each element in $P^+_{cl,2}(\Lambda)$ can be written in the form $\Lambda_i + \Lambda_j$ with $0 \le i \le j \le \ell$ and $i + j \equiv_2 a + b$. We define

$$C_s(\Lambda) := \{\Lambda_i + \Lambda_j \mid 0 \le i \le j \le \ell,\ j - i = s,\ i + j \equiv_2 a + b\} \subseteq P^+_{cl,2}(\Lambda).$$

Then, $P^+_{cl,2}(\Lambda) = \bigsqcup_{s \ge 0} C_s(\Lambda)$. We draw $\vec{C}(\Lambda)$ on the plane by putting elements of $C_s(\Lambda)$ in the same column and arranging $C_s(\Lambda)$'s as columns in increasing order from left to right. In this way, the leftmost column of $\vec{C}(\Lambda)$ is $C_0(\Lambda)$ if $b - a \equiv_2 0$ and $C_1(\Lambda)$ if $b - a \equiv_2 1$. Once a, b are fixed, it is easy to verify whether an arrow (or a vertex) exists or not by Definition 3.17. Similar to the case of $2\Lambda_a$, the representation type of $R^{\Lambda_a + \Lambda_b}(\beta_{\Lambda'})$ is mentioned by the superscript in the upper right corner of each vertex. Also, all other remaining cases are wild by Corollary 4.5.



We start with (iv′) in Theorem 8.11 (1). Then

$$\beta_{\Lambda'} = \alpha_a + \cdots + \alpha_{b-1} + 2\alpha_b + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

If $a = \ell - 1$ and $b = \ell$, it is (f3). If $1 \le a \le \ell - 2$ and $b = \ell$, it is (t6).

Proposition 8.12. *Let $\Lambda = \Lambda_a + \Lambda_b$ and $\Lambda' = \Lambda_{a-1} + \Lambda_{b-1}$, for $1 \le a < b \le \ell - 1$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. Suppose $1 \le a < b \le \ell - 1$, we choose a suitable $A := (e_1 + e_2) R^{\Lambda_a + \Lambda_b}(\beta_{\Lambda'})(e_1 + e_2)$ that is wild. Recall that $\nu_b = (b, b+1, \ldots, \ell-1, \ell, \ell-1, \ldots, b+1, b, b-1)$.

- If $1 \le a = b - 1$, $b \le \ell - 1$, we have $\ell \ge 3$ and

  $$\beta_{\Lambda'} = \alpha_{b-1} + 2(\alpha_b + \cdots + \alpha_{\ell-1}) + \alpha_\ell.$$

  We set $e_1 := e(\nu_b)$ and $e_2 := e(b-1, b, b+1, \ldots, \ell-1, \ell, \ell-1, \ldots, b+1, b)$.
- If $1 \le a \le b - 2$, $b \le \ell - 1$, we have $\ell \ge 4$ and

  $$\beta_{\Lambda'} = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{b-1} + 2(\alpha_b + \cdots + \alpha_{\ell-1}) + \alpha_\ell.$$

  We set $e_1 := e(a, a+1, \ldots b-3, b-2, \nu_b)$ and $e_2 := (a, a+1, \ldots, \ell-2, \ell-1, \ell, \ell-1, \ldots, b+1, b)$.

In both cases, we have

$$\begin{aligned}
\dim_q e_1 A e_1 &= 1 + q^2 + q^4, \\
\dim_q e_2 A e_2 &= 1 + 2q^2 + q^4, \\
\dim_q e_1 A e_2 &= \dim_q e_2 A e_1 = q^2.
\end{aligned}$$

It gives that A is wild by Lemma 2.15. □

The Case (iv″) is obtained by symmetry. The Case (v) is $\beta_{\Lambda'} = \alpha_a + \cdots + \alpha_b$, for $1 \le a < b \le \ell - 1$. This is (f4). Now we show that (vi) is wild. If $a > 0$ and $b < \ell$, then $R^\Lambda(\beta_{\Lambda'})$ is wild by Proposition 8.12 since there is an arrow from $\Lambda_{a-1} + \Lambda_{b-1}$ to $\Lambda_{a+1} + \Lambda_{b-1}$. Thus, we may assume $a = 0$ or $b = \ell$.

Proposition 8.13. *Let $\Lambda = \Lambda_a + \Lambda_b$ and $\Lambda' = \Lambda_{a+1} + \Lambda_{b-1}$ with $a = 0$ or $b = \ell$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. We have three cases.

Case 1: $a = 0$ and $b = \ell$. In this case, $\beta_{\Lambda'} = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell$.

Suppose $\ell > 2$. Let $e_1 := e(0, 1, 2, \ldots, \ell-2, \ell-1, \ell)$ and $e_2 = e(0, \ell, 1, 2, \ldots, \ell-3, \ell-2, \ell-1)$. Then, we have

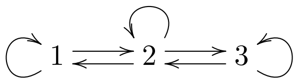
$$\begin{aligned}
\dim_q e_1 R^\Lambda(\beta_{\Lambda'}) e_1 &= 1 + q^2 + q^4 + q^6, \\
\dim_q e_2 R^\Lambda(\beta_{\Lambda'}) e_2 &= 1 + 2q^2 + 2q^4 + q^6, \\
\dim_q e_1 R^\Lambda(\beta_{\Lambda'}) e_2 &= \dim e_2 R^\Lambda(\beta_{\Lambda'}) e_1 = q^2 + q^4.
\end{aligned}$$

We deduce that $R^\Lambda(\beta_{\Lambda'})$ is wild by Lemma 2.15.

Suppose $\ell = 2$. Let $e := e_1 + e_1 + e_3$ with $e_1 := e(012)$, $e_2 := e(021)$ and $e_3 := (210)$. Then, we have

$$\dim_q e_i R^\Lambda(\beta_{\Lambda'}) e_i = 1 + q^2 + q^4 + q^6,$$
$$\dim_1 e_i R^\Lambda(\beta_{\Lambda'}) e_j = \begin{cases} q^2 + q^4 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This implies the quiver of $R^\Lambda(\beta_{\Lambda'})$ is of the form



and hence, it is wild by [27, I.10.8 (iv)]. □

Case 2: $a > 0$ and $b = \ell$. In this case, $\beta_{\Lambda'} = \alpha_0 + 2(\alpha_1 + \cdots + \alpha_a) + \alpha_{a+1} + \cdots + \alpha_\ell$. If $a \le b-4$, then $R^\Lambda(\beta_{\Lambda'})$ is wild by Proposition 8.15 since there is an arrow from $\Lambda_{a+2}+\Lambda_{b-2}$ to $\Lambda_{a+1} + \Lambda_{b-1}$. It remains to consider $a = b-2 = \ell-2$ or $a = b-3 = \ell-3$.

Let $\square_a := (a, a-1, a-2, \ldots, 2, 1)$. If $a = \ell - 2$, we set $e_1 := e(\square_a, 0, a+1, \square_a, \ell)$ and $e_2 := e(\square_a, 0, \ell, a+1, \square_a)$. If $a = \ell - 3$, we set $e_1 := e(\square_a, 0, a+1, a+2, \square_a, \ell)$ and $e_2 := e(\square_a, 0, \ell, a+1, a+2, \square_a)$. In both cases, we have the following graded dimensions such that $R^\Lambda(\beta_{\Lambda'})$ is wild, see Lemma 2.15.

$$\dim_q e_1 R^\Lambda(\beta_{\Lambda'}) e_1 = 1 + q^2 + q^4 + q^6,$$
$$\dim_q e_2 R^\Lambda(\beta_{\Lambda'}) e_2 = 1 + 2q^2 + 2q^4 + q^6,$$
$$\dim_q e_1 R^\Lambda(\beta_{\Lambda'}) e_2 = \dim e_2 R^\Lambda(\beta_{\Lambda'}) e_1 = q^2 + q^4.$$

Case 3: $a = 0$ and $b < \ell$. In this case, $\beta_{\Lambda'} = \alpha_0 + \alpha_1 + \cdots + \alpha_{b-1} + 2(\alpha_b + \cdots + \alpha_{\ell-1}) + \alpha_\ell$. Using the isomorphism in Proposition 2.4, we conclude that $R^\Lambda(\beta_{\Lambda'})$ is wild.

We have completed the proof of Proposition 8.13. □

The Case (vii′) is (f6) because

$$\beta_{\Lambda'} = \alpha_{b-1} + 2\alpha_b + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

The Case (vii″) is (f5). It remains to show that (viii″) is wild. The Case (viii′) is obtained by symmetry.

Proposition 8.14. *Let $\Lambda = \Lambda_a + \Lambda_b$ and $\Lambda' = \Lambda_{a-2} + \Lambda_b$ with $2 \le a < b \le \ell$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. If $b < \ell$, then $R^\Lambda(\beta_{\Lambda'})$ is wild by Proposition 8.12 since there is an arrow from $\Lambda_{a-1} + \Lambda_{b-1}$ to $\Lambda_{a-2} + \Lambda_b$. We assume $b = \ell$ in the following.

Case 1: $a = \ell - 1$ and $b = \ell$. In this case, $\beta_{\Lambda'} = \alpha_{\ell-2} + 2\alpha_{\ell-1} + \alpha_\ell$. We set

$$e_1 := (\ell-1, \ell, \ell-2, \ell-1) \quad \text{and} \quad e_2 := (\ell, \ell-1, \ell-1, \ell-2).$$

Then,

$$P_1 = f_{\ell-1} f_{\ell-2} f_\ell f_{\ell-1} L(0), \quad P_2 = f_{\ell-2} f_{\ell-1}^{(2)} f_\ell L(0).$$

 https://doi.org/10.5802/art.34

Then we may compute the graded dimensions as follows.

$$\begin{aligned}\dim_q \operatorname{End}(P_1) &= 1 + q^2 + q^4 + q^6,\\ \dim_q \operatorname{End}(P_2) &= 1 + 2q^2 + 2q^4 + q^6,\\ \dim_q \operatorname{Hom}(P_1, P_2) = \dim_q \operatorname{Hom}(P_2, P_1) &= q^2 + q^4.\end{aligned}$$

This implies that the algebra $R^\Lambda(\beta_{\Lambda'})$ is wild.

Case 2: $a < \ell - 1$ and $b = \ell$. In this case, $\beta_{\Lambda'} = \alpha_{a-1} + 2(\alpha_a + \cdots + \alpha_{\ell-1}) + \alpha_\ell$. Set

$$e := e(\ell, \ell-1, \ldots, a+2, a+1, a, a-1, a, a+1, a+2, \ldots, \ell-2, \ell-1).$$

Then, $\dim_q eR^\Lambda(\beta_{\Lambda'})e = 1 + 3q^2 + 3q^4 + q^6$ and $R^\Lambda(\beta_{\Lambda'})$ is wild by Lemma 2.12.

The proof is completed. □

Next, we prove Theorem 8.11 (2). If $a = 0$ and $b = \ell$, then it is (t12), and we already know that it is tame. Thus, we may assume $a > 0$ or $b < \ell$.

Proposition 8.15. *Let $\Lambda = \Lambda_a + \Lambda_b$ and $\Lambda' = \Lambda_{a+2} + \Lambda_{b-2}$ with $0 \le a \le b-4$, $4 \le b \le \ell$ such that $a > 0$ or $b < \ell$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. If $a = 0$, $b \le \ell - 1$, then $\beta_{\Lambda'} = \alpha_0 + \alpha_1 + \alpha_{b-1} + 2(\alpha_b + \cdots + \alpha_{\ell-1}) + \alpha_\ell$. We define $e_1 := e(0, 1, \nu_b)$ and $e_2 := e(0, 1, \nu_b')$ with

$$\begin{aligned}\nu_b &:= (b, b+1, \ldots, \ell-1, \ell, \ell-1, \ldots, b+1, b, b-1),\\ \nu_b' &:= (b, b-1, b+1, b+2, \ldots, \ell-1, \ell, \ell-1, \ldots, b+1, b).\end{aligned}$$

Setting $A = eR^\Lambda(\beta_{\Lambda'})e$ with $e = e_1 + e_2$. We obtain

$$\dim_q e_iAe_i = 1 + 2q^2 + q^4 \text{ for } i = 1, 2, \quad \dim_q e_1Ae_2 = \dim_q e_2Ae_1 = q + q^3.$$

Let $k = 2(\ell - b) + 4$. Direct computation as above shows that $x_1e_i = x_2^2e_i = 0$, $i = 1, 2$, and

$$x_je_1 = 0, x_he_2 = 0 \quad \text{for} \quad 3 \le j \le \ell - b + 3, 3 \le h \le \ell - b + 4. \tag{8.1}$$

We also show that

$$x_je_i = x_k^2e_i = 0 \quad \text{for} \quad i = 1, 2, 3 \le j \le k - 1. \tag{8.2}$$

Suppose that $b = \ell - 1$. Then $k = 6$ and $x_6^2e_2 = 0$ by $\psi_5e_2 = 0$ and (8.1). Using $\psi_3e_1 = 0 = \psi_4e_1$ shows that $(x_3 + x_5)e_1 = 0$ and hence $x_5e_1 = 0$ by (8.1). Moreover, $\psi_5^2e_1 = (x_5 - x_6)e_1$ and $x_6\psi_5^2e_1 = 0$ imply that $x_6^2e_1 = 0$. This completes the proof of (8.2) when $b = \ell - 1$. The case $b < \ell - 1$ can be checked similarly by using $\psi_{\ell-b+2}e_1 = 0 = \psi_{\ell-b+3}e_1$ and $\psi_{\ell-b+3}e_2 = 0 = \psi_{\ell-b+4}e_2$. Furthermore, $e_i\psi_we_i \ne 0$ only if $\psi_w = 1$. This together with (8.2) implies that the basis of e_iAe_h is given as follows.

$$\begin{aligned}e_iAe_i &= \Bbbk\text{-span}\{x_2^mx_k^ne_i \mid 0 \le m,\ n \le 1\}, \quad i = 1, 2,\\ e_1Ae_2 &= \Bbbk\text{-span}\{x_2^m\psi_{k-1}\psi_{k-2}\ldots\psi_4e_2 \mid 0 \le m \le 1\},\\ e_2Ae_1 &= \Bbbk\text{-span}\{x_2^m\psi_4\ldots\psi_{k-2}\psi_{k-1}e_1 \mid 0 \le m \le 1\}.\end{aligned}$$

By setting $\alpha = x_2e_1$, $\beta = x_2e_2$, $\mu = \psi_{k-1}\psi_{k-2}\ldots\psi_4e_2$ and $\nu = \psi_4\ldots\psi_{k-2}\psi_{k-1}e_1$, A is isomorphic to the bound quiver algebra defined by

$$\alpha \circlearrowright 1 \underset{\nu}{\overset{\mu}{\rightleftarrows}} 2 \circlearrowleft \beta \quad \text{and} \quad \left\langle \alpha^2, \beta^2, \mu\nu\mu, \nu\mu\nu, \alpha\mu - \mu\beta, \beta\nu - \nu\alpha \right\rangle.$$

Then, $A/\langle\nu\alpha\rangle$ is a wild algebra by [31, (32)].

If $a \geq 1$, $b = \ell$, then $\beta_{\Lambda'} = \alpha_0 + 2(\alpha_1 + \cdots + \alpha_a) + \alpha_{a+1} + \alpha_{\ell-1} + \alpha_\ell$. Similar to the case of $a = 0, b \leq \ell - 1$, one may show that $R^\Lambda(\beta_{\Lambda'})$ is wild.

If $a \geq 1$, $b \leq \ell - 1$, then we have

$$\beta_{\Lambda'} = \alpha_0 + 2(\alpha_1 + \cdots + \alpha_a) + \alpha_{a+1} + \alpha_{b-1} + 2(\alpha_b + \cdots + \alpha_{\ell-1}) + \alpha_\ell.$$

We choose $e_1 = e(\nu_a, \nu_b)$ and $e_2 = e(\nu'_a, \nu'_b)$, where

$$\nu_a := (a, a-1, \ldots, 1, 0, 1, \ldots, a-1, a, a+1),$$
$$\nu'_a := (a, a+1, a-1, a-2, \ldots, 1, 0, 1, \ldots, a-1, a).$$

and ν_b, ν'_b are defined in the case of $a = 0, b \leq \ell - 1$. Set $A := R^\Lambda(\beta_{\Lambda'})$, we obtain

$$\dim_q e_i A e_i = 1 + 2q^2 + q^4 \quad \text{for} \quad i = 1, 2, \ \dim_q e_1 A e_2 = \dim_q e_2 A e_1 = q^2.$$

Then, $R^\Lambda(\beta_{\Lambda'})$ is wild by Lemma 2.15. □

In order to show that all the other cyclotomic KLR algebras in level two are wild, we construct a neighborhood of Λ whose rim are all wild. For this, it suffices to show the wildness for

$$\Lambda' \in \{\Lambda_{a-2} + \Lambda_{b+2},\ \Lambda_{a+3} + \Lambda_{b+1},\ \Lambda_{a+4} + \Lambda_b,\ \Lambda_a + \Lambda_{b-4},\ \Lambda_{a-1} + \Lambda_{b-3}\}.$$

Proposition 8.16. *Let $\Lambda = \Lambda_a + \Lambda_b$ and $\Lambda' = \Lambda_{a-2} + \Lambda_{b+2}$ with $2 \leq a < b \leq \ell - 2$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. In this case, we have $\beta_{\Lambda'} = \alpha_{a-1} + 2\alpha_a + \cdots + 2\alpha_b + \alpha_{b+1}$. Then,

$$R^\Lambda(\alpha_{a-1} + 2\alpha_a + \cdots + 2\alpha_b + \alpha_{b+1}) \cong R^{\Lambda_A}(\alpha_{a-1} + 2\alpha_a + \cdots + 2\alpha_b + \alpha_{b+1}),$$

and the result follows from [15]. □

We prove the case $\Lambda' = \Lambda_{a-1} + \Lambda_{b-3}$ as follows. The case $\Lambda_{a+3} + \Lambda_{b+1}$ is obtained by symmetry.

Proposition 8.17. *Let $\Lambda = \Lambda_a + \Lambda_b$ and $\Lambda' = \Lambda_{a-1} + \Lambda_{b-3}$ with $0 \leq a \leq b - 2$, $2 \leq b \leq \ell$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. Since $b \leq \ell - 3$, $\Lambda_{a-1} + \Lambda_{b-1}$ is wild by (iv$'$) of Theorem 8.11. Then the result holds since we have an arrow $\Lambda_{a-1} + \Lambda_{b-1}$ to $\Lambda_{a-1} + \Lambda_{b-3}$. □

Finally, we consider the case $\Lambda' = \Lambda_a + \Lambda_{b-4}$. The case $\Lambda' = \Lambda_{a+4} + \Lambda_b$ is obtained by symmetry.

Proposition 8.18. *Let $\Lambda = \Lambda_a + \Lambda_b$ and $\Lambda' = \Lambda_a + \Lambda_{b-4}$ with $0 \leq a \leq b - 4$, $4 \leq b \leq \ell$. Then, $R^\Lambda(\beta_{\Lambda'})$ is wild.*

Proof. In this case, we have

$$\beta_{\Lambda'} = \alpha_{b-3} + 2\alpha_{b-2} + 3\alpha_{b-1} + 4\alpha_b + \cdots + 4\alpha_{\ell-1} + 2\alpha_\ell.$$

Thus, we have an isomorphism of algebras $R^\Lambda(\beta_{\Lambda'}) \cong R^{\Lambda_b}(\beta_{\Lambda'})$, and $R^\Lambda(\beta_{\Lambda'})$ is wild by Theorem 4.2. □

9. First neighbors in higher level cases

We consider higher level $R^\Lambda(\beta_{\Lambda'})$, for the first neighbors Λ' of Λ. We write $\Lambda = \sum_{i=0}^{\ell} m_i \Lambda_i$. As we have completed level two in the previous section, we assume that the level is $k \geq 3$ hereafter.

 https://doi.org/10.5802/art.34

9.1. (i′) $\Lambda = 2\Lambda_a + \widetilde{\Lambda}$ $(1 \le a \le \ell)$ **and** $\Lambda' = 2\Lambda_{a-1} + \widetilde{\Lambda}$**.** In this case,

$$\beta_{\Lambda'} = 2\alpha_a + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

If $1 \le a \le \ell - 2$, then $R^\Lambda(\beta_{\Lambda'})$ is wild by Theorem 8.2 (1)(i′). On the other hand, $R^\Lambda(\beta_{\Lambda'})$ is (f1) if $a = \ell$.

Suppose $a = \ell - 1$. Then $\beta = 2\alpha_{\ell-1} + \alpha_\ell$ and $R^\Lambda(2\alpha_{\ell-1} + \alpha_\ell)$ is (t2) if $m_{\ell-1} = 2$ and $m_\ell = 0$. We show that $R^\Lambda(2\alpha_{\ell-1} + \alpha_\ell)$ is wild if $m_{\ell-1} \ge 3$ or $m_\ell \ge 1$. To see this, it suffices to show that

$$R^{3\Lambda_{\ell-1}}(2\alpha_{\ell-1} + \alpha_\ell) \quad \text{and} \quad R^{2\Lambda_{\ell-1}+\Lambda_\ell}(2\alpha_{\ell-1} + \alpha_\ell)$$

are wild.

Lemma 9.1. *The algebra* $R^{2\Lambda_{\ell-1}+\Lambda_\ell}(2\alpha_{\ell-1} + \alpha_\ell)$ *is wild.*

Proof. Let $A = R^{2\Lambda_{\ell-1}+\Lambda_\ell}(2\alpha_{\ell-1} + \alpha_\ell)$ and $e_i = e(\nu_i)$, for

$$\nu_1 = (\ell-1, \ell-1, \ell), \quad \nu_2 = (\ell-1, \ell, \ell-1), \quad \nu_3 = (\ell, \ell-1, \ell-1).$$

By crystal computation, the number of simples is three. Moreover, computation of

$$f_\ell f_{\ell-1}^{(2)}(\emptyset, \emptyset, \emptyset), \quad f_{\ell-1} f_\ell f_{\ell-1}(\emptyset, \emptyset, \emptyset), \quad f_{\ell-1}^{(2)} f_\ell(\emptyset, \emptyset, \emptyset)$$

shows that

$$\begin{aligned}
\dim_q \operatorname{End}_A(P_1) &= 1 + q^4 + q^8, \\
\dim_q \operatorname{Hom}_A(P_1, P_2) &= 2q^2 + q^5 + q^7, \\
\dim_q \operatorname{Hom}_A(P_1, P_3) &= 0, \\
\dim_q \operatorname{End}_A(P_2) &= 1 + 2q^2 + 6q^4 + 2q^6 + q^8, \\
\dim_q \operatorname{Hom}_A(P_2, P_3) &= q + 2q^3 + q^5 + 2q^6, \\
\dim_q \operatorname{End}_A(P_3) &= 1 + q^2 + 2q^4 + q^6 + q^8.
\end{aligned}$$

Let $e = e_1 + e_2$ and consider $B = eAe$. Then, we observe the following.

- There are two degree two homomorphisms in $\operatorname{Hom}_A(P_1, P_2)$ and they cannot be linear combination of composition of two arrows of degree one.
- Next we consider $\operatorname{End}_A(P_2)$. There are two endomorphisms of degree two. The composition of arrows $P_2 \to P_3$ and $P_3 \to P_2$ of degree one gives one endomorphism of degree two, but there exists another endomorphism of degree two which is not linear combination of composition of two arrows of degree one.

Hence, the Gabriel quiver of B has a loop on vertex 2, and two arrows from vertex 1 to vertex 2. Hence, $A = R^{2\Lambda_{\ell-1}+\Lambda_\ell}(2\alpha_{\ell-1} + \alpha_\ell)$ is wild. □

Lemma 9.2. *The algebra* $R^{3\Lambda_{\ell-1}}(2\alpha_{\ell-1} + \alpha_\ell)$ *is wild.*

Proof. The readers may refer to the arXiv version [9] for the proof. □

9.2. (i″) $\Lambda = 2\Lambda_a + \widetilde{\Lambda}$ $(0 \le a \le \ell - 1)$ **and** $\Lambda' = 2\Lambda_{a+1} + \widetilde{\Lambda}$**.** In this case,

$$\beta_{\Lambda'} = \alpha_0 + 2(\alpha_1 + \cdots + \alpha_a).$$

By symmetry, we obtain the result for Case (i″).

9.3. (ii) $\Lambda = 2\Lambda_a + \widetilde{\Lambda}$ $(1 \le a \le \ell - 1)$ **and** $\Lambda' = \Lambda_{a-1} + \Lambda_{a+1} + \widetilde{\Lambda}$**.** In this case, $\beta_{\Lambda'} = \alpha_a$ and $R^\Lambda(\beta_{\Lambda'})$ is (f1).

9.4. (iii$''$) $\Lambda = 2\Lambda_a + \widetilde{\Lambda}$ $(0 \le a \le \ell - 2)$ **and** $\Lambda' = \Lambda_a + \Lambda_{a+2} + \widetilde{\Lambda}$**.** In this case,

$$\beta_{\Lambda'} = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_a + \alpha_{a+1}.$$

If $1 \le a \le \ell - 2$ then $R^{\Lambda}(\beta_{\Lambda'})$ is wild by Theorem 8.2 (1)(iii$''$). The case $a = 0$ follows from the general result for $R^{\Lambda}(\alpha_0 + \alpha_1)$ which we will give now.

Recall that $R^{\Lambda}(\alpha_0 + \alpha_1)$ is (f2) if $m_0 \ge 1$ and $m_1 = 0$, or $m_0 = m_1 = 1$, and (t3) or (t7) if $m_0 \ge 2$ and $m_1 = 1$, or $m_0 = 1$ and $m_1 = 2$. Note that $m_0 = 0$ cannot happen because $\langle \alpha_0^\vee, \Lambda - \alpha_0 - \alpha_1 \rangle = -1 < 0$. We show that $R^{\Lambda}(\alpha_0 + \alpha_1)$ is wild if $m_0 \ge 2$ and $m_1 \ge 2$ or $m_0 = 1$ and $m_1 \ge 3$.

Lemma 9.3. *The algebra* $R^{2\Lambda_0 + 2\Lambda_1}(\alpha_0 + \alpha_1)$ *is wild.*

Proof. Set $A = R^{2\Lambda_0+2\Lambda_1}(\alpha_0 + \alpha_1)$ and $B = e(10)Ae(10)$. Then

$$\dim_q B = 1 + q^2 + q^4 + q^6 + q^8 + q^{10}.$$

We have $x_1^2 e(10) = 0$ and $x_1^2 e(01) = 0$, which imply

$$0 = -\psi_1 x_1^2 e(01)\psi_1 = -x_2^2 \psi_1^2 e(10) = -x_2^2(x_1^2 - x_2)e(10) = x_2^3 e(10).$$

This together with $x_1^2 e(10) = 0$, the graded dimension shows that B has a basis

$$\left\{ x_1^a x_2^b e(10) \,\middle|\, 0 \le a \le 1,\ 0 \le b \le 2 \right\}.$$

Further, $B/(x_1 x_2^2 e(10)) \cong \Bbbk[X,Y]/(X^2, Y^3, XY^2)$ by sending $x_1 e(10)$ and $x_2 e(10)$ to X and Y, respectively. This implies B is wild and so is A. ☐

Lemma 9.4. *The algebra* $R^{\Lambda_0 + 3\Lambda_1}(\alpha_0 + \alpha_1)$ *is wild.*

Proof. Recall the algebra A' in Lemma 7.1 which is isomorphic to $R^{\Lambda_0+3\Lambda_1}(\alpha_0 + \alpha_1)$. It has the algebra (31) in [31, Table W] as a quotient algebra. The assertion follows. ☐

9.5. (iii$'$) $\Lambda = 2\Lambda_a + \widetilde{\Lambda}$ $(2 \le a \le \ell)$ **and** $\Lambda' = \Lambda_{a-2} + \Lambda_a + \widetilde{\Lambda}$**.** In this case,

$$\beta_{\Lambda'} = \alpha_{a-1} + 2\alpha_a + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

By symmetry, we have the result for this case from (iii$'$) .

9.6. (iv$'$) $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda}$ $(1 \le a < b \le \ell)$ **and** $\Lambda' = \Lambda_{a-1} + \Lambda_{b-1} + \widetilde{\Lambda}$**.** In this case,

$$\beta_{\Lambda'} = \alpha_a + \cdots + \alpha_{b-1} + 2\alpha_b + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

If $1 \le a < b \le \ell - 1$ then $R^{\Lambda}(\beta_{\Lambda'})$ is wild by Theorem 8.11 (iv$'$).

Suppose $1 \le a \le \ell - 2$ and $b = \ell$. If $m_i = \delta_{ai}$, for $a \le i \le \ell - 1$, then $R^{\Lambda}(\beta_{\Lambda'})$ is (t6). We show that $R^{\Lambda}(\beta_{\Lambda'})$ is wild if $m_a \ge 2$ or $m_i \ge 1$, for some $a < i < \ell$.

Lemma 9.5. *Suppose that* $\Lambda = 2\Lambda_a + \Lambda_\ell$ *and* $\Lambda' = \Lambda_{a-1} + \Lambda_a + \Lambda_{\ell-1}$. *Then* $R^{\Lambda}(\beta_{\Lambda'})$ *is wild.*

Proof. Set $e = e(\ell\, \ell{-}1\ \dots\ a{+}1\ a)$ and $A = eR^{\Lambda}(\beta_{\Lambda'})e$. Then $\dim_q A = 1 + 2q^2 + 2q^4 + q^6$. We have $x_1 e = 0$ and $\psi_i e = 0$ for $1 \le i \le \ell - a - 1$. This implies that

$$x_2^2 e = 0, x_i e = x_2 e, \quad \text{for } 3 \le i \le \ell - a.$$

Therefore, the degree 2 and the degree 4 components of A have bases

$$\{x_2 e, x_{\ell-a+1} e\} \quad \text{and} \quad \left\{x_2 x_{\ell-a+1} e, x_{\ell-a+1}^2 e\right\},$$

respectively. We conclude that $A/\operatorname{Rad}^3 A \cong \Bbbk[X,Y]/(X^2, Y^3, XY^2)$, which is wild. ☐

Lemma 9.6. *Suppose that* $\Lambda = \Lambda_a + \Lambda_i + \Lambda_\ell$ *and* $\Lambda' = \Lambda_{a-1} + \Lambda_i + \Lambda_{\ell-1}$ *for some* $a < i < \ell$. *Then* $R^\Lambda(\beta)$ *is wild.*

Proof. Set $A = eR^\Lambda(\beta)e$, where $e = e_1 + e_2$ with $e_1 = e(\ell, \ell-1, \dots, a+1, a)$ and $e_2 = e(i, \ell, \ell-1, \dots, i+1, i-1, i-2, \dots, a+1, a)$. If $i < \ell-1$, then $\dim_q e_1 A e_1 = 1+3q^2+3q^4+q^6$. If $i = \ell - 1$, then

$$\dim_q e_1 A e_1 = 1 + 2q^2 + 2q^4 + q^6, \quad \dim_q e_2 A e_2 = 1 + q^2 + q^4 + q^6,$$
$$\dim_q e_1 A e_2 = \dim_q e_2 A e_1 = q^2 + q^4.$$

In any case, we have that A is wild. □

It remains to consider the case $a = \ell - 1$ and $b = \ell$. If $m_\ell \geq 2$, it is already considered in (iii′). Thus we assume $m_{\ell-1} \geq 1$ and $m_\ell = 1$. $R^\Lambda(\beta_{\Lambda'})$ is (f3) if $m_{\ell-1} = 1$. If $m_{\ell-1} \geq 2$, we have an isomorphism of algebras

$$R^\Lambda(\beta_{\Lambda'}) \cong R^{m_{\ell-1}\Lambda_{\ell-1}+\Lambda_\ell}(\alpha_{\ell-1} + \alpha_\ell).$$

This is the algebra we analyzed at the beginning of Section 7. Thus, it is (t8) if $m_{\ell-1} = 2$, wild if $m_{\ell-1} \geq 3$.

9.7. (iv″) $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda}$ $(0 \leq a < b \leq \ell - 1)$ **and** $\Lambda' = \Lambda_{a+1} + \Lambda_{b+1} + \widetilde{\Lambda}$**.** In this case,

$$\beta_{\Lambda'} = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_a + \alpha_{a+1} + \cdots + \alpha_b.$$

By symmetry, we have the result from Case (iv′).

9.8. **The cases** (v)**,** (vi)**,** (viii′)**,** (viii″)**.**

(v) $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda}$ $(1 \leq a < b \leq \ell - 1)$ and $\Lambda' = \Lambda_{a-1} + \Lambda_{b+1} + \widetilde{\Lambda}$. In this case,

$$\beta_{\Lambda'} = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b.$$

Then the result from [15] for type $A^{(1)}_\ell$ shows that $R^\Lambda(\beta_{\Lambda'})$ is
- finite if $m_i = \delta_{ai} + \delta_{bi}$, for $a \leq i \leq b$, namely (f4),
- tame if $m_a \geq 2$ and $m_i = \delta_{bi}$, for $a < i \leq b$, or $m_b \geq 2$ and $m_i = \delta_{ai}$, for $a \leq i < b$, namely (t9),
- wild otherwise.

(vi) If $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda}$ $(0 \leq a < b \leq \ell)$ and $\Lambda' = \Lambda_{a+1} + \Lambda_{b-1} + \widetilde{\Lambda}$, where $a \leq b-2$, the level two result Theorem 8.11 (vi) implies that $R^\Lambda(\beta_{\Lambda'})$ is wild for $0 \leq a < b \leq \ell$ with $a \neq b - 1$.

(viii′) If $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda}$ $(0 \leq a < b \leq \ell - 2)$ and $\Lambda' = \Lambda_a + \Lambda_{b+2} + \widetilde{\Lambda}$,

$$\beta_{\Lambda'} = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_b + \alpha_{b+1}.$$

Then $R^\Lambda(\beta_{\Lambda'})$ is wild, for $0 \leq a < b \leq \ell - 2$, by Theorem 8.11 (viii′).

(viii″) If $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda}$ $(2 \leq a < b \leq \ell)$ and $\Lambda' = \Lambda_{a-2} + \Lambda_b + \widetilde{\Lambda}$, then

$$\beta_{\Lambda'} = \alpha_{a-1} + 2\alpha_a + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

By symmetry, Theorem 8.11 (viii″) implies that $R^\Lambda(\beta_{\Lambda'})$ is wild, for $2 \leq a < b \leq \ell$.

9.9. **The remaining cases.**

(vii′) If $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda}$ $(0 \le a < b \le \ell, b \ge 2)$ and $\Lambda' = \Lambda_a + \Lambda_{b-2} + \widetilde{\Lambda}$, it suffices to assume $a \le b-2$, because if $a = b-1$ then $\Lambda_a + \Lambda_{b-2} = \Lambda_{a-1} + \Lambda_{b-1}$ and it is already treated in (iv′). We have

$$\beta_{\Lambda'} = \alpha_{b-1} + 2\alpha_b + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

If $m_i = \delta_{bi}$, for $b-1 \le i \le \ell$, it is (f6). If $m_{b-1} \ge 1$, the arrow is

$$\Lambda = \Lambda_{b-1} + \Lambda_b + \widetilde{\Lambda}' \longrightarrow \Lambda' = \Lambda_{b-2} + \Lambda_{b-1} + \widetilde{\Lambda}',$$

and it is already treated in (iv′). If $m_b \ge 2$, the arrow is of the form

$$\Lambda = 2\Lambda_b + \widetilde{\Lambda}' \longrightarrow \Lambda' = \Lambda_{b-2} + \Lambda_b + \widetilde{\Lambda}',$$

and it is already treated in (iii′). If $m_i \ge 1$, for some $b+1 \le i \le \ell$, the arrow is

$$\Lambda = \Lambda_b + \Lambda_i + \widetilde{\Lambda}' \longrightarrow \Lambda' = \Lambda_{b-2} + \Lambda_i + \widetilde{\Lambda}',$$

and $R^\Lambda(\beta_{\Lambda'})$ is wild by (viii″).

(vii″) If $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda}$ $(0 \le a < b \le \ell, a \le \ell-2)$ and $\Lambda' = \Lambda_{a+2} + \Lambda_b + \widetilde{\Lambda}$, we may assume $a \le b-2$, because if $a = b-1$ then $\Lambda_{a+2} + \Lambda_b = \Lambda_{a+1} + \Lambda_{b+1}$ and it is already treated in (iv″). We have

$$\beta_{\Lambda'} = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_a + \alpha_{a+1}.$$

Then, by symmetry, we see that no new non-wild algebra appears.

10. Second neighbors in higher level cases

By the result on the first neighbors, it suffices to check the representation type of $R^\Lambda(\beta_{\Lambda''})$ for $\Lambda \to \Lambda' \to \Lambda''$ in the following cases in the second neighbors.

(1) $\Lambda = 2\Lambda_\ell + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_{\ell-1} + \widetilde{\Lambda}$ and $\Lambda = 2\Lambda_0 + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_1 + \widetilde{\Lambda}$.
(2) $\Lambda = 2\Lambda_{\ell-1} + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_{\ell-2} + \widetilde{\Lambda}$ and $\Lambda = 2\Lambda_1 + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_2 + \widetilde{\Lambda}$.
(3) $\Lambda = 2\Lambda_a + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_{a+1} + \widetilde{\Lambda}$ $(1 \le a \le \ell-1)$.
(4) $\Lambda = 2\Lambda_\ell + \widetilde{\Lambda} \to \Lambda' = \Lambda_{\ell-2} + \Lambda_\ell + \widetilde{\Lambda}$ and $\Lambda = 2\Lambda_0 + \widetilde{\Lambda} \to \Lambda' = \Lambda_0 + \Lambda_2 + \widetilde{\Lambda}$.
(5) $\Lambda = \Lambda_a + \Lambda_\ell + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_{\ell-1} + \widetilde{\Lambda}$ $(1 \le a \le \ell-1)$ and

$$\Lambda = \Lambda_0 + \Lambda_b + \widetilde{\Lambda} \longrightarrow \Lambda' = \Lambda_1 + \Lambda_{b+1} + \widetilde{\Lambda}(1 \le b \le \ell-1).$$

(6) $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_{b+1} + \widetilde{\Lambda}$ $(1 \le a < b \le \ell-1)$.
(7) $\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda} \to \Lambda' = \Lambda_a + \Lambda_{b-2} + \widetilde{\Lambda}$ $(0 \le a < b \le \ell,\ a \le b-2)$ and

$$\Lambda = \Lambda_a + \Lambda_b + \widetilde{\Lambda} \longrightarrow \Lambda' = \Lambda_{a+2} + \Lambda_b + \widetilde{\Lambda}(0 \le a < b \le \ell,\ a \le b-2).$$

The aim of this section is to show that no new non-wild algebra appears in the above seven cases. Our strategy for the proof is that we check the wildness of the algebras case by case. Basically, most algebras $R^\Lambda(\beta_{\Lambda''})$ in each case will belong to the following three patterns. Since we will use similar arguments repeatedly in each pattern, we adopt the following style of writing in order to avoid repetition.

(I) Λ'' is already in the first neighbors and hence already done in the previous section. By the definition of arrows, it is easy to see that Λ'' can be reached from Λ with one move. We list Λ'' in this pattern without further proof.

(II) Λ'' is not in the first neighbors but there is an arrow $\Lambda_{\text{mid}} \to \Lambda''$ such that we may know that $R^\Lambda(\beta_{\Lambda_{\text{mid}}})$ is wild, by the results of the first neighbors or level two results. Then $R^\Lambda(\beta_{\Lambda''})$ is wild. In this pattern, we will write the arrow (or just Λ_{mid} for each Λ'') and refer to the previous sections for the wildness of $R^\Lambda(\beta_{\Lambda_{\text{mid}}})$.

A variant of this argument is that $R^\Lambda(\beta_{\Lambda_{\text{mid}}})$ is not wild, but we know by results in the previous sections that $R^\Lambda(\beta_{\Lambda''})$ is wild for the path $\Lambda_{\text{mid}} \to \Lambda''$.

(III) We may use Lemma 2.18(tensor product lemma) to show that $R^\Lambda(\beta_{\Lambda''})$ is Morita equivalent to the tensor product of two algebras. Then the wildness of the tensor product is easy to see. For this pattern, we will just write the tensor product of two algebras without referring to Lemma 2.18 explicitly.

For the new non-wild algebras, we will see that they all belong to the tame cases listed in Main Theorem A.

In the following, we only list the arguments for Case (5) to showcase an example of the strategy. For the detailed arguments of the remaining cases, we refer to [9, Section 10].

10.1. **Case** (5)**.** This case studies $\Lambda = \Lambda_0 + \Lambda_b + \widetilde{\Lambda} \to \Lambda' = \Lambda_1 + \Lambda_{b+1} + \widetilde{\Lambda} \to \Lambda''$, for $1 \le b \le \ell - 1$, and $\beta_{\Lambda'} = \alpha_0 + \cdots + \alpha_b$.

10.1.1. *The case of changing* $\Lambda_1 + \Lambda_{b+1}$. First, cases $\Delta_{(b+1)^-}$, $\Delta_{1^+,(b+1)^-}$ and $\Delta_{1^-,(b+1)^+}$ are in pattern (I). Second, for the remaining cases $\Delta_{(b+1)^+}$, $\Delta_{1^+,(b+1)^+}$, and Δ_{1^+} are all in pattern (II) with $\Lambda_{\text{mid}} = \Lambda_0 + \Lambda_{b+2} + \widetilde{\Lambda}$, $\Lambda_0 + \Lambda_{b+2} + \widetilde{\Lambda}$ and $\Lambda_4 + \Lambda_b + \widetilde{\Lambda}$, respectively. For the first two, $R^\Lambda(\beta_{\Lambda_{\text{mid}}})$ is wild by (viii$'$) in the first neighbors. Finally, $R^\Lambda(\beta_{\Lambda_{\text{mid}}})$ for the last one is also wild since Theorem 4.2 shows that $R^{\Lambda_0}(\beta_{\Lambda_4})$ is wild.

10.1.2. *The case of changing* $\Lambda_1 + \Lambda_i$ *or* $\Lambda_{b+1} + \Lambda_i$. Here, we consider the path

$$\Lambda \longrightarrow \Lambda' = \Lambda_1 + \Lambda_{b+1} + \Lambda_i + \widetilde{\Lambda} \longrightarrow \Lambda''$$

and we must change Λ_i. First, we have cases in pattern (I):

- $\Delta_{i^-,(b+1)^-}$, $\Delta_{i^+,(b+1)^-}$, $\Delta_{i^-,1^-}$, $\Delta_{i^+,1^-}$,
- Δ_{i^-} for $2 = i \le b-1$, or $i = b+2, b+1$,
- Δ_{1^+,i^-} for $1 \le b = i-1$, or $1 = b = i$, or $i = 1, 2 \le b \le \ell - 1$ or $i = 2, 3 \le b \le \ell - 1$,
- $\Delta_{(b+1)^+,i^-}$ for $1 \le b = i-2$ or $1 \le b = i-1$.

Second, we have the following cases in pattern (II):

(Δ_{i^+}) with $1 \le i \le \ell - 2$: $\Lambda_{\text{mid}} = \Lambda_0 + \Lambda_{i+2} + \Lambda_b + \widetilde{\Lambda}$, by (viii$'$) in the first neighbors.

(Δ_{i^-}) with $2 \le i = b$ or $3 \le i \le b-1$: $\Lambda_{\text{mid}} = \Lambda_0 + \Lambda_{b-2} + \Lambda_b + \widetilde{\Lambda}$ and $\Lambda_0 + \Lambda_{i-2} + \Lambda_b + \widetilde{\Lambda}$, respectively, by Theorem 8.2 (iii$'$) and Theorem 8.11 (viii$''$), respectively.

(Δ_{1^+,i^+}) with $i \ne 0, b$ or $2 \le i = b$: $\Lambda_{\text{mid}} = \Lambda_0 + \Lambda_{b+1} + \Lambda_{i+1} + \widetilde{\Lambda}$ and $\Lambda_0 + 2\Lambda_{b+1} + \widetilde{\Lambda}$, respectively, by Theorem 8.11 (iv$''$), and by Theorem 8.2 (ii) respectively.

($\Delta_{(b+1)^+,i^+}$) $\Lambda_{\text{mid}} = \Lambda_0 + \Lambda_{b+2} + \Lambda_i + \widetilde{\Lambda}$, by (viii$'$) in the first neighbors.

(Δ_{1^+,i^-}) with $1 \le b \le i-2$ and ($\Delta_{(b+1)^+,i^-}$) with $1 \le b \le i-3$. For both cases, $\Lambda_{\text{mid}} = \Lambda_1 + \Lambda_b + \Lambda_{i-1} + \widetilde{\Lambda}$, by (vi) in the first neighbors.

Other than patterns (I) and (II), we have the following cases.

(Δ_{i^+}) We have $\Lambda'' = \Lambda_1 + \Lambda_{b+1} + \Lambda_{i+2} + \widetilde{\Lambda}$, for $0 \le i \le \ell - 2$. Here, it remains to consider the following subcases.

$(i=0, 2 \leq b \leq \ell-1)$ We choose $[P] = f_2 f_1^{(2)} f_0^{(2)} f_3 \dots f_b v_\Lambda \in V(\Lambda_0) \otimes V(\Lambda_0) \otimes V(\Lambda_b)$. Then $[P] = f_2 f_1^{(2)}((1),(1),(1^{b-2}))$ is obtained by applying f_2 to

$$\Big((1),(2,1),(1^{b-2})\Big) + q\Big((1^2),(1^2),(1^{b-2})\Big) + q^2\Big((1^2),(2),(1^{b-2})\Big) \\ + q^2\Big((2),(1^2),(1^{b-2})\Big) + q^3\Big((2),(2),(1^{b-2})\Big) + q^4\Big((2,1),(1),(1^{b-2})\Big).$$

Each 3-partition has three addable 2-nodes and no removable 2-node. Hence,

$$\begin{aligned}\dim_q \operatorname{End}(P) &= (1+q^2+q^4)(1+q^2+2q^4+q^6+q^8) \\ &= 1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12},\end{aligned}$$

and $P = f_2 f_1^{(2)} f_0^{(2)} f_3 \dots f_b R^\Lambda(0)$ is an indecomposable projective module. We apply Lemma 2.14 to conclude that $R^{2\Lambda_0+\Lambda_b}(\beta_{\Lambda''})$ is wild.

$(i=0, b=1)$ We have $\Lambda = 2\Lambda_0 + \Lambda_1 + \widetilde{\Lambda}$ and $\Lambda'' = \Lambda_1 + 2\Lambda_2 + \widetilde{\Lambda}$, $\beta_{\Lambda''} = 2\alpha_0 + 2\alpha_1$. We already proved in Subsection 10.1.1 that this algebra is wild.

(Δ_{i^-}) We have $\Lambda'' = \Lambda_1 + \Lambda_{b+1} + \Lambda_{i-2} + \widetilde{\Lambda}$, for $2 \leq i \leq \ell$. It remains to consider the case $b+3 \leq i \leq \ell$. We have $\Lambda'' = \Lambda_1 + \Lambda_{b+1} + \Lambda_{i-2} + \widetilde{\Lambda}$ and

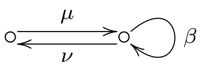
$$\beta_{\Lambda''} = \alpha_0 + \dots + \alpha_b + \alpha_{i-1} + 2\alpha_i + \dots + 2\alpha_{\ell-1} + \alpha_\ell.$$

Thus Lemma 2.18 implies that $R^{\Lambda_0+\Lambda_b+\Lambda_i}(\beta_{\Lambda''})$ is Morita equivalent to

$$R^{\Lambda_0+\Lambda_b}(\alpha_0 + \dots + \alpha_b) \otimes R^{\Lambda_i}(\alpha_{i-1} + 2\alpha_i + \dots + 2\alpha_{\ell-1} + \alpha_\ell),$$

which is $R^{\Lambda_0+\Lambda_b}(\beta_{\Lambda_1+\Lambda_{b+1}}) \otimes R^{\Lambda_i}(\beta_{\Lambda_{i-2}})$. In [22, Proposition 4.1], it was proved that $R^{\Lambda_i}(\beta_{\Lambda_{i-2}})$ is the Brauer line algebra whose number of simple modules is $\ell - i + 1$. Thus, we may choose an idempotent e such that $eR^{\Lambda_i}(\beta_{\Lambda_{i-2}})e \cong \Bbbk[X]/(X^2)$.

On the other hand, $R^{\Lambda_0+\Lambda_b}(\beta_{\Lambda_1+\Lambda_{b+1}})$ is (t5) and the number of simples is $b+1 \geq 2$. Thus, by considering the three leftmost vertices of the Brauer graph, we may obtain an idempotent truncation whose Gabriel quiver is



Therefore, an idempotent truncation of $R^{\Lambda_0+\Lambda_b+\Lambda_i}(\beta_{\Lambda''})$ has the Gabriel quiver which is obtained by adding one loop to each vertex. Hence, $R^{\Lambda_0+\Lambda_b+\Lambda_i}(\beta_{\Lambda''})$ is wild, which implies that $R^\Lambda(\beta_{\Lambda''})$ is wild.

(Δ_{1^+,i^+}) We have $\Lambda'' = \Lambda_2 + \Lambda_{i+1} + \Lambda_{b+1} + \widetilde{\Lambda}$. Then, the following are the remaining cases.

$(i=0)$ $\Lambda = 2\Lambda_0 + \Lambda_b + \widetilde{\Lambda}$, $\Lambda'' = \Lambda_1 + \Lambda_2 + \Lambda_{b+1} + \widetilde{\Lambda}$, and

$$\beta_{\Lambda''} = 2\alpha_0 + 2\alpha_1 + \alpha_2 + \dots + \alpha_b.$$

If $b=1$, we already showed that $R^{2\Lambda_0+\Lambda_1}(2\alpha_0+2\alpha_1)$ is wild in (Δ_+). Thus, we assume $b \geq 2$ and choose

$$[P] = f_0 f_1^{(2)} f_2 \dots f_b f_0 v_\Lambda \in V(\Lambda_0) \otimes V(\Lambda_0) \otimes V(\Lambda_b).$$

We then obtain $[P]$ by applying f_0 to

$$\begin{aligned} f_1^{(2)}\Big((0),(1),(1^{b-1})\Big) + q^2\Big((1),(0),(1^{b-1})\Big) \\ = \Big((0),(1^2),(1^b)\Big) + q\Big((0),(2),(1^b)\Big) + q^2\Big((0),(2,1),(1^{b-1})\Big) \\ + q^2\Big((1^2),(0),(1^b)\Big) + q^3\Big((2),(0),(1^b)\Big) + q^4\Big((2,1),(0),(1^{b-1})\Big). \end{aligned}$$

Each 3-partition has two addable 0-nodes and no removable 0-node. Thus,

$$\begin{aligned} \dim_q \operatorname{End}(P) &= (1+q^4)(1+q^2+2q^4+q^6+q^8) \\ &= 1+q^2+3q^4+2q^6+3q^8+q^{10}+q^{12} \end{aligned}$$

and we apply Lemma 2.13 to conclude that $\operatorname{End}(P)$ and $R^\Lambda(\beta_{\Lambda''})$ are wild.

$(i=b=1)$ $\Lambda = \Lambda_0 + 2\Lambda_1 + \widetilde{\Lambda}$, $\Lambda'' = 3\Lambda_2 + \widetilde{\Lambda}$ and $\beta_{\Lambda''} = 2\alpha_0 + 3\alpha_1$. We consider $R^{\Lambda_0+2\Lambda_1}(2\alpha_0+3\alpha_1)$ and choose $[P] = f_1^{(2)} f_0^{(2)} f_1 v_\Lambda$. Then

$$\dim_q \operatorname{End}(P) = 1 + 2q^2 + 3q^4 + 3q^6 + 2q^8 + q^{10}$$

by the similar computation above. Hence, Lemma 2.14 applies.

$(\Delta_{(b+1)^+,i^-})$ We have $\Lambda'' = \Lambda_1 + \Lambda_{b+2} + \Lambda_{i-1} + \widetilde{\Lambda}$. Then, we consider the following remaining cases.

$(2 \le b = i)$ $\Lambda = \Lambda_0 + 2\Lambda_b + \widetilde{\Lambda} \to \Lambda'' = \Lambda_1 + \Lambda_{b-1} + \Lambda_{b+2} + \widetilde{\Lambda}$ and

$$\beta_{\Lambda''} = \alpha_0 + \cdots + \alpha_{b-1} + 2\alpha_b + \alpha_{b+1}.$$

We choose $[P] = f_b f_{b-1} \dots f_0 f_{b+1} f_b v_\Lambda \in V(\Lambda_0) \otimes V(\Lambda_b) \otimes V(\Lambda_b)$. Then $[P]$ is obtained by applying $f_b f_{b-1}$ to

$$\Big((1^{b-1}),(0),(2)\Big) + q((b-1),(0),(2)) + q\Big((1^{b-1}),(2),(0)\Big) + q^2((b-1),(2),(0)).$$

Hence, we obtain

$$\dim_q \operatorname{End}(P) = 1 + 4q^2 + 6q^4 + 4q^6 + q^8$$

and $R^\Lambda(\beta_{\Lambda''})$ is wild by Lemma 2.14.

$(1 = b = i)$ This case is similar to the previous case. We choose $[P] = f_2 f_1 f_0 f_1 v_\Lambda$ and compute graded dimensions. Then,

$$\begin{aligned} \dim_q \operatorname{End}(P) &= (1+q^2)(1+3q^2+2q^4+3q^6+q^8) \\ &= 1+4q^2+5q^4+5q^6+4q^8+q^{10}. \end{aligned}$$

Hence, $R^\Lambda(\beta_{\Lambda''})$ is wild.

$(1 \le i < b \le \ell-1)$ In this case, we have

$$\beta_{\Lambda''} = \alpha_0 + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_b + \alpha_{b+1}.$$

We choose $[P] \in V(\Lambda_0) \otimes V(\Lambda_i) \otimes V(\Lambda_b)$ as

$$[P] = f_i (f_{i+1} f_i)(f_{i+2} \dots f_{b+1})(f_{i+1} \dots f_b)(f_{i-1} \dots f_0) v_\Lambda.$$

Then, one can show

$$\begin{aligned} \dim_q \operatorname{End}(P) &= (1+q^2)(1+q^2+2q^4+q^6+q^8) \\ &= 1+2q^2+3q^4+3q^6+2q^8+q^{10}. \end{aligned}$$

Hence, $R^\Lambda(\beta_{\Lambda''})$ is wild by Lemma 2.14.

10.1.3. *The case of changing* $\Lambda_i + \Lambda_j$. Here, we consider $\Lambda = \Lambda_0 + \Lambda_b + \Lambda_i + \Lambda_j + \widetilde{\Lambda}$, for $0 \le i \le j \le \ell$, and the path

$$\Lambda \longrightarrow \Lambda' = \Lambda_1 + \Lambda_{b+1} + \Lambda_i + \Lambda_j + \widetilde{\Lambda} \longrightarrow \Lambda''.$$

In the path, we must change $\Lambda_i + \Lambda_j$ in the second step. Cases in pattern (I) are:

(Δ_{i^-,j^+}) $i = j = b+1$, or $1 \le i < j \le \ell - 1$ and $i = 1$, or $1 \le i < j \le \ell - 1$ and $i = b+1$.

(Δ_{i^-,j^-}) $i = j = b = \ell-1$, or $i = j = b+1 = \ell$, or $i = b = 1$ and $j = \ell$, or $i = 1 < j \le \ell-1$.

Thus, their representation types have already been determined.

Next, we consider cases in pattern (II). Let Λ_{mid} be the dominant integral weight which is obtained by changing $\Lambda_i + \Lambda_j$ in Λ. We shall check when $R^\Lambda(\beta_{\Lambda_{\mathrm{mid}}})$ is wild, and whether there is an arrow $\Lambda_{\mathrm{mid}} \to \Lambda''$.

The following is the list of Λ_{mid} such that $R^\Lambda(\beta_{\Lambda_{\mathrm{mid}}})$ is wild. Then, we check whether $\beta_{\Lambda_{\mathrm{mid}}} + (\alpha_0 + \cdots + \alpha_b) - \delta \notin Q_+$, in order to know the existence of the arrow. The numbering in the list follows Theorem 8.2 (1) and Theorem 8.11 (1) as before.

(i′) $\Lambda - \Lambda_{\mathrm{mid}} = 2\Lambda_i - 2\Lambda_{i-1}$, for $2 \le i = j \le \ell - 2$. Then,

$$\beta_{\Lambda_{\mathrm{mid}}} = 2\alpha_i + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

Hence, we need to treat the cases $i = j = \ell - 1$ and $i = j = \ell$ below. Note that $i = j = 1$ implies $\Lambda'' = \Lambda$ and it does not occur.

(i″) $\Lambda - \Lambda_{\mathrm{mid}} = 2\Lambda_i - 2\Lambda_{i+1}$, for $2 \le i = j \le \ell - 1$. Then,

$$\beta_{\Lambda_{\mathrm{mid}}} = \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i.$$

Hence, we need to treat the cases $i = j = 0$ and $i = j = 1$ below.

(iv′) $\Lambda - \Lambda_{\mathrm{mid}} = \Lambda_i - \Lambda_{i-1} + \Lambda_j - \Lambda_{j-1}$, for $2 \le i < j \le \ell - 1$. Then,

$$\beta_{\Lambda_{\mathrm{mid}}} = (\alpha_i + \cdots + \alpha_{\ell-1}) + (\alpha_j + \cdots + \alpha_{\ell-1}) + \alpha_\ell.$$

Hence, we need to treat the case $j = \ell$ below. Note that the arrow $\Lambda' \to \Lambda''$ does not exist when $i = 1$.

(iv″) $\Lambda - \Lambda_{\mathrm{mid}} = \Lambda_i - \Lambda_{i+1} + \Lambda_j - \Lambda_{j+1}$, for $1 \le i < j \le \ell - 1$. Then,

$$\beta_{\Lambda_{\mathrm{mid}}} = \alpha_0 + (\alpha_1 + \cdots + \alpha_i) + (\alpha_1 + \cdots + \alpha_j).$$

Hence, we need to treat the case $i = 0 < j$ below.

(vi) $\Lambda - \Lambda_{\mathrm{mid}} = \Lambda_i - \Lambda_{i+1} + \Lambda_j - \Lambda_{j-1}$, for $0 \le i < j \le \ell$ and $b, i \le j - 2$.

$$\beta_{\Lambda_{\mathrm{mid}}} = (\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i) + (\alpha_{i+1} + \cdots + \alpha_{j-1}) + (2\alpha_j + \cdots + 2\alpha_{\ell-1} + \alpha_\ell).$$

We do not need to consider (iii′), (iii″), (viii′) and (viii″), because there are only three changes. Below, we handle the cases that $R^\Lambda(\beta_{\Lambda_{\mathrm{mid}}})$ is not wild.

(Δ_{--}) (i) Suppose that $i = j = \ell-1$. Then, $R^\Lambda(\beta_{\Lambda_{\mathrm{mid}}})$ is the Case (i′) with $i = j = \ell-1$, which is not wild.

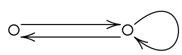
$$\Lambda = \Lambda_0 + \Lambda_b + 2\Lambda_{\ell-1} + \widetilde{\Lambda}, \quad \Lambda'' = \Lambda_1 + \Lambda_{b+1} + 2\Lambda_{\ell-2} + \widetilde{\Lambda}$$

and $\beta_{\Lambda''} = (\alpha_0 + \cdots + \alpha_b) + (2\alpha_{\ell-1} + \alpha_\ell) = \beta_{\Lambda'} + \beta_{\Lambda_{\mathrm{mid}}}$.

($1 \le b \le \ell - 3$) Lemma 2.18 implies that $R^{\Lambda_0+\Lambda_b+2\Lambda_{\ell-1}}(\beta_{\Lambda''})$ is Morita equivalent to

$$R^{\Lambda_0+\Lambda_b}(\alpha_0 + \cdots + \alpha_b) \otimes R^{2\Lambda_{\ell-1}}(2\alpha_{\ell-1} + \alpha_\ell).$$

We know that $R^{\Lambda_0+\Lambda_b}(\alpha_0 + \cdots + \alpha_b)$ is the Brauer graph algebra such that the Gabriel quiver of an idempotent truncation contains



 https://doi.org/10.5802/art.34

and that $R^{2\Lambda_{\ell-1}}(2\alpha_{\ell-1}+\alpha_\ell)$ has an indecomposable projective module P with $\operatorname{End}(P)^{\mathrm{op}} \cong \Bbbk[X]/(X^2)$. Thus, $R^{\Lambda_0+\Lambda_b+2\Lambda_{\ell-1}}(\beta_{\Lambda''})$ is wild.

($b=\ell-1$) We have $\Lambda_{\mathrm{mid}} = (\Lambda_1+\Lambda_{\ell-2}) + 2\Lambda_{\ell-1} + \widetilde{\Lambda}$. If $\ell \geq 3$ there is a path

$$\Lambda = \Lambda_0 + 3\Lambda_{\ell-1} + \widetilde{\Lambda} \longrightarrow \Lambda_{\mathrm{mid}} \longrightarrow \Lambda'' = \Lambda_1 + 2\Lambda_{\ell-2} + \Lambda_\ell + \widetilde{\Lambda},$$

since $\beta_{\Lambda_{\mathrm{mid}}} = \alpha_0 + \cdots + \alpha_{\ell-2} + 2\alpha_{\ell-1} + \alpha_\ell$ and $\beta_{\Lambda''} = \beta_{\Lambda_{\mathrm{mid}}} + \alpha_{\ell-1}$. Thus, it is wild because $R^\Lambda(\beta_{\Lambda_{\mathrm{mid}}})$ is wild. If $\ell = 2$, we have the arrow

$$\Lambda = \Lambda_0 + 3\Lambda_1 + \widetilde{\Lambda} \longrightarrow \Lambda'' = 2\Lambda_0 + \Lambda_1 + \Lambda_2 + \widetilde{\Lambda},$$

which is in the first neighbors and $\beta_{\Lambda''} = \alpha_1$. Hence, it is (f1) if $\ell = 2$.

(ii) Next, we consider the case $i = j = \ell$, for $1 \leq b \leq \ell-2$. Then, $R^\Lambda(\beta_{\Lambda'})$ is from Case (i′) with $i = j = \ell$, which is not wild. Recall

$$\Lambda = \Lambda_0 + \Lambda_b + 2\Lambda_\ell + \widetilde{\Lambda}, \quad \Lambda'' = \Lambda_1 + \Lambda_{b+1} + 2\Lambda_{\ell-1} + \widetilde{\Lambda}$$

and $\beta_{\Lambda''} = (\alpha_0 + \cdots + \alpha_b) + \alpha_\ell$. Lemma 2.18 implies that $R^{\Lambda_0+\Lambda_b+2\Lambda_\ell}(\beta_{\Lambda''})$ is Morita equivalent to $R^{\Lambda_0+\Lambda_b}(\alpha_0+\cdots+\alpha_b) \otimes R^{2\Lambda_\ell}(\alpha_\ell)$, which is wild.

(Δ_{++}) (1) Suppose that $i = j = 1$. Then, $R^\Lambda(\beta_{\Lambda_{\mathrm{mid}}})$ is the algebra from Case (i″) with $i = j = 1$, which is not wild. In this case,

$$\Lambda = \Lambda_0 + \Lambda_b + 2\Lambda_1 + \widetilde{\Lambda}, \quad \Lambda'' = \Lambda_1 + \Lambda_{b+1} + 2\Lambda_2 + \widetilde{\Lambda}$$

and there is a path

$$\Lambda_0 + \Lambda_1 + \Lambda_b \longrightarrow \Lambda_0 + \Lambda_2 + \Lambda_{b+1} \longrightarrow 2\Lambda_2 + \Lambda_{b+1}.$$

If $2 \leq b \leq \ell-1$, $R^{\Lambda_1+\Lambda_b}(\beta_{\Lambda_2+\Lambda_{b+1}})$ is wild. If $b = 1$, then we already computed in Case (5) (Δ_{++})($i = b = 1$) that $R^{\Lambda_0+2\Lambda_1}(2\alpha_0+3\alpha_1)$ is wild. To see this, we computed $\dim_q \operatorname{End}(P)$, for $[P] = f_1^{(2)} f_0^{(2)} f_1 v_\Lambda$. Thus, $R^\Lambda(\beta_{\Lambda''})$ is wild.

(2) Next, we consider the case $i = j = 0$. This $R^\Lambda(\beta_{\Lambda'})$ is a non-wild algebra from Case (i″) with $i = j = 0$. Then,

$$\Lambda = 3\Lambda_0 + \Lambda_b + \widetilde{\Lambda}, \quad \Lambda'' = 3\Lambda_1 + \Lambda_{b+1} + \widetilde{\Lambda}.$$

and $\beta_{\Lambda''} = 2\alpha_0 + \alpha_1 + \cdots + \alpha_b$.

($b=1$) We consider projective $R^{3\Lambda_0+\Lambda_1}(2\alpha_0+\alpha_1)$-modules $[P_1] = f_1 f_0^{(2)} v_\Lambda$ and $[P_2] = f_0^{(2)} f_1 v_\Lambda$ in $V(\Lambda_0)^{\otimes 3} \otimes V(\Lambda_b)$. Then,

$$\begin{aligned}
\dim_q \operatorname{End}(P_1) &= 1 + q^2 + 2q^4 + 2q^6 + 3q^8 + 2q^{10} + 2q^{12} + q^{14} + q^{16},\\
\dim_q \operatorname{End}(P_2) &= 1 + q^4 + 2q^8 + q^{12} + q^{16},\\
\dim_q \operatorname{Hom}(P_1,P_2) &= q^4 + q^8 + q^{12}.
\end{aligned}$$

Since $\dim_q \operatorname{Hom}(P_1,P_2) = \dim_q \operatorname{Hom}(P_2,P_1)$ starts with degree 4, we have one loop of degree 2 and one loop of degree 4 on vertex 1, one loop of degree 4 on vertex 2. Hence, $R^{3\Lambda_0+\Lambda_1}(2\alpha_0+\alpha_1)$ is wild.

($2 \leq b \leq \ell-1$) Set $[P] = f_b \ldots f_1 f_0^{(2)} v_\Lambda \in V(\Lambda_0)^{\otimes 3} \otimes V(\Lambda_b)$. Then

$$\dim_q \operatorname{End}(P) = 1 + 2q^2 + 3q^4 + 4q^6 + 4q^8 + 4q^{10} + 3q^{12} + 2q^{14} + q^{16}.$$

Thus, Lemma 2.14 implies that $R^{3\Lambda_0+\Lambda_b}(2\alpha_0+\alpha_1+\cdots+\alpha_b)$ is wild.

($\Delta_{+-} = \Delta_{-+}$) We consider the case $1 \le i = j \le \ell - 1$ here. We have

$$\Lambda = \Lambda_0 + \Lambda_b + 2\Lambda_i + \widetilde{\Lambda}, \quad \Lambda'' = \Lambda_1 + \Lambda_{b+1} + \Lambda_{i-1} + \Lambda_{i+1} + \widetilde{\Lambda}.$$

and $\beta_{\Lambda''} = (\alpha_0 + \cdots + \alpha_b) + \alpha_i$.

($b + 2 \le i \le \ell - 1$) By Lemma 2.18, $R^{\Lambda_0+\Lambda_b+2\Lambda_i}(\alpha_0 + \cdots + \alpha_b + \alpha_i)$ is Morita equivalent to

$$R^{\Lambda_0+\Lambda_b}(\alpha_0 + \cdots + \alpha_b) \otimes R^{2\Lambda_i}(\alpha_i),$$

which is wild.

($i = b$) In this case, we have $\Lambda - \Lambda'' = (\Lambda_0 + 3\Lambda_b) - (\Lambda_1 + \Lambda_{b-1} + 2\Lambda_{b+1})$ and $\beta_{\Lambda''} = \alpha_0 + \cdots + \alpha_{b-1} + 2\alpha_b$. We set

$$[P] = f_{b-1} \dots f_0 f_b^{(2)} v_\Lambda \in V(\Lambda_0) \otimes V(\Lambda_b)^{\otimes 3}.$$

Then

$$\begin{aligned} f_{b-2} \dots f_0 f_b^{(2)} v_\Lambda = &\Big((1^{b-1}), (0), (1), (1)\Big) + q((b-1), (0), (1), (1)) \\ &+ q\Big((1^{b-1}), (1), (0), (1)\Big) + q^2((b-1), (1), (0), (1)) \\ &+ q^2\Big((1^{b-1}), (1), (1), (0)\Big) + q^3((b-1), (1), (1), (0)) \end{aligned}$$

and each 4-partition has 3 addable $(b-1)$-nodes and no removable $(b-1)$-node. Therefore,

$$\begin{aligned} \dim_q \mathrm{End}(P) &= \big(1 + q^2 + q^4\big)\big(1 + 2q^2 + 2q^4 + q^6\big) \\ &= 1 + 3q^2 + 5q^4 + 5q^6 + 3q^8 + q^{10} \end{aligned}$$

and the Gabriel quiver of $\mathrm{End}(P)$ has three loops. Hence $R^\Lambda(\beta_{\Lambda''})$ is wild.

($1 \le i \le b - 1$) $\beta_{\Lambda''} = \alpha_0 + \cdots + \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_b$. We set

$$[P] = f_i^{(2)} f_{i-1} \dots f_0 f_{i+1} \dots f_b v_\Lambda \in V(\Lambda_0) \otimes V(\Lambda_i)^{\otimes 2} \otimes V(\Lambda_b).$$

Then $f_{i-1} \dots f_0 f_{i+1} \dots f_b v_\Lambda$ is equal to

$$\begin{aligned} &\Big((1^i), (0), (0), (1^{b-i})\Big) + q\Big((1^i), (0), (1^{b-i}), (0)\Big) + q^2\Big((1^i), (1^{b-i}), (0), (0)\Big) \\ &\quad + q\Big((i), (0), (0), (1^{b-i})\Big) + q^2\Big((i), (0), (1^{b-i}), (0)\Big) + q^3\Big((i), (1^{b-i}), (0), (0)\Big) \end{aligned}$$

and each 4-partition has 4 addable i-nodes and no removable i-node. Hence,

$$\begin{aligned} \dim_q \mathrm{End}(P) &= \big(1 + q^2 + 2q^4 + q^6 + q^8\big)\big(1 + 2q^2 + 2q^4 + q^6\big) \\ &= 1 + 3q^2 + 6q^4 + 8q^6 + 8q^8 + 6q^{10} + 3q^{12} + q^{14} \end{aligned}$$

and it is wild.

(Δ_{--}) We consider the case $2 \le i < j = \ell$. These $R^\Lambda(\beta_{\Lambda'})$ are the non-wild algebras from Case (iv′). We have

$$\Lambda = \Lambda_0 + \Lambda_b + \Lambda_i + \Lambda_\ell + \widetilde{\Lambda}, \quad \Lambda'' = \Lambda_1 + \Lambda_{b+1} + \Lambda_{i-1} + \Lambda_{\ell-1} + \widetilde{\Lambda}.$$

(i) First, we consider the case $1 \le b \le i - 2$. We set

$$\Lambda_{\mathrm{mid}} = \Lambda_0 + \Lambda_{b+1} + \Lambda_{i-1} + \Lambda_\ell + \widetilde{\Lambda}.$$

Then, there is a path $\Lambda \to \Lambda_{\mathrm{mid}} \to \Lambda''$ because

$$\begin{aligned}\beta_{\Lambda_{\mathrm{mid}}} &= \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_b + \alpha_{b+1} + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{\ell-1} + \alpha_\ell,\\ \beta_{\Lambda''} &= 2\alpha_0 + 3\alpha_1 + \cdots + 3\alpha_b + \alpha_{b+1} + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.\end{aligned}$$

Hence, the wildness of $R^\Lambda(\beta_{\Lambda''})$ follows.

(ii) Second, we consider the case $b = i$ and set $\Lambda_{\mathrm{mid}} = \Lambda_1 + 2\Lambda_b + \Lambda_{\ell-1} + \widetilde{\Lambda}$. Then, we have

$$\begin{aligned}\beta_{\Lambda_{\mathrm{mid}}} &= \alpha_0 + \cdots + \alpha_{\ell-1},\\ \beta_{\Lambda''} &= \alpha_0 + \cdots + \alpha_{b-1} + 2\alpha_b + \alpha_{b+1} + \cdots + \alpha_\ell.\end{aligned}$$

(iii) Third, we consider the case $b = i + 1$. In this case, we have

$$\begin{aligned}\Lambda &= \Lambda_0 + \Lambda_{b-1} + \Lambda_b + \Lambda_\ell + \widetilde{\Lambda},\\ \Lambda'' &= \Lambda_1 + \Lambda_{b-2} + \Lambda_{b+1} + \Lambda_{\ell-1} + \widetilde{\Lambda},\end{aligned}$$

and

$$\beta_{\Lambda''} = \alpha_0 + \cdots + \alpha_{b-2} + 2\alpha_{b-1} + 2\alpha_b + \alpha_{b+1} + \cdots + \alpha_\ell.$$

Define an indecomposable $R^{\Lambda_0+\Lambda_{b-1}+\Lambda_b+\Lambda_\ell}(\beta_{\Lambda''})$-module P by

$$[P] = f_{b-1}^{(2)} f_b^{(2)} f_{b+1} \dots f_\ell f_{b-2} \dots f_0 v_\Lambda \in V(\Lambda_0) \otimes V(\Lambda_{b-1}) \otimes V(\Lambda_b) \otimes V(\Lambda_\ell).$$

Then, $f_b^{(2)} f_{b+1} \dots f_\ell f_{b-2} \dots f_0 v_\Lambda$ is equal to

$$\begin{aligned}&\Big((1^{b-1}), (0), (1), (1^{\ell-b+1})\Big) + q^2\Big((b-1), (0), (1), (1^{\ell-b+1})\Big)\\ &\qquad + q^2\Big((1^{b-1}), (0), (1), (\ell-b+1)\Big) + q^4\big((b-1), (0), (1), (\ell-b+1)\big).\end{aligned}$$

Each 4-partition has 4 addable $(b-1)$-nodes and no removable $(b-1)$-node. Thus,

$$\begin{aligned}\dim_q \operatorname{End}(P) &= \big(1 + 2q^4 + q^8\big)\big(1 + q^2 + 2q^4 + q^6 + q^8\big)\\ &= 1 + q^2 + 4q^4 + 3q^6 + 6q^8 + 3q^{10} + 4q^{12} + q^{14} + q^{16},\end{aligned}$$

and both Lemma 2.13 and Lemma 2.14 implies that it is wild.

(iv) Finally, we consider the case $i + 2 \le b \le \ell - 1$.

$$\Lambda - \Lambda'' = (\Lambda_0 - \Lambda_1 + \Lambda_b - \Lambda_{b+1}) + (\Lambda_i - \Lambda_{i-1} + \Lambda_\ell - \Lambda_{\ell-1})$$

and $\beta_{\Lambda''} = (\alpha_0 + \cdots + \alpha_b) + (\alpha_i + \cdots + \alpha_\ell)$. Then, Lemma 2.18 implies that $R^{\Lambda_0+\Lambda_i+\Lambda_b+\Lambda_\ell}(\beta_{\Lambda''})$ is Morita equivalent to

$$R^{\Lambda_0+\Lambda_b}(\alpha_0 + \cdots + \alpha_b) \otimes R^{\Lambda_i+\Lambda_\ell}(\alpha_i + \cdots + \alpha_\ell).$$

Both algebras are Brauer graph algebras we already computed, which implies that $R^{\Lambda_0+\Lambda_i+\Lambda_b+\Lambda_\ell}(\beta_{\Lambda''})$ is wild.

(Δ_{++}) (1) We consider the case $1 \le i < j = \ell - 1$. These $R^\Lambda(\beta_{\Lambda'})$ are the non-wild algebras from Case (iv$''$). We have

$$\Lambda = \Lambda_0 + \Lambda_b + \Lambda_i + \Lambda_j + \widetilde{\Lambda}, \quad \Lambda'' = \Lambda_1 + \Lambda_{b+1} + \Lambda_{i+1} + \Lambda_{j+1} + \widetilde{\Lambda}.$$

We choose $\Lambda_{\mathrm{mid}} = \Lambda_0 + \Lambda_b + \Lambda_{i+1} + \Lambda_{j+1} + \widetilde{\Lambda}$. Then

$$\beta_{\Lambda_{\mathrm{mid}}} = \alpha_0 + (\alpha_1 + \cdots + \alpha_i) + (\alpha_1 + \cdots + \alpha_j).$$

Since $\Lambda - \Lambda_{\mathrm{mid}} = \Lambda_i - \Lambda_{i+1} + \Lambda_j - \Lambda_{j+1}$ and

$$\Lambda - \Lambda'' = \Lambda - \Lambda_{\mathrm{mid}} + \Lambda_0 - \Lambda_1 + \Lambda_b - \Lambda_{b+1},$$

we have $\beta_{\Lambda''} = \beta_{\Lambda_{\mathrm{mid}}} + (\alpha_0 + \cdots + \alpha_b)$.

(2) Next we consider the case $i = 0 < j = \ell - 1$. These $R^{\Lambda}(\beta_{\Lambda'})$ are the other non-wild algebras from Case (iv″). We have

$$\Lambda = 2\Lambda_0 + \Lambda_b + \Lambda_j + \widetilde{\Lambda}, \quad \Lambda'' = 2\Lambda_1 + \Lambda_{b+1} + \Lambda_{j+1} + \widetilde{\Lambda}.$$

Then, $\beta_{\Lambda''} = 2\alpha_0 + (\alpha_1 + \cdots + \alpha_b) + (\alpha_1 + \cdots + \alpha_j)$.
We define $[P_1], [P_2] \in V(\Lambda_0)^{\otimes 2} \otimes V(\Lambda_b) \otimes V(\Lambda_j)$ by

$$[P_1] = f_1^{(2)} f_2^{(2)} \dots f_{\min(b,j)}^{(2)} f_0^{(2)} f_{\min(b,j)+1} \cdots f_{\max(b,j)} v_\Lambda,$$
$$[P_2] = f_0^{(2)} f_1^{(2)} \dots f_{\min(b,j)}^{(2)} f_{\min(b,j)+1} \cdots f_{\max(b,j)} v_\Lambda.$$

Then, we have the following.

- $[P_1] = f_1^{(2)}((1),(1),(1^{b-1}),(1^{j-1}))$ and $((1),(1),(1^{b-1}),(1^{j-1}))$ has 6 addable 1-nodes and no removable 1-node.
- $[P_2] = f_0^{(2)}((0),(0),(1^b),(1^j))$ and $((0),(0),(1^b),(1^j))$ has 4 addable 0-nodes and no removable 0-node.

Then, we may find that

$$\dim_q \mathrm{End}(P_1) = 1 + q^2 + 2q^4 + 2q^6 + 3q^8 + 2q^{10} + 2q^{12} + q^{14} + q^{16},$$
$$\dim_q \mathrm{End}(P_2) = 1 + q^4 + 2q^8 + q^{12} + q^{16},$$
$$\dim_q \mathrm{Hom}(P_1, P_2) = \dim \mathrm{Hom}(P_2, P_1) = q^8.$$

Hence, there are 2 loops, one is of degree 2 and the other is of degree 4, on vertex 1, and one loop of degree 4 on vertex 2. Thus, it is wild.

(Δ_{i^-,j^+}) We consider the case $2 \le i < j = \ell - 1$. These $R^{\Lambda}(\beta_{\Lambda'})$ are the non-wild algebras from Case (v). We have

$$\Lambda = \Lambda_0 + \Lambda_b + \Lambda_i + \Lambda_j + \widetilde{\Lambda}, \;\; \Lambda'' = \Lambda_1 + \Lambda_{b+1} + \Lambda_{i-1} + \Lambda_{j+1} + \widetilde{\Lambda}$$

and $\beta_{\Lambda''} = (\alpha_0 + \cdots + \alpha_b) + (\alpha_i + \cdots + \alpha_j)$.

($1 \le b \le i - 2$) In this case, $R^{\Lambda_0 + \Lambda_b + \Lambda_i + \Lambda_j}(\beta_{\Lambda''})$ is Morita equivalent to

$$R^{\Lambda_0 + \Lambda_b}(\alpha_0 + \cdots + \alpha_b) \otimes R^{\Lambda_i + \Lambda_j}(\alpha_i + \cdots + \alpha_j).$$

Both are Brauer graph algebras which we have computed. Then, we see that $R^{\Lambda_0 + \Lambda_b + \Lambda_i + \Lambda_j}(\beta_{\Lambda''})$ is wild.

($i \le b \le \ell - 1$) In this case, we have

$$\Lambda = \Lambda_0 + \Lambda_i + \Lambda_b + \Lambda_j + \widetilde{\Lambda}, \quad \Lambda'' = \Lambda_1 + \Lambda_{i-1} + \Lambda_{b+1} + \Lambda_{j+1} + \widetilde{\Lambda},$$

$$\beta_{\Lambda''} = (\alpha_0 + \cdots + \alpha_{i-1}) + \Big(2\alpha_i + \cdots + 2\alpha_{\min(b,j)}\Big) + \Big(\alpha_{\min(b,j)+1} + \cdots + \alpha_{\max(b,j)}\Big).$$

We define $[P] \in V(\Lambda_0) \otimes V(\Lambda_i) \otimes V(\Lambda_b) \otimes V(\Lambda_j)$ by

$$[P] = f_b^{(2)} f_{b-1}^{(2)} \cdots f_i^{(2)} f_{i-1} \cdots f_0 f_{\min(b,j)+1} \cdots f_{\max(b,j)} v_\Lambda.$$

Then, one can show that

$$\begin{aligned}\dim_q \mathrm{End}(P) &= (1 + q^2 + 2q^4 + q^6 + q^8)(1 + q^4)\\ &= 1 + q^2 + 3q^4 + 2q^6 + 3q^8 + q^{10} + q^{12}.\end{aligned}$$

Lemma 2.13 implies that it is wild.

(Δ_{i^+,j^-}) We consider the case $0 \le i < j = \ell$, $i \le j-2$. These $R^\Lambda(\beta_{\Lambda'})$ are the non-wild algebras from Case (vi). We have

$$\Lambda = \Lambda_0 + \Lambda_b + \Lambda_i + \Lambda_j + \widetilde{\Lambda}, \quad \Lambda'' = \Lambda_1 + \Lambda_{b+1} + \Lambda_{i+1} + \Lambda_{j-1} + \widetilde{\Lambda}.$$

Recall that the arrow $\Lambda' \to \Lambda''$ does not exist if $1 \le j-1 \le b$.

($1 \le b \le j-2$) We choose $\Lambda_{\mathrm{mid}} = \Lambda_0 + \Lambda_b + \Lambda_{i+1} + \Lambda_{j-1} + \widetilde{\Lambda}$. Then,

$$\begin{aligned}\beta_{\Lambda_{\mathrm{mid}}} &= (\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i) + (\alpha_{i+1} + \cdots + \alpha_{j-1}) + (2\alpha_j + \cdots + 2\alpha_{\ell-1} + \alpha_\ell)\\ \beta_{\Lambda''} &= \beta_{\Lambda_{\mathrm{mid}}} + (\alpha_0 + \cdots + \alpha_b).\end{aligned}$$

Then, we see that $R^\Lambda(\beta_{\Lambda_{\mathrm{mid}}})$ is wild.

11. Third neighbors in higher level cases

11.1. **New non-wild cases in the second neighbors.** Note that we do not need to consider those non-wild algebras that have already appeared in the first neighbors as we have treated them. Therefore, we only list the new non-wild cases in the second neighbors (and not in the first neighbors). By the result of the second neighbors, we see that there are no new non-wild algebras in Cases (2), (4), (5), and (6). So, the non-wild cases we have to consider in this section are those listed in Sections 11.1.1, 11.1.2 and 11.1.3 below.

11.1.1. *New non-wild cases in the second neighbors of Case* (7).

(i) $\Lambda = \Lambda_0 + \Lambda_\ell + \widetilde{\Lambda}$, $\Lambda' = \Lambda_2 + \Lambda_\ell + \widetilde{\Lambda}$, $\Lambda'' = \Lambda_2 + \Lambda_{\ell-2} + \widetilde{\Lambda}$ with $m_0 = m_\ell = 1$, $m_1 = m_{\ell-1} = 0$ and $\ell \ge 4$. In this case,

$$\beta_{\Lambda''} = \alpha_0 + \alpha_1 + \alpha_{\ell-1} + \alpha_\ell.$$

(ii) $\Lambda = \Lambda_0 + 2\Lambda_i + \widetilde{\Lambda}$, $\Lambda' = \Lambda_2 + 2\Lambda_i + \widetilde{\Lambda}$, $\Lambda'' = \Lambda_2 + \Lambda_{i-1} + \Lambda_{i+1} + \widetilde{\Lambda}$ with $m_0 = 1$, $m_1 = 0$, $m_i = 2$ and $2 < i \le \ell - 1$. In this case,

$$\beta_{\Lambda''} = \alpha_0 + \alpha_1 + \alpha_i.$$

(iii) $\Lambda = \Lambda_0 + 2\Lambda_\ell + \widetilde{\Lambda}$, $\Lambda' = \Lambda_2 + 2\Lambda_\ell + \widetilde{\Lambda}$, $\Lambda'' = \Lambda_2 + 2\Lambda_{\ell-1} + \widetilde{\Lambda}$ with $m_0 = 1$, $m_1 = 0$, $m_\ell = 2$ and $\ell \ge 3$. In this case,

$$\beta_{\Lambda''} = \alpha_0 + \alpha_1 + \alpha_\ell.$$

11.1.2. *New non-wild cases in the second neighbors of Case* (1). The path we consider is

$$\Lambda = 2\Lambda_0 + \widetilde{\Lambda} \longrightarrow \Lambda' = 2\Lambda_1 + \widetilde{\Lambda} \longrightarrow \Lambda''.$$

(i) $\Lambda = 2\Lambda_0 + \Lambda_\ell + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_1 + \Lambda_\ell + \widetilde{\Lambda} \to \Lambda'' = 2\Lambda_1 + \Lambda_{\ell-2} + \widetilde{\Lambda}$ and $m_0 = 2, m_{\ell-1} = 0$, $m_\ell = 1$. In this case, $\beta_{\Lambda''} = \alpha_0 + \alpha_{\ell-1} + \alpha_\ell$. This also appears in Case (7).

(ii) $\Lambda = 2\Lambda_0 + 2\Lambda_\ell + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_1 + 2\Lambda_\ell + \widetilde{\Lambda} \to \Lambda'' = 2\Lambda_1 + 2\Lambda_{\ell-1}$, $m_0 = 2 = m_\ell$. In this case, $\beta_{\Lambda''} = \alpha_0 + \alpha_\ell$.

(iii) $\Lambda = 2\Lambda_0 + 2\Lambda_i + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_1 + 2\Lambda_i + \widetilde{\Lambda} \to \Lambda'' = 2\Lambda_1 + \Lambda_{i-1} + \Lambda_{i+1} + \widetilde{\Lambda}$, $2 \le i \le \ell - 1$, $m_0 = m_i = 2$. In this case, $\beta_{\Lambda''} = \alpha_0 + \alpha_i$.

(iv) $\Lambda = 2\Lambda_0 + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_1 + 2\widetilde{\Lambda} \to \Lambda'' = 2\Lambda_2 + \widetilde{\Lambda}$, $m_0 = 2, m_1 = 0$, $\operatorname{char}\Bbbk \neq 2$. In this case, $\beta_{\Lambda''} = 2\alpha_0 + 2\alpha_1$.

(v) $\Lambda = 2\Lambda_\ell + \widetilde{\Lambda} \to \Lambda' = 2\Lambda_{\ell-1} + 2\widetilde{\Lambda} \to \Lambda'' = 2\Lambda_{\ell-2} + \widetilde{\Lambda}$, $m_\ell = 2, m_{\ell-1} = 0$, $\operatorname{char}\Bbbk \neq 2$. In this case, $\beta_{\Lambda''} = 2\alpha_{\ell-1} + 2\alpha_\ell$. Note that by symmetry, this case is equivalent to Case (1) (iv).

11.1.3. *New non-wild cases in the second neighbors of Case* (3).

(i) $\Lambda = 2\Lambda_a + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_{a+1} + \widetilde{\Lambda} \to \Lambda'' = \Lambda_{a-2} + \Lambda_{a+2} + \widetilde{\Lambda}$, $2 \leq a \leq \ell - 2$, $m_a = 2$, $m_{a-1} = m_{a+1} = 0$, $\operatorname{char}\Bbbk \neq 2$. We have $\beta_{\Lambda''} = \alpha_{a-1} + 2\alpha_a + \alpha_{a+1}$.

(ii) $\Lambda = 3\Lambda_a + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_a + \Lambda_{a+1} + \widetilde{\Lambda} \to \Lambda'' = 2\Lambda_{a-1} + \Lambda_{a+2} + \widetilde{\Lambda}$, $1 \leq a \leq \ell-2$, $m_a = 3, m_{a+1} = 0$, $\operatorname{char}\Bbbk \neq 3$. We have $\beta_{\Lambda''} = 2\alpha_a + \alpha_{a+1}$.

(iii) $\Lambda = 3\Lambda_a + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_a + \Lambda_{a+1} + \widetilde{\Lambda} \to \Lambda'' = \Lambda_{a-2} + 2\Lambda_{a+1} + \widetilde{\Lambda}$, $2 \leq a \leq \ell-1$, $m_a = 3, m_{a-1} = 0$ and $\operatorname{char}\Bbbk \neq 3$. We have

$$\beta_{\Lambda''} = \alpha_{a-1} + 2\alpha_a.$$

This case is equivalent to Case (3) (ii) by symmetry.

(iv) $\Lambda = 2\Lambda_a + 2\Lambda_b + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_{a+1} + 2\Lambda_b + \widetilde{\Lambda} \to \Lambda'' = \Lambda_{a-1} + \Lambda_{a+1} + \Lambda_{b-1} + \Lambda_{b+1} + \widetilde{\Lambda}$, $1 \leq a < b - 1$, $b \leq \ell - 1$, $m_a = m_b = 2$. We have $\beta_{\Lambda''} = \alpha_a + \alpha_b$.

(v) $\Lambda = 4\Lambda_a + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_{a+1} + 2\Lambda_a + \widetilde{\Lambda} \to 2\Lambda_{a-1} + 2\Lambda_{a+1} + \widetilde{\Lambda}$, $1 \leq a \leq \ell - 1$, $m_a = 4$ and $\operatorname{char}\Bbbk \neq 2$. We have $\beta_{\Lambda''} = 2\alpha_a$.

(vi) $\Lambda = 2\Lambda_a + \Lambda_0 + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_{a+1} + \Lambda_0 + \widetilde{\Lambda} \to \Lambda'' = \Lambda_2 + \Lambda_{a-1} + \Lambda_{a+1} + \widetilde{\Lambda}$, $3 \leq a \leq \ell - 1$, $m_a = 2$, $m_0 = 1$, $m_1 = 0$. In this case,

$$\beta_{\Lambda''} = \alpha_0 + \alpha_1 + \alpha_a.$$

This case also appears in Case (7).

(vii) $\Lambda = 2\Lambda_a + \Lambda_\ell + \widetilde{\Lambda} \to \Lambda' = \Lambda_{a-1} + \Lambda_{a+1} + \Lambda_\ell + \widetilde{\Lambda} \to \Lambda'' = \Lambda_{\ell-1} + \Lambda_{a-1} + \Lambda_{a+1} + \widetilde{\Lambda}$, $1 \leq a \leq \ell - 3$, $m_a = 2$, $m_\ell = 1$, $m_{\ell-1} = 0$. In this case,

$$\beta_{\Lambda''} = \alpha_a + \alpha_{\ell-1} + \alpha_\ell.$$

This case also appears in Case (7).

(viii) $\Lambda = 2\Lambda_0 + 2\Lambda_a + \widetilde{\Lambda} \to \Lambda_{a-1} + \Lambda_{a+1} + 2\Lambda_0 + \widetilde{\Lambda} \to 2\Lambda_1 + \Lambda_{a-1} + \Lambda_{a+1} + \widetilde{\Lambda}$, $2 \leq a \leq \ell-1$, $m_0 = m_a = 2$. In this case,

$$\beta_{\Lambda''} = \alpha_0 + \alpha_a.$$

This case also appears in Case (1).

(ix) $\Lambda = 2\Lambda_\ell + 2\Lambda_a + \widetilde{\Lambda} \to \Lambda_{a-1} + \Lambda_{a+1} + 2\Lambda_\ell + \widetilde{\Lambda} \to 2\Lambda_{\ell-1} + \Lambda_{a-1} + \Lambda_{a+1} + \widetilde{\Lambda}$, $1 \leq a \leq \ell - 2$, $m_\ell = m_a = 2$. In this case,

$$\beta_{\Lambda''} = \alpha_\ell + \alpha_a.$$

This case also appears in Case (1).

We may show that the algebras associated with the third neighbors in these cases are either wild or belong to the first or the second neighbors. Details may be found in [9, Section 11].

Appendix A.

This is a proof of Lemma 2.18, which works for general Lie type [43]. We thank him for the permission to include the proof.

Suppose that $A = (a_{ij})_{i,j\in I}$ is a symmetrizable Cartan matrix, and we have a partition $I = I_1 \cup I_2$ such that $a_{ij} = 0$ for $(i,j) \in I_1 \times I_2$, and we consider $R^\Lambda(\beta)$, for $\beta = \beta_1 + \beta_2$ with

$$\beta_1 \in \bigoplus_{i\in I_1} \mathbb{Z}_{\geq 0}\alpha_i, \quad \beta_2 \in \bigoplus_{i\in I_2} \mathbb{Z}_{\geq 0}\alpha_i.$$

We want to show

$$e(\beta_1 * \beta_2)R^\Lambda(\beta)e(\beta_1 * \beta_2) \cong R^{\Lambda_1}(\beta_1) \otimes R^{\Lambda_2}(\beta_2),$$

where

$$\Lambda_1 = \sum_{i\in I_1} \langle \alpha_i^\vee, \Lambda\rangle \Lambda_i, \quad \Lambda_2 = \sum_{i\in I_2} \langle \alpha_i^\vee, \Lambda\rangle \Lambda_i.$$

Lemma A.1. *If L_1, L_2 are a simple $R(\beta_1)$-module and a simple $R(\beta_2)$-module, respectively, then $L_1 \circ L_2$ is a simple $R(\beta)$-module.*

Proof. Suppose $e(\beta_1 * \beta_2)K \neq 0$, for a submodule K of $L_1 \circ L_2$. Then

$$e(\beta_1 * \beta_2)K \subseteq e(\beta_1 * \beta_2)L_1 \circ L_2 = L_1 \otimes L_2,$$

which implies that $L_1 \otimes L_2$ generates K. Hence, $e(\beta_1 * \beta_2)K = 0$, for any proper submodule of $L_1 \circ L_2$. Since $e(\beta_1 * \beta_2)L_1 \circ L_2 \neq 0$, we may conclude that $\mathrm{Top}(L_1 \circ L_2)$ is a simple module. In particular, $\mathrm{Top}(L_1 \circ L_2) \cong \mathrm{Soc}(L_2 \circ L_1)$.

From Kang–Kashiwara–Kim–Oh's [34, 2.2], we have an $R(\beta_1) \otimes R(\beta_2)$-module homomorphism $L_1 \otimes L_2 \to L_2 \circ L_1$ defined by $u \otimes v \mapsto \psi_{w[n_2,n_1]}(v \otimes u)$. Then, it induces $f : L_1 \circ L_2 \to L_2 \circ L_1$. Similarly, we have $g : L_2 \circ L_1 \to L_1 \circ L_2$. Now,

$$gf(u \otimes v) = g\Big(\psi_{w[n_2,n_1]}(v \otimes u)\Big) = \psi_{w[n_2,n_1]}g(v \otimes u) = \psi^2_{w[n_2,n_1]}u \otimes v,$$

which implies $gf = \mathrm{id}$ by the assumption $a_{ij} = 0$ for $(i,j) \in I_1 \times I_2$. Hence f splits. Since $\mathrm{Top}(L_2 \circ L_1)$ is simple, it implies that $L_1 \circ L_2 \cong L_2 \circ L_1$. If $L_2 \circ L_1$ was not simple, $\mathrm{Im}(f) \subseteq \mathrm{Rad}(L_2 \circ L_1)$, so that it would contradict $gf = \mathrm{id}$. ☐

Remark A.2. By [39, Theorem 2.2], we have

$$(L_1 \circ L_2)^* \cong L_2^* \circ L_1^* \langle (\alpha, \beta)\rangle.$$

Lemma A.3. *Every simple $R(\beta)$-module is isomorphic to $L_1 \circ L_2$, for a simple $R(\beta_1)$-module L_1 and a simple $R(\beta_2)$-module L_2.*

Proof. Let $n = |\beta|$, $n_1 = |\beta_1|$, $n_2 = |\beta_2|$. Observe that $L_1 \otimes L_2 = e(\beta_1 * \beta_2)L_1 \otimes L_2$ and

$$e(\beta_1 * \beta_2)L_1 \circ L_2 = \sum_{w\in S_n} e(\beta_1 * \beta_2)\psi_w \Bbbk[x_1, \ldots, x_n]e(\beta_1 * \beta_2)L_1 \otimes L_2.$$

Since $I_1 \cap I_2 = \emptyset$ implies that $w \in \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ if $e(\beta_1 * \beta_2)\psi_w e(\beta_1 * \beta_2) \neq 0$,

$$e(\beta_1 * \beta_2)L_1 \circ L_2 = L_1 \otimes L_2.$$

Hence, the map $\mathrm{Irr}(R(\beta_1)) \times \mathrm{Irr}(R(\beta_2)) \to \mathrm{Irr}(R(\beta))$ given by $(L_1, L_2) \mapsto L_1 \circ L_2$, which is well-defined by the previous lemma, is injective.

On the other hand, if we consider the span of

$$\{f_{i_1} \dots f_{i_n} \mid \alpha_{i_1} + \dots + \alpha_{i_n} = \beta\},$$

the assumption $a_{ij} = 0$, for $(i,j) \in I_1 \times I_2$, and the categorification theorem implies

$$\begin{aligned} |\operatorname{Irr}(R(\beta)| &= \dim U(\mathfrak{g})_{-\beta} \\ &= \dim U(\mathfrak{g})_{-\beta_1} \times \dim U(\mathfrak{g})_{-\beta_2} \\ &= |\operatorname{Irr}(R(\beta_1)| \times |\operatorname{Irr}(R(\beta_2)|. \end{aligned}$$

Hence, the map $\mathrm{Irr}(R(\beta_1)) \times \mathrm{Irr}(R(\beta_2)) \to \mathrm{Irr}(R(\beta))$ is bijective. □

Lemma A.4. *$R(\beta)e(\beta_1 * \beta_2) \cong R(\beta_1) \circ R(\beta_2)$ as $(R(\beta), R(\beta_1) \otimes R(\beta_2))$-bimodule.*

Proof. Observe that

$$R(\beta)e(\beta_1 * \beta_2) = \bigoplus_{w \in S_n / S_{n_1} \times S_{n_2}} \psi_w (R(\beta_1) \otimes R(\beta_2)).$$

Hence, we have the equality. □

Lemma A.5. *$R^\Lambda(\beta)e(\beta_1 * \beta_2)$ is a progenerator of $R^\Lambda(\beta)$ and*

$$R^\Lambda(\beta)e(\beta_1 * \beta_2) \cong R^{\Lambda_1}(\beta_1) \circ R^{\Lambda_2}(\beta_2)$$

as an $(R^\Lambda(\beta), R^{\Lambda_1}(\beta_1) \otimes R^{\Lambda_2}(\beta_2))$-bimodule.

Proof. The proof that $R^\Lambda(\beta)e(\beta_1 * \beta_2)$ is a progenerator is the same as the proof of Lemma 2.18. Next, we have a surjective $R(\beta)$-module homomorphism

$$\begin{aligned} R(\beta)e(\beta_1 * \beta_2) &\longrightarrow R(\beta_1) \circ R(\beta_2) \otimes_{R(\beta_1) \otimes R(\beta_2)} R^{\Lambda_1}(\beta_1) \otimes R^{\Lambda_2}(\beta_2) \\ &= R^{\Lambda_1}(\beta_1) \circ R^{\Lambda_2}(\beta_2) \\ &= \bigoplus_{w \in \mathfrak{S}_n / \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}} \psi_w R^{\Lambda_1}(\beta_1) \otimes R^{\Lambda_2}(\beta_2). \end{aligned}$$

Further, $w^{-1}(1) = 1$ or $n_1 + 1$ and the first entry of $w^{-1}\nu$ or the $(n_1+1)^{\text{th}}$ entry is ν_1, respectively. Thus,

$$x_1^{\langle \alpha_{\nu_1}^\vee, \Lambda \rangle} e(\nu) \psi_w = \psi_w x_{w^{-1}(1)}^{\langle \alpha_{\nu_1}^\vee, \Lambda \rangle} e\left(w^{-1}\nu\right) = 0.$$

It implies that $R^{\Lambda_1}(\beta_1) \circ R^{\Lambda_2}(\beta_2)$ is an $R^\Lambda(\beta)$-module. We have obtained a surjective $R(\beta)$-module homomorphism

$$R^\Lambda(\beta)e(\beta_1 * \beta_2) \longrightarrow R^{\Lambda_1}(\beta_1) \circ R^{\Lambda_2}(\beta_2).$$

On the other hand, we have a surjective $R(\beta)$-module homomorphism

$$R(\beta_1) \circ R(\beta_2) \cong R(\beta)e(\beta_1 * \beta_2) \longrightarrow R^\Lambda(\beta)e(\beta_1 * \beta_2).$$

If $\nu_1 \in I^{\beta_1}$ and $\nu_2 = (i, \nu') \in I^{\beta_2}$, then

$$e(\nu_1 * \nu_2) x_{n_1+1}^{\langle \alpha_i^\vee, \Lambda \rangle} = e(\nu_1 * \nu_2) \psi_{n_1} \dots \psi_1 x_1^{\langle \alpha_i^\vee, \Lambda \rangle} \psi_1 \dots \psi_{n_1} = 0.$$

It implies that $R^\Lambda(\beta)e(\beta_1 * \beta_2)$ is a right $R^{\Lambda_1}(\beta_1) \otimes R^{\Lambda_2}(\beta_2)$-module, so that we have a surjective $R(\beta)$-module homomorphism

$$R^{\Lambda_1}(\beta_1) \circ R^{\Lambda_2}(\beta_2) \longrightarrow R^\Lambda(\beta)e(\beta_1 * \beta_2).$$

 https://doi.org/10.5802/art.34

Comparing dimensions, we have the desired isomorphism. □

Multiplying $e(\beta_1 * \beta_2)$ on the left of

$$R^\Lambda(\beta)e(\beta_1 * \beta_2) \cong R^{\Lambda_1}(\beta_1) \circ R^{\Lambda_2}(\beta_2),$$

we obtain

$$e(\beta_1 * \beta_2)R^\Lambda(\beta)e(\beta_1 * \beta_2) \cong R^{\Lambda_1}(\beta_1) \otimes R^{\Lambda_2}(\beta_2).$$

Acknowledgments

The second author thanks Liron Speyer for his helpful comments and conversations.

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