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Tropical coefficient dynamics for higher-dimensional cluster categories

Steffen Oppermann and Hugh Thomas *

Dedicated to the memory of Idun Reiten

Abstract. We show that the index in higher-dimensional cluster categories mutates according to a higher-dimensional version of tropical coefficient dynamics.

1. Introduction

The simplest cluster algebras are those which admit a categorification built from a 1-representation finite (i.e., representation finite and hereditary) algebra. There is a theory of d-representation finite algebras for $d > 1$ defined in [8]. This seems a promising place to go looking for "higher" analogues of cluster phenomena.

In a previous paper [13], we focussed on the $(d-1)$-higher Auslander algebra of linearly oriented A_n, written A_n^d, a particular example of d-representation finite algebras. We showed that the basic cluster tilting objects in $\mathscr{C}_n^d$, the d-cluster tilting category of A_n^d correspond to triangulations of a $2d$-dimensional cyclic polytope with $n+2d+1$ vertices. (This generalizes the well-known description of the basic cluster tilting objects in $\mathscr{C}_n^1$ as corresponding to triangulations of an $(n+3)$-gon.) Mutation of cluster tilting objects corresponds to bistellar flip of triangulations, the appropriate higher-dimensional generalization of diagonal flips. We did not find an analogue of a cluster algebra associated to this model. However, we showed that a tropical version of the cluster algebra exchange relations has a geometrically meaningful interpretation.

Within the network of ideas connected to cluster algebras, as well as the cluster variable dynamics (X-dynamics), there is also the system of coefficient dynamics (Y-dynamics),

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which can likewise be tropicalized. It turns out that tropical coefficient dynamics play an important role in the study of cluster algebras, because they govern the evolution of g-vectors, which are used to define bases of cluster algebras. The goal of this paper is to exhibit a higher-dimensional version of tropical coefficient dynamics. It turns out that while some of our construction goes through for any positive d, we get the best results when d is odd. We therefore assume this for the remainder of the introduction. Note that for our main results, we do not assume that our ambient category is associated to the higher Auslander algebra A_n^d — our results apply to any 2-Calabi–Yau $(d+2)$-angulated category, including $\mathscr{C}_n^d$ as a special case.

Let $\mathscr{C}$ be a 2-Calabi–Yau $(d+2)$-angulated category. Let T be a cluster tilting object. Any object X in $\mathscr{C}$ admits a resolution whose terms are in $\operatorname{add} T$, i.e., a $(d+2)$-angle of the form:

$$T_d \longrightarrow T_{d-1} \longrightarrow \cdots \longrightarrow T_0 \longrightarrow X \longrightarrow \Sigma T_d$$

with $T_i \in \operatorname{add} T$ for all i. We define

$$\operatorname{index}_T(X) = \sum_{i=0}^{d} (-1)^i [T_i]$$

as an element of $\mathrm{K}_0^{\mathrm{split}}(\operatorname{add} T)$ (i.e., the free abelian group on the indecomposable summands of T).

Now suppose that T^* is a cluster tilting object obtained from T by a single mutation. The main result of this paper shows how to calculate $\operatorname{index}_{T^*}(X)$ from $\operatorname{index}_T(X)$ (using no other information about X). The result can be summarized in a phrase by saying that the index satisfies a higher analogue of tropical coefficient dynamics.

Theorem 1.1. *Let T be a cluster tilting object in a 2-Calabi–Yau $(d+2)$-angulated category, for some odd d. Let T^* be obtained from T by a single mutation, that is by replacing some indecomposable summand E_m of T by a different indecomposable $E_m^\star$. Assume there is no loop in the quiver of T at the vertex corresponding to E_m.*

- *There are exchange $(d+2)$-angles*

$$\Sigma^{-1} E_m \longrightarrow E_m^\star \longrightarrow \overline{T}_d \longrightarrow \overline{T}_{d-1} \longrightarrow \cdots \longrightarrow \overline{T}_1 \longrightarrow E_m$$

and

$$E_m \longrightarrow \overline{T}^1 \longrightarrow \overline{T}^2 \longrightarrow \cdots \longrightarrow \overline{T}^d \longrightarrow E_m^\star \longrightarrow \Sigma E_m$$

with the $\overline{T}_i$ and $\overline{T}_i$ in $\operatorname{add} T / E_m$.

- *For any rigid object X, the index of X with respect to T^* is obtained from the index of X with respect to T by replacing $[E_m]$ by*

$$-[E_m^\star] - \sum_i (-1)^i [\overline{T}_i] \qquad \text{if the coefficient of } [E_m] \text{ is positive,}$$

$$-[E_m^\star] - \sum_i (-1)^i [\overline{T}^i] \qquad \text{if the coefficient of } [E_m] \text{ is negative.}$$

In Section 2 we explain the usual meaning of tropical coefficient dynamics, and the result of Dehy and Keller [2] showing that, in a 2-Calabi–Yau triangulated category, the index satisfies tropical coefficient dynamics.

In Section 3 we recall the definition of $(d+2)$-angulated categories, and the fact that we can construct a 2-Calabi–Yau $(d+2)$-angulated cluster category for any d-representation finite algebra.

In Section 4 we investigate the notion of mutation for cluster tilting objects in 2-Calabi–Yau $(d+2)$-angulated categories. In particular we give a criterion for the mutability of indecomposable summands of cluster tilting objects (this is a new phenomenon in higher dimension, as all summands are mutable in dimension 1).

In Section 5 we prove our main theorem, giving an explicit formula for the dynamics of the index under mutation.

In Section 6, we return to the setting of [13] and apply the index to define a notion of higher shear coordinates for laminations in a cyclic polytope. We end with an example of the mutation of indices worked out in this setting in Section 7.

2. Coefficient dynamics and the index

This section is motivational. The reader who does not find the prospect of a link to cluster algebraic considerations to be motivational may safely skip it.

2.1. **Coefficient dynamics and tropical coefficient dynamics.**

We follow the presentation in [4], since it is convenient for our purposes. Coefficient dynamics are defined in a semifield $(\mathscr{F}, \cdot, +)$. By definition, $(\mathscr{F}, \cdot)$ is an abelian group, and $+$ is a commutative, associative operation such that multiplication distributes over addition. To orient the reader, one example of a semifield is the positive real numbers equipped with the usual operations of multiplication and addition.

A Y-seed is an $n \times n$ skew-symmetric matrix B together with an n-tuple $y = (y_1, \ldots, y_n)$ of elements of $\mathscr{F}$.

For $1 \leq m \leq n$, we define a mutation operation μ_m on Y-seeds. $\mu_m(B)$ is the result of the usual mutation operation on $n \times n$ skew-symmetric matrices; we do not recall it here. The operation on the coefficients is given as follows:

$$\mu_m^B(y)_i = \begin{cases} y_m^{-1} & \text{if } i = m, \\ y_i \left(1 + y_m^{-\operatorname{sign}(b_{mi})}\right)^{-b_{mi}} & \text{otherwise.} \end{cases}$$

Here $\operatorname{sign}(x)$ is 1, 0, or -1 as x is positive, zero, or negative. We include B in the notation for $\mu_m^B(y)$ to record the dependence of this operation on the matrix B.

We obtain tropical coefficient dynamics by specializing to a tropical semifield. Let $\mathbb{P} = \mathbb{Z}^r$. Coordinatewise addition defines an abelian group structure on $\mathbb{P}$. Define an operation $\oplus$ on $\mathbb{P}$ by

$$(a_1, \ldots, a_r) \oplus (b_1, \ldots, b_r) = (\min(a_1, b_1), \ldots, \min(a_r, b_r))$$

Then $(\mathbb{P}, +, \oplus)$ defines a semifield structure on $\mathbb{P}$. Such semifields are known as *tropical semifields.* (Note that the operation $+$ on $\mathbb{P}$ plays the role of multiplication in the definition of a semifield, while $\oplus$ plays the role of addition.) Specializing the coefficient dynamics to a tropical semifield, we obtain tropical coefficient dynamics.

For a tuple $v = (v_1, \ldots, v_n) \in \mathbb{P}^n$, the mutation μ_m^B is by definition

$$\mu_m^B(v)_i = \begin{cases} -v_m & \text{if } i = m \\ v_i + [0 \oplus (-\operatorname{sign}(b_{mi})v_m)](-b_{mi}) & \text{otherwise,} \end{cases}$$

Note that for a vector x, the term $[0 \oplus x]$ just describes the vector which one obtains from x by replacing all positive entries by zero. If we write $[x]_- = -[0 \oplus x]$, and similarly

$[x]_+ = -[0 \oplus (-x)]$, then the above simplifies to

$$\mu_m^B(v)_i = \begin{cases} -v_m & \text{if } i = m \\ v_i + [b_{mi}]_+[v_m]_+ - [b_{mi}]_-[v_m]_- & \text{otherwise.} \end{cases} \tag{2.1}$$

(Here, we use the notion $[\ ]_+, [\ ]_-$ both on vectors in $\mathbb{Z}^r$ and on scalars in $\mathbb{Z}$.)

2.2. **Indices and tropical coefficient dynamics.** Let $\mathscr{C}$ be a 2-Calabi–Yau category, and let X be a rigid object in the category. For T any cluster tilting object in $\mathscr{C}$, there is a triangle

$$T_1 \longrightarrow T_0 \longrightarrow X \longrightarrow \Sigma T_1$$

with $T_0, T_1 \in \operatorname{add} T$. We define $\operatorname{index}_T(X)$, the index of X with respect to T, to be $[T_0] - [T_1]$ in $\mathrm{K}_0^{\mathrm{split}}(\operatorname{add} T)$.

Let T and T' be cluster tilting objects related by a mutation. We will explain in this section the relationship between $\operatorname{index}_T(X)$ and $\operatorname{index}_{T'}(X)$, and then make the connection to tropical coefficient mutation.

Let $T = E_1 \oplus \cdots \oplus E_n$ be a cluster tilting object in $\mathscr{C}$. For $E_m \in \{E_1, \ldots, E_n\}$ there is a unique other indecomposable $E_m^\star$ such that replacing E_m with $E_m^\star$ gives a new cluster tilting object $T^\star$. This object is given by the following two exchange triangles:

$$E_m \longrightarrow \overline{T}^{\mathrm{left}} \longrightarrow E_m^* \longrightarrow \Sigma E_m$$
$$E_m^* \longrightarrow \overline{T}^{\mathrm{right}} \longrightarrow E_m \longrightarrow \Sigma E_m'$$

with $\overline{T}^{\mathrm{left}}$ and $\overline{T}^{\mathrm{right}}$ in $\operatorname{add} \bigoplus_{j \neq m} E_j$.

The following result explains how the index changes under mutation of cluster tilting objects.

Theorem 2.1 ([2, Theorem 3])**.** *Let T and T^* be cluster tilting objects in $\mathscr{C}$ differing by a single mutation at E_m, as above. Let X be a rigid object in $\mathscr{C}$. Then $\operatorname{index}_{T^*}(X)$ is obtained from $\operatorname{index}_T(X)$ by substituting for $[E_m]$:*

$$-[E_m^*] + \left[\overline{T}^{\mathrm{right}}\right] \qquad \textit{if the coefficient of } [E_m] \textit{ in } \operatorname{index}_T(X) \textit{ is positive}$$
$$-[E_m^*] + \left[\overline{T}^{\mathrm{left}}\right] \qquad \textit{if the coefficient of } [E_m] \textit{ in } \operatorname{index}_T(X) \textit{ is negative}$$

We now relate the previous theorem to tropical coefficient dynamics.

First we turn the indices into integer vectors. For an indecomposable summand E_i of T we denote by $[G : E_i]$ the coefficient of $[E_i]$ in $G \in \mathrm{K}_0^{\mathrm{split}}(T)$. Allowing ourselves a slight abuse of notation, we use the same notation if G is an object in $\operatorname{add} T$, rather than an element of the Grothendieck group.

We then set

$$\overrightarrow{\operatorname{index}}_T(X) = ([\operatorname{index}_T(X) : E_1], \ldots, [\operatorname{index}_T(X) : E_n])$$

Note that the definition of $\overrightarrow{\operatorname{index}}_T(X)$ depends on the an ordering of the summands of T, which we have implicitly fixed by calling them $E_1, \ldots, E_n$. We order the summands of T^* by replacing E_m by E_m^* in its position in the order, that is, the m^{th} coordinate of $\overrightarrow{\operatorname{index}}_{T^*}$ is the coefficient of E_m^*.

With this convention the mutation rule of Theorem 2.1 turns into

$$\overrightarrow{\operatorname{index}}_{T^*}(X) = \mu_m^T(\overrightarrow{\operatorname{index}}_T(X)),$$

with

$$\mu_m^T(v)_i = \begin{cases} -v_m & \text{if } i = m \\ v_i + [\overline{T}^{\text{right}} : E_i]v_m & \text{if } i \neq m \text{ and } v_m \geq 0 \\ v_i + [\overline{T}^{\text{left}} : E_i]v_m & \text{if } i \neq m \text{ and } v_m \leq 0. \end{cases} \tag{2.2}$$

Let us now assume that there are no loops or 2-cycles adjacent to E_m in the quiver of T. Let B be the skew-symmetric matrix whose entry b_{jk} is the number of arrows from k to j, minus the number from j to k.

Under these hypotheses, we have

$$\overline{T}^{\text{right}} \simeq \bigoplus_{j=1}^{n} E_j^{\oplus [b_{mj}]_+} \qquad \overline{T}^{\text{left}} \simeq \bigoplus_{j=1}^{n} E_j^{\oplus [b_{mj}]_-},$$

and then μ_m^T of Equation (2.2) and μ_m^B of Equation (2.1) coincide.

3. $(d+2)$-angulated cluster categories

Throughout, let k be a field. All categories appearing will be k-categories with finite dimensional morphism spaces and splitting idempotents. In particular they have the Krull–Schmidt property. We denote by $\mathrm{D} = \mathrm{Hom}_k(-, k)$ the standard duality of k-vector spaces.

Recall that a triangulated category is an additive category, equipped with an automorphism Σ (called *suspension*), and a class of 3-morphism sequences of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

The sequences in this chosen class are referred to as *triangles*. They are required to satisfy certain axioms. Vaguely, the triangles behave like short exact sequences, but provide a more symmetric language.

A $(d+2)$-angulated category is defined similarly to a triangulated category, just that its special sequences (the $(d+2)$-angles) mimic the behaviour of longer exact sequences.

Definition 3.1 ([5])**.** A $(d+2)$*-angulated category* is an additive category $\mathscr{D}$ with an automorphism Σ, and a class of $(d+2)$-morphism sequences of the form

$$D_0 \xrightarrow{f_0} \cdots \xrightarrow{f_d} D_{d+1} \xrightarrow{f_{d+1}} \Sigma D_0,$$

subject to the following four axioms. The sequences in this chosen class are referred to as *(distinguished)$(d+2)$-angles.*

(D1)
- Sums and summands of distinguished $(d+2)$-angles are distinguished $(d+2)$-angles again.
- For any object $D \in \mathscr{D}$ the trivial $(d+2)$-angle

$$D \xrightarrow{\text{id}} D \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma D$$

  is a distinguished $(d+2)$-angle.
- Any morphism is the first morphism in some distinguished $(d+2)$-angle.

(D2) The class of distinguished $(d+2)$-angles is closed under rotation. That is, for a $(d+2)$-angle as above also

$$D_1 \xrightarrow{f_1} \cdots \xrightarrow{f_d} D_{d+1} \xrightarrow{f_{d+1}} \Sigma D_0 \xrightarrow{(-1)^d \Sigma f_0} \Sigma D_1$$

and

$$\Sigma^{-1} D_{d+1} \xrightarrow{(-1)^d \Sigma^{-1} f_{d+1}} D_0 \xrightarrow{f_0} \cdots \xrightarrow{f_d} D_{d+1}$$

are distinguished $(d+2)$-angles.

(D3) Given the solid part of the following commutative diagram, where the rows are $(d+2)$-angles

$$\begin{array}{ccccccccccc} D_0 & \xrightarrow{f_0} & D_1 & \xrightarrow{f_1} & D_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_d} & D_{d+1} & \xrightarrow{f_{d+1}} & \Sigma D_0 \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{d+1}} & & \downarrow{\scriptstyle \Sigma\varphi_0} \\ E_0 & \xrightarrow{g_0} & E_1 & \xrightarrow{g_1} & E_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_d} & E_{d+1} & \xrightarrow{g_{d+1}} & \Sigma E_0 \end{array}$$

it is possible to find the dashed morphisms φ_2 to φ_{d+1} such that the entire diagram commutes.

(D4) In (D3) it is possible to choose the dashed morphisms in such a way that also the cone of the vertical morphism

$$E_0 \oplus D_1 \xrightarrow{\begin{pmatrix} g_0 & \varphi_1 \\ 0 & -f_1 \end{pmatrix}} E_1 \oplus D_2 \xrightarrow{\begin{pmatrix} g_1 & \varphi_2 \\ 0 & -f_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} g_d & \varphi_{d+1} \\ 0 & -f_{d+1} \end{pmatrix}} E_{d+1} \oplus \Sigma D_0 \xrightarrow{\begin{pmatrix} g_{d+1} & \Sigma\varphi_0 \\ 0 & -\Sigma f_0 \end{pmatrix}} \Sigma E_0 \oplus \Sigma D_1$$

is a distinguished $(d+2)$-angle. (Such a choice of the dashed arrows is sometimes called *good*).

The following elementary properties of $(d+2)$-angulated categories are immediate generalizations of the same properties for triangulated categories, with the exact same proof.

Lemma 3.2. *Any morphism in a distinguished $(d+2)$-angle is a weak kernel of the following morphism, and a weak cokernel of the previous morphism.*

Any monomorphism or epimorphism in a $(d+2)$-angulated category splits. In particular any $(d+2)$-angle in which at least one term is 0 is contractible.

While we did *not* require uniqueness for the $(d+2)$-angle starting with a given morphism in (D2), we do have the following weak uniqueness.

Lemma 3.3. *Let $f_0 \colon D_0 \to D_1$ be a morphism in a $(d+2)$-angulated category $\mathcal{D}$. If we have two completions to $(d+2)$-angles*

$$D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1^i} D_2^i \xrightarrow{f_2^i} \cdots \xrightarrow{f_d^i} D_{d+1}^i \xrightarrow{f_{d+1}^i} \Sigma D_0 \quad i \in \{1, 2\},$$

such that all f_j^i with $2 \leq j \leq d$ are radical morphisms, then the two $(d+2)$-angles are isomorphic.

Proof. By (D4) we can find morphisms $\varphi_2, \ldots, \varphi_{d+1}$ such that the diagram

$$\begin{array}{ccccccccccc} D_0 & \xrightarrow{f_0} & D_1 & \xrightarrow{f_1^1} & D_2^1 & \xrightarrow{f_2^1} & \cdots & \xrightarrow{f_d^1} & D_{d+1}^1 & \xrightarrow{f_{d+1}^1} & \Sigma D_0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{d+1}} & & \downarrow{\scriptstyle \mathrm{id}} \\ D_0 & \xrightarrow{f_0} & D_1 & \xrightarrow{f_1^2} & D_2^2 & \xrightarrow{f_2^2} & \cdots & \xrightarrow{f_d^2} & D_{d+1}^2 & \xrightarrow{f_{d+1}^2} & \Sigma D_0 \end{array}$$

commutes, and such that the cone as in the first row below is a distinguished $(d+2)$-angle again. We can simplify the maps by applying isomorphisms in all positions as depicted

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in the following diagram, where the isomorphisms hidden in the dots in the middle are identities.

$$\begin{array}{ccccccccc}
D_0 \oplus D_1 & \xrightarrow{\begin{pmatrix} f_0 & \mathrm{id} \\ 0 & -f_1^1 \end{pmatrix}} & D_1 \oplus D_2^1 & \xrightarrow{\begin{pmatrix} f_1^2 & \varphi_2 \\ 0 & -f_2^1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} f_d^2 & \varphi_{d+1} \\ 0 & -f_{d+1}^1 \end{pmatrix}} & D_{d+1}^2 \oplus \Sigma D_0 & \xrightarrow{\begin{pmatrix} f_{d+1}^2 & \mathrm{id} \\ 0 & -\Sigma f_0 \end{pmatrix}} & \Sigma D_0 \oplus \Sigma D_1 \\
\downarrow{\scriptstyle \begin{pmatrix} \mathrm{id} & 0 \\ f_0 & \mathrm{id} \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \mathrm{id} & 0 \\ f_1^1 & \mathrm{id} \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \mathrm{id} & 0 \\ f_{d+1}^2 & \mathrm{id} \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \mathrm{id} & 0 \\ \Sigma f_0 & \mathrm{id} \end{pmatrix}} \\
D_0 \oplus D_1 & \xrightarrow{\begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}} & D_1 \oplus D_2^1 & \xrightarrow{\begin{pmatrix} 0 & \varphi_2 \\ 0 & -f_2^1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} f_d^2 & \varphi_{d+1} \\ 0 & 0 \end{pmatrix}} & D_{d+1}^2 \oplus \Sigma D_0 & \xrightarrow{\begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}} & \Sigma D_0 \oplus \Sigma D_1
\end{array}$$

This lower $(d+2)$-angle is isomorphic to the direct sum of the two trivial $(d+2)$-angles $D_1 \to D_1 \to 0 \to \cdots \to 0 \to \Sigma D_1$ and $D_0 \to 0 \to \cdots \to \Sigma D_0 \to \Sigma D_0$, plus

$$0 \longrightarrow D_2^1 \xrightarrow{\begin{pmatrix} \varphi_2 \\ -f_2^1 \end{pmatrix}} D_2^2 \oplus D_3^1 \xrightarrow{\begin{pmatrix} f_2^2 & \varphi_3 \\ 0 & -f_3^1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} f_d^2 & \varphi_{d+1} \end{pmatrix}} D_{d+1}^2 \longrightarrow 0.$$

By the first part of axiom (D1) this last sequence is also a $(d+2)$-angle, and since it contains a zero object it is contractible. Let $h_i\colon D_i^2 \oplus D_{i+1}^1 \to D_{i-1}^2 \oplus D_i^1$ be a homotopy witnessing this contractability, that is

$$\begin{pmatrix} \mathrm{id}_{D_i^2} & 0 \\ 0 & \mathrm{id}_{D_{i+1}^1} \end{pmatrix} = \begin{pmatrix} h_{i+1}^{11} & h_{i+1}^{12} \\ h_{i+1}^{21} & h_{i+1}^{22} \end{pmatrix} \begin{pmatrix} f_i^2 & \varphi_{i+1} \\ 0 & -f_{i+1}^1 \end{pmatrix} + \begin{pmatrix} f_{i-1}^2 & \varphi_i \\ 0 & -f_i^1 \end{pmatrix} \begin{pmatrix} h_i^{11} & h_i^{12} \\ h_i^{21} & h_i^{22} \end{pmatrix}$$

By assumption all the f_j^i are radical morphisms, so $h_{i+1}^{21}\varphi_{i+1}$ and $\varphi_i h_i^{21}$ need to be invertible. Thus φ_{i+1} is split mono and φ_i is split epi. Since this holds for any i (with obvious simplifications at the end of the complex), it follows that all the φ_i are actually isomorphisms. □

One situation that lends itself to applying the theory of $(d+2)$ angulated categories particularly nicely is d-dimensional representation theory in the sense of [6, 7, 8].

Definition 3.4. A subcategory $\mathscr{X}$ of an abelian or triangulated category $\mathscr{C}$ is called *d-cluster tilting* if it is functorially finite, and

$$\begin{aligned} \mathscr{X} &= \left\{ C \in \mathscr{C} \,\middle|\, \mathrm{Ext}^i(C, \mathscr{X}) = 0 \;\; \forall\, i \in \{1, \ldots, d-1\} \right\} \\ &= \left\{ C \in \mathscr{C} \,\middle|\, \mathrm{Ext}^i(\mathscr{X}, C) = 0 \;\; \forall\, i \in \{1, \ldots, d-1\} \right\}. \end{aligned}$$

An object X is called d-cluster tilting if the subcategory $\mathscr{X} = \operatorname{add} X$ is. (One may note that if $\mathscr{C}$ has finite dimensional Hom-spaces then the condition that $\operatorname{add} X$ is functorially finite is automatic.)

Definition 3.5. A finite dimensional algebra Λ is called *d-representation finite* if its global dimension is at most d, and the category $\operatorname{mod} \Lambda$ admits a d-cluster tilting module M.

In particular the summands of M do not have any extensions of degrees smaller than d, but they do admit a nice theory of d-extensions. See [8]. This leads to them admitting a $(d+2)$-angulated version of derived categories.

Definition 3.6. Let Λ be a d-representation finite algebra, and M as in the definition above. The *$(d+2)$-angulated derived category* of Λ, denoted by $\mathscr{D}$, is the additive category given as follows:

- The indecomposable objects of $\mathscr{D}$ are symbols $\Sigma^i X$, where $i \in \mathbb{Z}$ and X is an indecomposable summand of M.

- The morphisms of $\mathscr{D}$ are given as

$$\operatorname{Hom}_{\mathscr{D}}\left(\Sigma^i X, \Sigma^j Y\right) = \begin{cases} \operatorname{Hom}_\Lambda(X,Y) & i = j \\ \operatorname{Ext}^d_\Lambda(X,Y) & i+1 = j \\ 0 & \text{otherwise.} \end{cases}$$

This should be seen as a generalization of the description of derived categories of hereditary algebras ($d = 1$). As in that case, any object is a sum of shifts of modules, and morphisms are given as Hom and Ext^d from the module category.

Theorem 3.7. *Let Λ be a d-representation finite algebra. Then its $(d+2)$-angulated derived category $\mathscr{D}$ is $(d+2)$-angulated, with suspension Σ (i.e. sending the object $\Sigma^i X$ to $\Sigma^{i+1} X$).*

Proof. By [7, Theorem 1.21], the subcategory

$$\operatorname{add}\left\{\Sigma_{\text{triang}}^{id} M \,\middle|\, i \in \mathbb{Z}\right\} \subseteq \mathrm{D}^b(\operatorname{mod}\Lambda)$$

is d-cluster tilting, where Σ_{triang} denotes the suspension in the triangulated category $\mathrm{D}^b(\operatorname{mod}\Lambda)$. By construction this subcategory is closed under Σ^d_{triang}. Thus, by [5], it is $(d+2)$-angulated. □

Remark 3.8. The suspension of the $(d+2)$-angulated category $\mathscr{D}$ is given as $\Sigma = \Sigma^d_{\text{triang}}$. In particular one needs to be careful to not confuse Σ with Σ_{triang}, the latter of which doesn't even define an endofunctor of $\mathscr{D}$.

Definition 3.9. A *Serre functor* on a k-category $\mathscr{C}$ is an endofunctor $S\colon \mathscr{C} \to \mathscr{C}$ giving rise to a functorial isomorphism

$$\mathrm{D}\operatorname{Hom}_{\mathscr{C}}(X,Y) \cong \operatorname{Hom}_{\mathscr{C}}(Y, SX).$$

If a Serre functor exists, then it is unique up to natural isomorphism.

Example 3.10. Let Λ be a finite dimensional algebra of finite global dimension. Then the functor $- \otimes^{\mathbb{L}}_\Lambda \mathrm{D}\Lambda$ is a Serre functor of $\mathrm{D}^b(\operatorname{mod}\Lambda)$.

When realising the $(d+2)$-angulated derived category of a d-representation finite algebra Λ as a subcategory of its ordinary derived category as in the proof of Theorem 3.7 above, then $- \otimes^{\mathbb{L}}_\Lambda \mathrm{D}\Lambda$ restricts to an endofunctor of $\mathscr{D}$ [9, Theorem 3.1]. Thus, in particular, it also defines a Serre functor on $\mathscr{D}$.

For the next proposition, recall that an algebra is called *triangular* if the endomorphism ring of every indecomposable projective module is the base field, and one can linearly order the indecomposable projectives in such a way that there are no non-zero morphisms against the order.

Remark 3.11. We do not know any examples of non-triangular d-representation finite algebras. Thus, from the point of view of treating examples, the assumption that Λ is triangular is rather mild.

Proposition 3.12. *Assume Λ is a triangular d-representation finite algebra. Then its $(d+2)$-angulated derived category $\mathscr{D}$ is directed. In particular, all non-zero endomorphisms of indecomposable objects are isomorphisms.*

Proof. By [7, Theorem 1.23 and the preceding discussion], we know that

$$\mathscr{D} = \operatorname{add} \left\{ \Sigma_{\text{triang}}^{-id} S^i \Lambda \,\middle|\, i \in \mathbb{Z} \right\} \subseteq \mathrm{D}^b(\operatorname{mod} \Lambda)$$

in its incarnation as a subcategory of the triangulated derived category, where $S = - \otimes_\Lambda^{\mathbb{L}} \mathrm{D}\Lambda$ is the Serre functor.

For $i > 0$ we have that

$$\Sigma_{\text{triang}}^{-id} S^i \Lambda = \Sigma_{\text{triang}}^{-id} S^{i-1} \mathrm{D}\Lambda$$

is concentrated in (homologically) positive degrees. Therefore

$$\operatorname{Hom}_{\mathscr{D}} \left(\Lambda, \Sigma_{\text{triang}}^{-id} S^i \Lambda \right) = 0.$$

It follows that any cycle of maps between indecomposable objects in $\mathscr{D}$ necessarily lies entirely inside one of the $\operatorname{add} \Sigma_{\text{triang}}^{-id} S^i \Lambda$. Since Σ_{triang} and S are autoequivalences we may as well consider the case $i = 0$, that is ask about cycles in $\operatorname{add} \Lambda$. These however don't exist by the assumption of Λ being triangular. □

Definition 3.13. A $(d+2)$-angulated category is called 2-*Calabi–Yau* if Σ^2 is a Serre functor.

Remark 3.14. The notion of Calabi–Yau triangulated categories is classical. Here we extend this notion (by using the same definition verbatim) to $(d+2)$-angulated categories.

The $(d+2)$-angulated categories we are most interested in this paper are versions of cluster categories. Let us first recall this concept in the classical situation. Briefly, the idea is to force the derived category to become 2-Calabi–Yau.

Definition 3.15. Let $\mathscr{C}$ be an additive category, $\mathtt{F} \colon \mathscr{C} \to \mathscr{C}$ an automorphism of $\mathscr{C}$. Then the orbit category of $\mathscr{C}$ with respect to $\mathtt{F}$, which is denoted by $\mathscr{C}/\mathtt{F}$, is given as follows:

- The objects of $\mathscr{C}/\mathtt{F}$ are the same as the objects of $\mathscr{C}$.
- The morphisms of $\mathscr{C}/\mathtt{F}$ are given by

$$\operatorname{Hom}_{\mathscr{C}/\mathtt{F}}(C, D) = \coprod_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathscr{C}}(C, \mathtt{F}^i D).$$

It follows that there is a natural functor $\pi \colon \mathscr{C} \to \mathscr{C}/\mathtt{F}$. For this functor, one observes that $\pi \circ \mathtt{F} \cong \pi$, and moreover π is universal amongst functors with this natural isomorphism.

Definition 3.16 ([1])**.** Let H be a hereditary algebra. The *cluster category* associated to H is the orbit category $\mathscr{C} = \mathrm{D}^b(\operatorname{mod} H)/\Sigma^{-2} S$, where $S = - \otimes_H^{\mathbb{L}} \mathrm{D}H$ is the Serre functor on $\mathrm{D}^b(\operatorname{mod} \Lambda)$.

Theorem 3.17 ([12])**.** *For a hereditary algebra H, the cluster category $\mathscr{C}$ as above is triangulated, and the canonical functor $\pi \colon \mathrm{D}^b(\operatorname{mod} \Lambda) \to \mathscr{C}$ is a triangle functor.*

The above definition of cluster category can easily be generalized to d-representation finite algebras.

Definition 3.18 ([13, Section 5])**.** Let Λ be d-representation finite, and $\mathscr{D}$ be its $(d+2)$-angulated derived category as in Definition 3.6.

The $(d+2)$*-angulated cluster category of* Λ is the orbit category

$$\mathscr{C} = \mathscr{D}/\Sigma^{-2} S,$$

where Σ denotes the $(d+2)$-angulated suspension on $\mathscr{D}$, and $S = - \otimes_\Lambda^{\mathbb{L}} \mathrm{D}\Lambda$ is the Serre functor.

This name is justified by the following result:

Theorem 3.19 ([13, Section 5]). *If Λ is d-representation finite, then the $(d+2)$-angulated cluster category is 2-Calabi–Yau $(d+2)$-angulated.*

Observation 3.20. Let Λ be d-representation finite. The subset

$$\operatorname{add}(M \oplus \Sigma\Lambda) \subseteq \mathscr{D}$$

is a *fundamental domain* of the $(d+2)$-angulated cluster category in the sense that the natural projection functor induces a bijection between isomorphism classes of objects in the fundamental domain and isomorphism classes of objects in the $(d+2)$-angulated cluster category.

For X and Y in our fundamental domain we observe that

$$\operatorname{Hom}_{\mathscr{C}}(X,Y) = \operatorname{Hom}_{\mathscr{D}}(X,Y) \oplus \operatorname{Hom}_{\mathscr{D}}\left(X, \Sigma^2 S^{-1} Y\right).$$

Proposition 3.21. *Let Λ be triangular d-representation finite. Let $X \in \mathscr{C}$ be an indecomposable object in the $(d+2)$-angulated cluster category of Λ. Then*

- *any non-zero endomorphism of X is an automorphism;*
- *for any $(d+2)$-angle*

$$X \longrightarrow C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_d \longrightarrow \Sigma X$$

  with all morphisms in the radical of $\mathscr{C}$, we have $X \notin \operatorname{add}\{C_0, \ldots, C_d\}$.

Proof. We may assume that X is indecomposable projective. Then any endomorphism of X in $\mathscr{C}$ is induced by an endomorphism in $\mathscr{D}$. In particular the first claim follows from Proposition 3.12.

For the second claim, assume that C_0 also lies in our fundamental domain. It follows that the given map from X to C_0 in $\mathscr{C}$ is the image of a map in $\mathscr{D}$. This map may be completed to a $(d+2)$-angle in $\mathscr{D}$, with all morphisms being radical morphisms. By the uniqueness of Lemma 3.3 we observe that the given $(d+2)$-angle is isomorphic to the image of the $(d+2)$-angle in $\mathscr{D}$. Now, by Proposition 3.12, X cannot appear in any other terms of the $(d+2)$-angle in $\mathscr{D}$. On the other hand we may observe that the entire $(d+2)$-angle in $\mathscr{D}$ lies in the fundamental domain (since both X and ΣX do), so it cannot contain any $(\Sigma^2 S^{-1})^i X$ for $i \neq 0$. It follows that the image in $\mathscr{C}$ does not contain any other copies of X either. □

4. Mutation of cluster tilting objects

Let $\mathscr{C}$ be a $(d+2)$-angulated category, with suspension Σ. Assume $\mathscr{C}$ is 2-Calabi–Yau.

Definition 4.1. An object $T \in \mathscr{C}$ is called *cluster tilting* if:

(1) it is *rigid*, that is $\operatorname{Hom}_{\mathscr{C}}(T, \Sigma T) = 0$, and
(2) it has the *resolving property*, that is for any object $K \in \mathscr{C}$ there is a $(d+2)$-angle

$$\Sigma^{-1} K \longrightarrow T_d \longrightarrow T_{d-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow K$$

with $T_i \in \operatorname{add} T$.

The basic idea of mutation is to replace an individual indecomposable summand of a cluster tilting object. Let us fix names for the cluster tilting object and the summand we want to replace, which we will use throughout this section.

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Notation 4.2. Throughout, $T \in \mathscr{C}$ denotes a basic cluster tilting object. We denote its indecomposable summands by E_i, so that $T = \bigoplus_{i=1}^{n} E_i$. We pick $m \in \{1, \ldots, n\}$ and try to replace the indecomposable summand E_m. For convenience, we write

$$\overline{T} = \bigoplus_{\substack{i=1 \\ i \neq m}}^{n} E_i$$

for the sum of the remaining summands.

Definition 4.3. A *mutation* of T at E_m is given by a different (up to isomorphism) basic cluster tilting object $T^\star = \overline{T} \oplus E_m^\star$. That is, we replace the indecomposable summand E_m by a different object $E_m^\star$ without destroying the cluster tilting property.

We will see below that $E_m^\star$ is unique and indecomposable if it exists. However, in contrast to the classical case ($d = 1$), there need not be such an $E_m^\star$. If it does exist, we call E_m *mutable.*

Theorem 4.4. *Let T, E_m and $\overline{T}$ be as in Notation 4.2 above. Assume that there is no loop at E_m in the quiver of T. Let $E_m^\star \ncong E_m$ be a non-zero object, such that $\overline{T} \oplus E_m^\star$ is basic. Then the following are equivalent:*

(1) $\overline{T} \oplus E_m^\star$ is cluster tilting. (So it is a mutation of T at E_m.)
(2) $\overline{T} \oplus E_m^\star$ is rigid.
(3) There is a $(d+2)$-angle

$$\Sigma^{-1} E_m \longrightarrow E_m^\star \longrightarrow \overline{T}_d \longrightarrow \overline{T}_{d-1} \longrightarrow \cdots \longrightarrow \overline{T}_1 \xrightarrow{f} E_m$$

with $\overline{T}_i \in \operatorname{add} \overline{T}$ such that f is a right $\overline{T}$-approximation.
Such a $(d+2)$-angle will be called a right exchange $(d+2)$-angle.

(4) There is a $(d+2)$-angle

$$E_m \xrightarrow{g} \overline{T}^1 \longrightarrow \overline{T}^2 \longrightarrow \cdots \longrightarrow \overline{T}^d \longrightarrow E_m^\star \longrightarrow \Sigma E_m$$

with $\overline{T}^i \in \operatorname{add} \overline{T}$ such that g is a left $\overline{T}$-approximation.
Such a $(d+2)$-angle will be called a left exchange $(d+2)$-angle.

In particular, (3) *and* (4) *show an object $E_m^\star$ satisfying these conditions is unique (by Lemma 3.3) and indecomposable, if it exists.*

Moreover, if the above equivalent conditions are satisfied then

(5) For any object $K \in \mathscr{C}$ there is a resolving $(d+2)$-angle

$$\Sigma^{-1} K \longrightarrow T_d \longrightarrow T_{d-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow K$$

with $T_i \in \operatorname{add} T$, such that "middle terms" $T_{d-1}, \ldots, T_1$ all lie in $\operatorname{add} \overline{T}$.

The theorem improves for the case of cluster categories of (certain)d-representation finite algebras. In that case, we need not worry about the existence of loops, and all the above conditions become equivalent:

Theorem 4.5. *Let Λ be a triangular d-representation finite algebra. Let $T \in \mathscr{C}_\Lambda$ be cluster tilting in the $(d+2)$-angulated cluster category of Λ, and E_m an indecomposable summand of T. If Theorem 4.4*(5) *holds, then there is $E_m^* \ncong E_m$ indecomposable satisfying* (1) *to* (4) *in Theorem 4.4.*

Theorem 4.5 gives a criterion for the summand E_m of T to be mutable: in the setting of the theorem, E_m is mutable if and only if E_m only ever appears in the first or last term of a T-resolving $(d+2)$-angle of any $K \in \mathscr{C}_\Lambda$.

In the setting of [13], which, combinatorially speaking, is the setting where cluster tilting objects correspond to triangulations of even-dimensional cyclic polytopes, Williams [17, Theorem 4.10] has given a combinatorial criterion for mutability of summands of a cluster tilting object. It would be interesting to relate the two criteria.

The remainder of this section is devoted to the proofs of Theorems 4.4 and 4.5.

Proof of Theorem 4.4.

$(2) \Longrightarrow (3)$. Let

$$E_m^\star \longrightarrow T_d \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow \Sigma E_m^\star$$

be a $(d+2)$-angle with $T_i \in \operatorname{add} T$ (this exists since T is cluster tilting). We may choose this in such a way that all the maps $T_{i+1} \to T_i$ lie in the radical of $\mathscr{C}$, by splitting off all isomorphisms between the T_i.

Since $\operatorname{Hom}_{\mathscr{C}}(\overline{T}, \Sigma E_m^\star) = 0$ we have $T_0 = (E_m)^n$ for some n. If T_0 was zero then the map $E_m^\star \to T_d$ would be split mono, contradicting $E_m^\star \notin \operatorname{add} T$. Thus $n \geq 1$.

For $i > 0$ we decompose $T_i = \overline{T}_i \oplus (E_m)^{n_i}$, with $\overline{T}_i \in \operatorname{add} \overline{T}$. Applying $\operatorname{Hom}_{\mathscr{C}}(-, \Sigma E_m^\star)$ to the $(d+2)$-angle above, we obtain the exact sequence

$$\begin{gathered}\operatorname{End}_{\mathscr{C}}(\Sigma E_m^\star) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(E_m, \Sigma E_m^\star)^n \longrightarrow \operatorname{Hom}_{\mathscr{C}}(E_m, \Sigma E_m^\star)^{n_1} \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Hom}_{\mathscr{C}}(E_m, \Sigma E_m^\star)^{n_{d-1}} \longrightarrow \operatorname{Hom}_{\mathscr{C}}(E_m, \Sigma E_m^\star)^{n_d} \longrightarrow 0.\end{gathered}$$

The final map is induced by a matrix of radical endomorphisms of E_m. In particular its image lies in $\operatorname{Rad}\operatorname{End}(E_m) \cdot \operatorname{Hom}_{\mathscr{C}}(E_m, \Sigma E_m^\star)^{n_d}$. By the Nakayama lemma this can only happen if $n_d = 0$. Once we know that $n_d = 0$ the same argument iteratively shows that $n_{d-1} = 0$, $n_{d-2} = 0$, until finally also $n_1 = 0$.

Now, since $\operatorname{Hom}_{\mathscr{C}}(E_m, \Sigma E_m^\star)^n$ is the epimorphic image of $\operatorname{End}_{\mathscr{C}}(\Sigma E_m^\star)$ as a module over that endomorphism ring, and this endomorphism ring is basic by assumption, we have $n \leq 1$. Together with the inequality from the start of the proof this means $T_0 = E_m$.

Finally, applying $\operatorname{Hom}_{\mathscr{C}}(\overline{T}, -)$ to the $(d+2)$-angle above, we obtain the exact sequence

$$\operatorname{Hom}_{\mathscr{C}}(\overline{T}, T_1) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(\overline{T}, T_0) \longrightarrow 0.$$

So the map $T_1 \to T_0 = E_m$ is a right $\overline{T}$-approximation.

$(3) \Longrightarrow (2)$. Applying $\operatorname{Hom}_{\mathscr{C}}(\overline{T}, -)$ to the right exchange $(d+2)$-angle, we obtain an exact sequence

$$\operatorname{Hom}_{\mathscr{C}}(\overline{T}, \overline{T}_1) \xrightarrow{f_*} \operatorname{Hom}_{\mathscr{C}}(\overline{T}, E_m) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(\overline{T}, \Sigma E_m^\star) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(\overline{T}, \Sigma \overline{T}_d).$$

Since f is a right $\overline{T}$ approximation the map f_* is surjective. Since T is cluster tilting the rightmost term above vanishes. So we see that $\operatorname{Hom}_{\mathscr{C}}(\overline{T}, \Sigma E_m^\star) = 0$.

By the 2-Calabi–Yau property this also means that

$$\operatorname{Hom}_{\mathscr{C}}\left(E_m^\star, \Sigma\overline{T}\right) = \operatorname{D}\operatorname{Hom}_{\mathscr{C}}\left(\Sigma\overline{T}, \Sigma^2 E_m^\star\right) = 0.$$

Using the right exchange $(d+2)$-angle again, in each argument, we obtain the following commutative diagram, where the three term row and the three term column are exact.

$$\begin{array}{ccccc}
\operatorname{Hom}_{\mathscr{C}}(\overline{T}_d, E_m) & \longrightarrow & \boxed{\operatorname{Hom}_{\mathscr{C}}(\overline{T}_d, \Sigma E_m^\star)} & & \\
\downarrow & & \downarrow & & \\
\operatorname{Hom}_{\mathscr{C}}(E_m^\star, E_m) & \longrightarrow & \operatorname{Hom}_{\mathscr{C}}(E_m^\star, \Sigma E_m^\star) & \longrightarrow & \boxed{\operatorname{Hom}_{\mathscr{C}}(E_m^\star, \Sigma \overline{T}_d)} \\
\downarrow & & & & \\
\boxed{\operatorname{Hom}_{\mathscr{C}}(\Sigma^{-1} E_m, E_m)} & & & &
\end{array}$$

Note that the spaces in the dashed boxes are zero by what we have seen above and the assumption that T is cluster tilting.

It follows that the composition $\operatorname{Hom}_{\mathscr{C}}(\overline{T}_d, E_m) \to \operatorname{Hom}_{\mathscr{C}}(E_m^\star, \Sigma E_m^\star)$ is both surjective and zero, and therefore $\operatorname{Hom}_{\mathscr{C}}(E_m^\star, \Sigma E_m^\star) = 0$.

$(2) \iff (4)$. This equivalence is dual to the equivalence of (2) and (3) which we have proven above.

$(3) \implies (5)$. Let $K \in \mathscr{C}$, and let

$$\Sigma K \longrightarrow T_d \longrightarrow \cdots \longrightarrow T_0 \longrightarrow K$$

be a resolving $(d+2)$-angle with $T_i \in \operatorname{add} T$ and such that all the maps $T_i \to T_{i-1}$ are radical morphisms.

Assume that E_m is a direct summand of T_i for some $i \in \{1, \dots, d-1\}$. Then, rotating the right exchange $(d+2)$-angle, we have two $(d+2)$-angles as in the following diagram.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & T_{i+1} & \longrightarrow & T_i & \longrightarrow & T_{i-1} & \longrightarrow & \cdots & & \\
 & & \downarrow & & \downarrow{\scriptstyle \text{split epi}} & \swarrow & \downarrow & & & & \\
\cdots & \longrightarrow & \overline{T}_1 & \longrightarrow & E_m & \longrightarrow & \Sigma E_m^\star & \longrightarrow & \Sigma \overline{T}_d & \longrightarrow & \cdots
\end{array}$$

The map $\overline{T}_1 \to E_m$ is a right $\overline{T}$-approximation of E_m. Since E_m does not have any loops attached to it, this map is also a radical T-approximation of E_m. Thus the composition $T_{i+1} \to T_i \to E_m$ factors through $\overline{T}_1$ as indicated above by the left dashed arrow. We then obtain a morphism of $(d+2)$-angles, and in particular the right dashed arrow.

Now the map $E_m \to \Sigma E_m^\star$ is a right T-approximation (since $\operatorname{Hom}_{\mathscr{C}}(T, \Sigma \overline{T}_d) = 0$), hence the right dashed map factors through it as indicated by the dotted arrow above. Hence we have that

$$\begin{aligned}
[E_m \longrightarrow \Sigma E_m^\star] &= [E_m \longrightarrow \Sigma E_m^\star] \circ [T_i \longrightarrow E_m] \circ \overbrace{[E_m \longrightarrow T_i]}^{\text{split mono}} \\
&= [T_{i-1} \longrightarrow \Sigma E_m^\star] \circ [T_i \longrightarrow T_{i-1}] \circ [E_m \longrightarrow T_i] \\
&= [E_m \longrightarrow \Sigma E_m^\star] \circ \underbrace{[T_{i-1} \longrightarrow E_m] \circ [T_i \longrightarrow T_{i-1}] \circ [E_m \longrightarrow T_i]}_{\in \operatorname{Rad} \operatorname{End}_{\mathscr{C}}(E_m)}
\end{aligned}$$

and so the map $E_m \to \Sigma E_m^\star$ vanishes. This however means that the map $\overline{T}_1 \to E_m$ is a split epimorphism, which cannot happen because $E_m \notin \operatorname{add} \overline{T}$.

$(1) \implies (2)$. Immediate.

(2) $\Longrightarrow$ (1). By the implications we already know, we may assume additionally that (3), (4) and (5) hold. That is, we have a right and a left exchange triangle connecting E_m and $E_m^\star$ and we know that any K has a resolving $(d+2)$-angle without E_m appearing in the middle terms.

Since rigidity is assumed in (2), we only need to show that $\overline{T} \oplus E_m^\star$ has the resolving property. So let $K \in \mathscr{C}$. Then there is a $(d+2)$-angle

$$\Sigma^{-1}K \longrightarrow T_d \longrightarrow T_{d-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow K$$

with $T_i \in \operatorname{add} T$, and we may assume by (5) that $T_i \in \operatorname{add} \overline{T}$ for $i \in \{1, \dots, d-1\}$. We write $T_0 = \widetilde{T}_0 \oplus (E_m)^a$ and $T_d = \widetilde{T}_d \oplus (E_m)^b$, with $\widetilde{T}_0$ and $\widetilde{T}_d \in \operatorname{add} \overline{T}$. First we consider the following diagram, where the top row is a sum of copies of the right exchange $(d+2)$-angle.

$$\begin{array}{ccccccccccccc}
(E_m^\star)^a & \longrightarrow & (\overline{T}_d)^a & \longrightarrow & (\overline{T}_{d-1})^a & \longrightarrow & \cdots & \longrightarrow & (\overline{T}_1)^a & \xrightarrow{(f)^a} & (E_m)^a & \longrightarrow & (\Sigma E_m^\star)^a \\
 & & & & & & & & \uparrow & & \uparrow{\scriptstyle (0\ 1)} & & \\
\Sigma^{-1}K & \longrightarrow & T_d & \longrightarrow & T_{d-1} & \longrightarrow & \cdots & \longrightarrow & T_1 & \longrightarrow & \widetilde{T}_0 \oplus (E_m)^a & \longrightarrow & K
\end{array}$$

Since f is a right $\overline{T}$-approximation, and $T_1 \in \operatorname{add} \overline{T}$ we get the factorization indicated by the dashed arrow.

We may complete these vertical morphisms to a good morphism of $(d+2)$-angles, and consider the cone $(d+2)$-angle. After splitting off the identity on $(E_m)^a$ this looks like the upper row in the following diagram.

$$\begin{array}{ccccccccccccc}
\Sigma^{-1}K & \longrightarrow & \widetilde{T}_d \oplus (E_m)^b \oplus (E_m^\star)^a & \to & T_{d-1} \oplus (\overline{T}_d)^a & \to & \cdots & \to & T_1 \oplus (\overline{T}_2)^a & \to & \widetilde{T}_0 \oplus (\overline{T}_1)^a & \longrightarrow & K \\
 & & {\scriptstyle (0\ 1\ 0)}\uparrow & & \uparrow & & & & & & & & \\
(\Sigma^{-1}E_m^\star)^b & \longrightarrow & (E_m)^b & \longrightarrow & (\overline{T}^1)^b & \longrightarrow & \cdots & \longrightarrow & (\overline{T}^{d-1})^b & \longrightarrow & (\overline{T}^d)^b & \longrightarrow & (E_m^\star)^b
\end{array}$$

The lower row consists of b copies of the left exchange $(d+2)$-angle. Dual to the first step, we observe that since g is a left $\overline{T}$-approximation and $T_{d-1} \oplus (\overline{T}_d)^a \in \operatorname{add} \overline{T}$ we can find the dashed arrow, and hence a good morphism of $(d+2)$-angles. Again we take the cone and split off the identity on $(E_m)^b$, obtaining a $(d+2)$-angle resolving K by terms in $\operatorname{add} \overline{T} \oplus E_m^\star$. □

Proof of Theorem 4.5. We first observe that, by Proposition 3.21, E_m has no loops in the quiver of T. Therefore Theorem 4.4 applies. Thus it suffices to show that (5) $\Longrightarrow$ (3).

Let f be a minimal right $\overline{T}$-approximation of E_m, and complete it to a $(d+2)$-angle

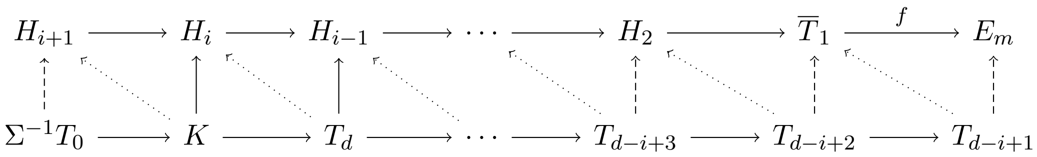
$$\Sigma^{-1}E_m \longrightarrow H_{d+1} \longrightarrow H_d \longrightarrow \cdots \longrightarrow H_2 \longrightarrow \overline{T}_1 \xrightarrow{f} E_m,$$

in which all maps lie in the radical of $\mathscr{C}$.

First assume that $H_i \notin \operatorname{add} T$ for some $i \in \{2, \dots, d\}$. Let i be minimal with this property, and let K be an indecomposable direct summand of H_i which does not lie in $\operatorname{add} T$. Since T is cluster tilting there is a $(d+2)$-angle

$$K \longrightarrow T_d \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow \Sigma K$$

with $T_i \in \operatorname{add} T$, and by (5) we may choose it such that E_m is not a direct summand of T_j for $j \in \{1, \dots, d-1\}$. We obtain the following diagram, where the rows are (parts of) rotations of these $(d+2)$-angles.



Here, the left solid vertical map is a split monomorphism (which exists by assumption). The right vertical solid map exists since the map $K \to T_d$ is a left T-approximation, and $H_{i-1} \in \operatorname{add} T$ by the minimality of i. (In the extreme case that $i = 2$ we use $\overline{T}_1$ instead of H_{i-1}, but the argument remains the same.) Then we get a morphism of $(d+2)$-angles as indicated by the dashed vertical maps. Note that since $i \in \{2, \ldots, d\}$ we have $d - i + 1 \in \{1, \ldots, d-1\}$, so $T_{d-i+1} \in \operatorname{add} \overline{T}$. Hence, since f is a right $\overline{T}$-approximation, the rightmost dashed vertical map factors through f, as indicated by the rightmost dotted arrow. Now we propagate this homotopy to the left, obtaining the other dotted arrows. Looking at the position of K we see that the split monomorphism $K \to H_i$ can now be written as the sum of the compositions $K \to H_{i+1} \to H_i$ and $K \to T_d \to H_i$, both of which lie in the radical of $\mathscr{C}$. This is a contradiction. Hence we have shown that for $i \in \{2, \ldots, d\}$ we have $H_i \in \operatorname{add} T$.

Finally note that by Proposition 3.21 we may choose the $(d+2)$-angle such that no other term has summands E_m. It follows that H_2 up to H_d are in $\operatorname{add} \overline{T}$, and thus that we have found a right mutation $(d+2)$-angle. □

5. The index

Definition 5.1. For an additive category $\mathscr{A}$ we denote by $\mathrm{K}_0^{\mathrm{split}}(\mathscr{A})$ the split Grothendieck group of $\mathscr{A}$, that is the free abelian group on the objects of $\mathscr{A}$ subject to the identifications $[A_1 \oplus A_2] = [A_1] + [A_2]$. If the Krull–Schmidt theorem holds for $\mathscr{A}$ then this is just the free abelian group on the indecomposables in $\mathscr{A}$.

By abuse of notation, we write $\mathrm{K}_0^{\mathrm{split}}(A)$ instead of $\mathrm{K}_0^{\mathrm{split}}(\operatorname{add} A)$ for an object $A \in \mathscr{A}$.

Definition 5.2. Let T be a cluster tilting object in a 2-Calabi–Yau $(d+2)$-angulated category $\mathscr{C}$. Then, for any object $K \in \mathscr{C}$ the *index* of K with respect to T is

$$\operatorname{index}_T(K) = \sum_{i=0}^{d} (-1)^i [T_i] \in \mathrm{K}_0^{\mathrm{split}}(T),$$

provided that the resolution of K in $\operatorname{add} T$ is

$$\Sigma^{-1} K \longrightarrow T_d \longrightarrow \cdots \longrightarrow T_0 \longrightarrow K.$$

Remark 5.3. Note that splitting off any isomorphisms between summands of two adjacents T_i does not affect the index, since this summand will appear twice with opposite signs. Since any resolution can be turned into one only having radical morphisms between the T_i by splitting off summands, and since a resolution only containing radical morphisms is unique by (a version of) Lemma 3.3, we see that the index is well-defined.

This definition of index was given by Jørgensen [10], and also plays an important role in [11, 14, 15, 16].

Suppose that the cluster tilting object T^* is obtained from the cluster tilting object T by a single mutation. Our goal in this section is to explain the relationship between $\operatorname{index}_T(K)$ and $\operatorname{index}_{T^*}(K)$.

The paper [10] is also concerned with relations among different indices, but with the important distinction that Jørgensen is relating indices of different modules with respect to the same cluster tilting object, whereas we are relating indices of the same object with respect to different cluster tilting objects. These two approaches produce quite different dynamics, as is already visible in the cluster algebraic setting ($d = 1$).

Our ultimate goal for this section is to show that (for odd d) the index satisfies higher tropical coefficient dynamics. Let us start by making precise what that means.

Definition 5.4. Let d be an odd positive integer, and let $\mathscr{C}$ be a 2-Calabi–Yau $(d+2)$-angulated category. Assume all cluster tilting objects in $\mathscr{C}$ have exactly n summands.

A function

$$f\colon \{\text{(ordered) cluster tilting objects in } \mathscr{C}\} \longrightarrow \mathbb{Z}^n$$

satisfies *higher tropical coefficient dynamics* if the following mutation formula holds:

For $T = \bigoplus_{i=1}^n E_i$ and $T^\star$ two cluster tilting objects related by a single mutation in E_m as in Definition 4.3, using the notation of Theorem 4.4 we have

$$f(T^\star)_i = \begin{cases} -f(T)_i & \text{if } i = m \\ f(T)_i - \sum_{j=1}^d (-1)^j [\overline{T}_j : E_i] f(T)_m & \text{if } f(T)_m \geq 0 \\ f(T)_i - \sum_{j=1}^d (-1)^j [\overline{T}^j : E_i] f(T)_m & \text{if } f(T)_m \leq 0, \end{cases}$$

generalizing the formula for μ_m^T of Section 2.2.

Remark 5.5. Note that the alternating sums in the mutation formula are very close to the ones appearing in the definition of the index of $\Sigma E_m^\star$ and $E_m^\star$, respectively. Thus the formula may also be written as

$$f(T^\star)_i = \begin{cases} -f(T)_i & \text{if } i = m \\ f(T)_i - [\operatorname{index}_T(\Sigma E_m^\star) : E_i] f(T)_m & \text{if } f(T)_m \geq 0 \\ f(T)_i + [\operatorname{index}_T(E_m^\star) : E_i] f(T)_m & \text{if } f(T)_m \leq 0, \end{cases}$$

We start our investigation of the effect of mutation on indices with the following proposition, which summarizes the information we obtain relatively directly from Theorem 4.4.

Proposition 5.6. *Let T be a cluster tilting object in $\mathscr{C}$, and let E_m be a mutable summand of $T = \overline{T} \oplus E_m$ without loops attached to it. Let $T^\star = \overline{T} \oplus E_m^\star$ be the mutated cluster tilting object.*

Let $K \in \mathscr{C}$ and let

$$\Sigma^{-1} K \longrightarrow T_d \longrightarrow \cdots \longrightarrow T_0 \longrightarrow K$$

be a $(d+2)$-angle with $T_i \in \operatorname{add} T$, and such that the maps between the T_i lie in the radical of $\mathscr{C}$.

Then

$$\begin{aligned} \operatorname{index}_{T^\star}(K) = \operatorname{index}_T(K) + [T_0 : E_m] \cdot \Big((-1)^d [E_m^\star] - \operatorname{index}_T(\Sigma E_m^\star) \Big) \\ + [T_d : E_m] \cdot ([E_m^\star] - \operatorname{index}_T(E_m^\star)) \end{aligned}$$

(In particular this includes the claim that the right hand side, which a priori is an element of $\mathrm{K}_0^{\mathrm{split}}(T^\star \oplus E_m)$, in fact lies in the subgroup $\mathrm{K}_0^{\mathrm{split}}(T^\star)$.)

Proof. This follows from the construction of a $(d+2)$-angle resolving K in the proof of Theorem 4.4. In that proof we wrote $a = [T_0 : E_m]$ and $b = [T_d : E_m]$, and constructed a $(d+2)$-angle showing that (in the notation of that proof)

$$\begin{aligned}
\operatorname{index}_{T^\star}(K) &= [\widetilde{T}_0] + a[\overline{T}_1] + b[E_m^\star] + \sum_{i=1}^{d-1}(-1)^i \left([T_i] + a[\overline{T}_{i+1}] + b\left[\overline{T}^{d+1-i}\right]\right) \\
&\qquad + (-1)^d \left(\widetilde{T}_d + a\,[E_m^\star] + b[\overline{T}^1]\right) \\
&= \operatorname{index}_T(K) - a[E_m] - (-1)^d b[E_m] \\
&\qquad - a\left(\operatorname{index}_T(\Sigma E_m^\star) - [E_m] - (-1)^d[E_m^\star]\right) \\
&\qquad - b\left(\operatorname{index}_T(E_m^\star) - (-1)^d[E_m] - [E_m^\star]\right) \\
&= \operatorname{index}_T(K) - a\left(\operatorname{index}_T(\Sigma E_m^\star) - (-1)^d[E_m^\star]\right) - b\left(\operatorname{index}_T(E_m^\star) - [E_m^\star]\right)
\end{aligned}$$

as claimed. □

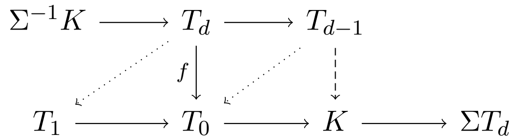
Lemma 5.7. *Let $K \in \mathscr{C}$ be rigid (i.e. $\operatorname{Hom}_{\mathscr{C}}(K, \Sigma K) = 0$), and let*

$$\Sigma^{-1}K \longrightarrow T_d \longrightarrow T_{d-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow K$$

be a resolving $(d+2)$-angle, such that $T_i \in \operatorname{add} T$, and all the maps $T_i \to T_{i-1}$ lie in the radical of $\mathscr{C}$.

Then T_d and T_0 have no common direct summands.

Proof. Assume T_d and T_0 have a common direct summand. Then we have a map $f\colon T_d \to T_0$ which does not lie in the radical of $\mathscr{C}$. We consider the following diagram.



Since the composition from $\Sigma^{-1}K$ to K is 0, we obtain the dashed arrow making the square commutative. Since $\operatorname{Hom}_{\mathscr{C}}(T_{d-1}, \Sigma T_d) = 0$ the dashed arrow factors through $T_0 \to K$ as indicated by the right dotted arrow. Now the compositions $T_d \xrightarrow{f} T_0 \to K$ and $T_d \to T_{d-1} \to T_0 \to K$ coincide, so the difference $f - [T_d \to T_{d-1} \to T_0]$ factors through $[T_1 \to T_0]$, as indicated by the left dotted arrow. However, both $[T_d \to T_{d-1}]$ and $[T_1 \to T_0]$ are radical morphisms, contradicting the fact that f is not a radical morphism. □

Corollary 5.8. *Let T be a cluster tilting object in $\mathscr{C}$, and let E_m be a mutable summand of $T = \overline{T} \oplus E_m$ without loops attached to it. Let $T^\star = \overline{T} \oplus E_m^\star$ be the mutated cluster tilting object.*

Let $K \in \mathscr{C}$, and let

$$\Sigma^{-1}K \longrightarrow T_d \longrightarrow \cdots \longrightarrow T_0 \longrightarrow K$$

be a resolving $(d+2)$-angle, with $T_i \in \operatorname{add} T$, and such that the maps between the T_i lie in the radical of $\mathscr{C}$. Then

$$\begin{aligned}
\operatorname{index}_{T^\star}(K) &= \operatorname{index}_T(K) + (-1)^d[\operatorname{index}_T(K) : E_m][E_m^\star] \\
&\qquad - |[\operatorname{index}_T(K) : E_m]| \cdot \begin{cases} \operatorname{index}_T(\Sigma E_m^\star) & \text{if } [T_d : E_m] = 0 \\ \operatorname{index}_T(E_m^\star) & \text{if } [T_0 : E_m] = 0 \end{cases}
\end{aligned}$$

Note that this formula only applies if (at least) one of the two "if" conditions holds. If $K \in \mathscr{C}$ is rigid ($\operatorname{Hom}_{\mathscr{C}}(K, \Sigma K) = 0$)*, then by Lemma 5.7 (at least) one of the "if" conditions holds.*

Proof. This immediately follows from Proposition 5.6. Note that if $[T_d : E_m] = 0$ then $[T_0 : E_m] = [\operatorname{index}_T(K) : E_m]$ and similarly in the other case. □

For odd d we obtain the following result as an immediate consequence of the previous corollary.

Corollary 5.9. *Assume d is odd. Let T be a cluster tilting object in $\mathscr{C}$, and let E_m be a mutable summand of $T = \overline{T} \oplus E_m$ without loops attached to it. Let $T^\star = \overline{T} \oplus E_m^\star$ be the mutated cluster tilting object.*

Let $K \in \mathscr{C}$ be rigid (i.e. $\operatorname{Hom}_{\mathscr{C}}(K, \Sigma K) = 0$*).*

Then

$$\operatorname{index}_{T^\star}(K) = \operatorname{index}_T(K) - [\operatorname{index}_T(K) : E_m][E_m^\star] \begin{cases} -[\operatorname{index}_T(K) : E_m] \operatorname{index}_T(\Sigma E_m^\star) & \text{if } [\operatorname{index}_T(K) : E_m] \geq 0 \\ +[\operatorname{index}_T(K) : E_m] \operatorname{index}_T(E_m^\star) & \text{if } [\operatorname{index}_T(K) : E_m] \leq 0 \end{cases}$$

Comparing to Remark 5.5, we see that Corollary 5.9 precisely says that the index obeys higher tropical coefficient dynamics.

The important advantage of Corollary 5.9 over Corollary 5.8 is that in Corollary 5.9, in order to know which case of the formula to apply, we ask whether $[\operatorname{index}_T(K) : E_m]$ is non-negative or non-positive, which, obviously, depends only on $\operatorname{index}_T(K)$. In contrast, when applying the Corollary 5.8, we must know whether $[T_d : E_m]$ is non-negative or $[T_0 : E_m]$ is non-negative, and when d is even, the index will not tell us this, since either term would contribute positively to the index.

In other words, Corollary 5.9 tells us how to determine $\operatorname{index}_{T^*}(K)$ knowing nothing about K other than $\operatorname{index}_T(K)$.

6. Higher shear coordinates

We now specialize to the situation studied in [13]. A_n^d is the $(d-1)$ higher Auslander algebra of linearly oriented A_n, and $\mathscr{C}_n^d$ is its $(d+2)$-angulated cluster category.

6.1. Reminders about $\mathscr{C}_n^d$.

The following description of $\mathscr{C}_n^d$ can be found in [13], in particular in Section 6. Define ${}^{\circlearrowleft}\mathbf{I}_n^d$ to be the set of $d+1$-sets from $\{0, 1, \ldots, n+2d\}$ with no two elements cyclically consecutive (i.e., if $i \in I$, then $i+1 \notin I$, with addition taken modulo $n+2d+1$).

The indecomposable objects of $\mathscr{C}_n^d$ are indexed by ${}^{\circlearrowleft}\mathbf{I}_n^d$. For $(i_0, \ldots, i_d) \in {}^{\circlearrowleft}\mathbf{I}_n^d$, we write $O_{(i_0, \ldots, i_d)}$ for the corresponding indecomposable object of $\mathscr{C}_n^d$.

We say that I and J in ${}^{\circlearrowleft}\mathbf{I}_n^d$ *intertwine* if the elements of I and J alternate cyclically; that is to say, it is possible to list the elements of I as $i_0 < \cdots < i_d$ and the elements of J as $j_0 < \cdots < j_d$ such that either $i_0 < j_0 < i_1 < j_1 \cdots < i_d < j_d$ or $j_0 < i_0 < j_1 < i_1 < \cdots < j_d < i_d$. In this case we say that I and J intertwine. We say that two subsets X and Y of ${}^{\circlearrowleft}\mathbf{I}_n^d$ do not intertwine if no $I \in X$ and $J \in Y$ intertwine.

For $I, J \in {}^{\circlearrowleft}\mathbf{I}_n^d$, we have that $\operatorname{Hom}(O_I, \Sigma O_J)$ is one-dimensional if I and J intertwine, and zero-dimensional otherwise.

It follows that the indices of the summands of a cluster tilting object are necessarily non-intertwining. In fact, such collection of indices are exactly the non-intertwining sets of maximal size.

The action of the suspension has a nice interpretation in terms of this indexing. For $I \in {}^{\circlearrowleft}\mathbf{I}_n^d$,

$$\Sigma O_{(1,\ldots,i_d)} \simeq \begin{cases} O_{(i_0-1,\ldots,i_d-1)} & \text{for } i_0 > 1 \\ O_{(i_1-1,\ldots,i_d-1,n+2d+1)} & \text{for } i_0 = 1. \end{cases}$$

The set ${}^{\circlearrowleft}\mathbf{I}_n^d$ also has an interpretation in terms of convex geometry. Choose real numbers $v_0 < \cdots < v_{n+2d}$. Consider the moment curve, defined by $p(t) = (t, t^2, \ldots, t^{2d})$. Take the convex hull of the points $p(v_i)$; this is a cyclic polytope P with vertices $p(v_i)$. An internal d-simplex of P is a simplex whose vertices are vertices of P and which does not lie entirely within the boundary of P. The internal d-simplices of P are the convex hulls of $p(i_0), \ldots, p\ (i_d)$ for $(i_0, \ldots, i_d) \in {}^{\circlearrowleft}\mathbf{I}_n^d$.

The cluster tilting objects in $\mathscr{C}_d^n$ are in bijective correspondence with triangulations of P (i.e., subdivisions of P into simplices, all of whose vertices are vertices of P). Under this correspondence, a triangulation corresponds to the direct sum of the indecomposable objects corresponding to the internal d-simplices appearing in the triangulation.

For $I \in {}^{\circlearrowleft}\mathbf{I}_n^d$, if O_I is a mutable summand of T, which mutates to O_J for $J = (j_0, \ldots, j_d)$, then the triangulation corresponding to T restricts to a triangulation of the convex hull of the points $p(i_0), \ldots, p(i_d), p(j_0), \ldots, p(j_d)$. There are two triangulations of a $2d$-dimensional cyclic polytope with $2d+2$ vertices, one with internal d-simplex I, the other with internal d-simplex J; on the level of the triangulation, mutating T at O_I replaces the one by the other.

6.2. **Shear coordinates for** $d = 1$**.** We begin in the $d = 1$ setting, where A_n^1 is simply the path algebra of linearly oriented A_n quiver, and $\mathscr{C}_n^1$ is the corresponding cluster category $D^b(\operatorname{mod} A_n^1)/\Sigma^{-2}S$. The combinatorial model for all d described in Subsection 6.1 restricts to something simpler here. The convex hull of $n+3$ points on the moment curve is a convex $(n+3)$-gon, and the choice to locate the vertices on the moment curve does not make any difference: we can simply think of any convex polygon in the plane, with vertices numbered counterclockwise from 0 to $n+2$.

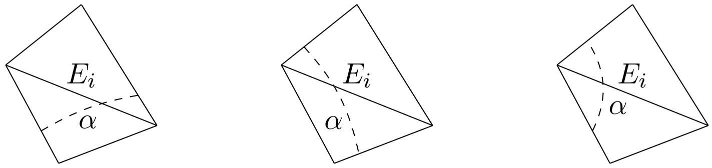
The indecomposable objects of $\mathscr{C}_n^1$ are of the form $O_{\{i_0,i_1\}}$, where the possible pairs $\{i_0, i_1\}$ correspond to the diagonals of the polygon.

For K an indecomposable object of $\mathscr{C}_n^1$, we write $\gamma(K)$ for the corresponding diagonal of the polygon.

Let $T = \bigoplus_{i=1}^n E_i$ be a cluster tilting object. The diagonals corresponding to summands of T define a triangulation.

We will be interested in what we refer to as "arcs" in the polygon. These are curves whose endpoints lie on two different boundary segments of the polygon (and, in particular, not at vertices). The shear coordinates of an arc α with respect to a triangulation T are given as follows. Each diagonal $\gamma(E_i)$ defines a quadrilateral composed of the two triangles with edges corresponding to summands of T or edges of the polygon, and having $\gamma(E_i)$ as an edge. In order for $\mathrm{sh}_T(\alpha)_i$ to be non-zero, α must enter and leave the quadrilateral surrounding $\gamma(E_i)$ on opposite sides. In this case, we have $\mathrm{sh}_T(\alpha)_i = 1$ or -1, according to the diagram below. (We also include an example of an arc α with $\mathrm{sh}_T(\alpha)_i = 0$.)

 https://doi.org/10.5802/art.33



$\mathrm{sh}_T(\alpha)_i = 1$ $\mathrm{sh}_T(\alpha)_i = -1$ $\mathrm{sh}_T(\alpha)_i = 0$

Note that our choice of signs in shear coordinates is the opposite of the usual one. This is a result of the choice we have made for the orientation of the numbering of the vertices of the polygon, which is consistent with the interpretation in terms of points on the moment curve.

For an arc α, we define α^- to be the line segment that results by moving its endpoints counterclockwise until they reach vertices of the polygon.

The following theorem follows immediately from putting together two facts that are well-known, though possibly to different sets of people.

Theorem 6.1. *Let T be a cluster tilting object, and let α be an arc. If α^- is not a diagonal of the polygon, then* $\mathrm{sh}_T(\alpha) = 0$. *Otherwise,* $\mathrm{sh}_T(\alpha) = \mathrm{index}_T(O_{\alpha^-})$.

Proof. If α^- is not a diagonal of the polygon, then α^- is an edge, and the endpoints of α are on adjacent edges of the polygon. In this case, it is clear that the shear coordinates of α are zero, as it can never enter and leave a quadrilateral by opposite edges.

Otherwise, $\mathrm{sh}_T(\alpha)$ agrees with the g-vector of the cluster variable corresponding to O_{α^-} with respect to the initial variables corresponding to the summands of T by Fomin–Thurston [3, Proposition 17.3].

The result of Dehy–Keller [2] mentioned earlier implies, as they explain, that the index of an indecomposable object K in $\mathcal{C}_n^1$ with respect to a cluster tilting object T agrees with the g-vector of the cluster variable corresponding to K with respect to the initial variables corresponding to the summands of T.

Putting these two facts together, the theorem is proved. □

This result can be extended to collections of arcs not intersecting in their interiors. These are called *laminations.* The corresponding objects are no longer indecomposable, but they are rigid.

For generalization to higher dimensions, it will be convenient to express the combinatorial essence of the geometric story we have been telling about arcs. Let $A = (a_0, a_1) \in \mathbb{R}^2$, where $a_0 < a_1$, and neither a_0 nor a_1 belongs to $\{v_0, v_1, \dots, v_{n+2}\}$. We think of A as corresponding to the arc α which is the intersection of the convex hull of $p(v_0), \dots, p(v_{n+2})$ with the line segment from $p(a_0)$ to $p(a_1)$.

To describe α^-, it is convenient to introduce some further notation. For a real number t, let t^- denote the largest v_i less than or equal to t. If $t < v_0$, define $t^- = v_{n+2d}$. Now define $A^- = \mathrm{sort}(a_0^-, a_1^-)$. (Here "sort" sorts the elements of the tuple to be weakly increasing.) Then $\alpha^- = A^-$.

For future use, let us similarly define t^+ to be the smallest v_i greater than or equal to t (or v_0 if there is none), and $A^+ = \mathrm{sort}(a_0^+, a_1^+)$.

6.3. **Higher shear coordinates.** Now consider the case that $d > 1$ is odd.

Pick A a $(d+1)$-tuple of real numbers, disjoint from $\{v_i\}$. Then A determines a $(d+1)$-tuple of points on the moment curve, avoiding the vertices of P. We consider slicing P by the d-dimensional plane which is the affine hull of $\{p(a) \mid a \in A\}$. We want to define shear coordinates for this slice with respect to T.

Let E_m be a summand of T, and suppose that we want to compute the shear coordinate of A corresponding to it. Suppose first that E_m is mutable. In this case, there is a $(2d+2)$-tuple $B = (b_0 < b_1 < \cdots < b_{2d+1})$ such that T restricts to a triangulation of the convex hull of $p(b_0), \dots, p(b_{2d+1})$, and E_m corresponds to one of the two internal simplices of this polytope, whose vertices are the points corresponding to some $R \subseteq B$, where either $R = (b_0, b_2, \dots, b_{2d})$ or $R = (b_1, b_3, \dots, b_{2d+1})$. The points of B divide the real line into $2d+2$ intervals (with the interval below b_0 and the interval above b_{2d+1} counting as the same interval). There are $d+1$ of these intervals which are immediately below some element of R, and there are $d+1$ which are immediately above some element of R. If there is an element of A which lies in each of the intervals immediately below an element of R, the shear coordinate in position E_m of A is -1. If there is an element of A which lies in each of the intervals immediately above an element of R, the shear coordinate is 1. Otherwise, the shear coordinate is zero.

Unfortunately, if E_m is not mutable, then we do not have control over the configuration of the triangulation T around the d-simplex corresponding to E_m, and it is not clear how to use the above approach to define shear coordinates.

We may however take inspiration from the fact that shear coordinates satisfy tropical coefficient dynamics, and make the following definition.

Definition 6.2. A function $\mathrm{sh}_\bullet(A)\colon \{\text{cluster tilting objects in } \mathscr{C}\} \to \mathbb{Z}^n$ defines higher shear coordinates of A if:

(1) $\mathrm{sh}_T(A)_m$ is the shear coordinate of A with respect to E_m as defined above whenever E_m is a mutable summand of T;
(2) $\mathrm{sh}_\bullet(A)$ satisfies higher tropical coefficient dynamics (see Definition 5.4).

One easily sees that there is at most one function defining shear coordinates for a given A: start with any two that differ in some coordinate on a given T. By the first property of shear coordinates they do agree on mutable summands. Then the second property implies that they still differ on the same coordinate of all mutations $T^\star$ of T. But under iterated mutation any summand will eventually become mutable, contradicting the first property we assumed.

We will show that a unique function defining shear coordinates does exist, by showing the following, where A^- is defined as at the end of Section 6.2.

Theorem 6.3. *There exists a (unique) function* $\mathrm{sh}_\bullet(A)$ *defining shear coordinates of* A*. This function is given by*

$$\mathrm{sh}_\bullet(A) = \mathrm{index}_\bullet(O_{A^-})$$

where we set $O_{A^-} = 0$ *whenever* $A^- \notin {}^{\circlearrowleft}\mathbf{I}_n^d$.

Proof. We have seen in Corollary 5.9 that the index satisfies higher tropical coefficient dynamics. Thus it only remains to show that the index computes shear coordinates with respect to mutable summands as described above.

To that end, assume E_m is mutable, and let B and R be as above. For notational purposes we fix $R = (b_0, b_2, \dots, b_{2d})$, and write $R^* = (b_1, b_3, \dots, b_{2d+1})$, but nothing changes if the roles are reversed.

Note that

$$\mathrm{Hom}(O_R, O_{A^-}) = \mathrm{Hom}(O_R, \Sigma O_{A^+}) = \begin{cases} k & R \text{ and } A^+ \text{ intertwine} \\ 0 & \text{otherwise.} \end{cases}$$

The left $\overline{T}$-approximation of E_m is given by the sum over all O_{R_i}, where R_i is obtained from R by replacing b_{2i} by b_{2i+1} (see [13, Theorem 6.3(3)]).

If A lies in the intervals directly above R, then R and A^+ intertwine, but none of the R_i intertwine with A^+. It follows that there is a map from E_m to O_{A^-} which does not factor through a left $\overline{T}$-approximation of E_m, and hence that E_m appears in the right T-approximation of O_{A^-}.

By Theorem 4.4(5) and Lemma 5.7 this is the only contribution of E_m to the index of O_{A^-}, so $[\operatorname{index}_T(O_{A^-}) : E_m] = 1$ as desired.

If A lies in the intervals directly below R, then it lies in the intervals directly above R^*. The same argument as above shows that $[\operatorname{index}_{T^*}(O_{A^-}) : E_m^*] = 1$, and it follows from our mutation formula that $[\operatorname{index}_T(O_{A^-}) : E_m] = -1$.

Next, we must consider other possible values of A, and show that in all other cases, $[\operatorname{index}_T(O_{A^-}) : E_m] = 0$. We shall do this by assuming that this quantity is either strictly positive or strictly negative, and showing that we must in fact be in one of the two previous cases.

Assume first that $[\operatorname{index}_T(O_{A^-}) : E_m] > 0$. Then E_m appears as a summand in the right T-approximation of O_{A^-}. In particular R and A^+ intertwine. Since the morphism $E_m \to O_{A^-}$ does not factor through $E_m \to \bigoplus_i O_{R_i}$ it necessarily induces a non-zero morphism $\Sigma^{-1} E_m^* \to O_{A^-}$. In particular $(b_1+1, b_3+1, \ldots, b_{2d+1}+1)$ and A^+ also intertwine. Thus, up to cyclic renumbering, either $b_{2i+1} + 1 < a_i^+ < b_{2i+2}$, so A lies in the intervals immediately below R, or $b_{2i} < a_i^+ < b_{2i+1} + 1$, so A lies in the intervals immediately above R. In the former case we have seen above that $[\operatorname{index}_T(O_{A^-}) : E_m] = -1$, contradicting our current assumption. In the latter case we have seen that indeed $[\operatorname{index}_T(O_{A^-}) : E_m] = 1$.

Finally, assume $[\operatorname{index}_T(O_{A^-}) : E_m] < 0$. Then $[\operatorname{index}_{T^*}(O_{A^-}) : E_m^*] > 0$, and thus A lies either in the intervals immediately below or the intervals immediately above R^*. But this is the exact same condition as being in the intervals immediately above or immediately below R. □

In fact, shear coordinates are also determined by the following different set of three properties, which may be easier to verify:

(1) If A^- does not define an internal d-simplex of P, then the shear coordinates of A with respect to any T are zero.
(2) If A^- does define an internal simplex of P, and the triangulation T contains that simplex as a face, then the shear coordinates of A with respect to T is the unit vector e_{A^-} (i.e., 1 in position A^- and zero elsewhere).
(3) If T^* is obtained from T by a mutation, the shear coordinates of A with respect to T^* are obtained from those with respect to A by tropical coefficient mutation.

For $d = 1$ it is clear that these three properties uniquely describe shear coordinates.

For higher d, it is clear that there is at most one collection of shear coordinates satisfying the three properties above, but it is not clear that there are any: on the face of it, it isn't clear that tropical coefficient dynamics define a consistent collection of shear coordinates. However, one easily sees that the index of O_{A^-} does satisfy (1), by defining the corresponding object to be zero, (2), by definition of the index, and (3) by Corollary 5.9.

As in the $d = 1$ case, instead of considering A, a single increasing $(d+1)$-tuple of real numbers, we could consider a collection $\mathcal{A}$ of increasing $(d+1)$-tuples all disjoint from $\{v_i\}$ and which are pairwise non-intertwining. Geometrically, this is equivalent to saying that the d-simplices corresponding to the elements of $\mathcal{A}$ do not intersect. The shear coordinates of such a collection of $(d+1)$-tuples can be defined as the sum of the shear coordinates of each of the $(d+1)$-tuples individually.

We then have the following result.

Corollary 6.4. *Let $\mathcal{A}$ be a collection of increasing $(d+1)$-tuples of real numbers all disjoint from $\{v_i\}$, and which are pairwise non-intertwining. Then $\mathrm{sh}_\bullet(\mathcal{A})$ satisfies higher tropical coefficient dynamics.*

Proof. We have that

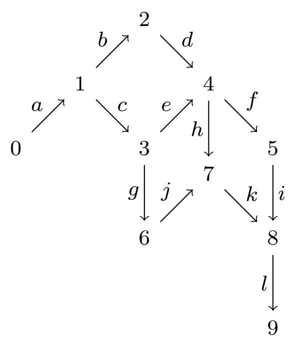
$$\mathrm{sh}_\bullet(\mathcal{A}) = \mathrm{index}_\bullet\left(\bigoplus_{A\in\mathcal{A}} O_{A^-}\right),$$

and since $\bigoplus_{A\in\mathcal{A}} O_{A^-}$ is rigid, its index satisfies higher tropical coefficient dynamics. □

7. An example in $\mathscr{C}_3^3$

In this section, we consider a particular case of the situation studied in [13]. A_3^3 is the second higher Auslander algebra of linearly oriented A_3, and $\mathscr{C}_3^3$ is its 5-angulated cluster category. We describe them both explicitly below.

Example 7.1. Consider the algebra given by the following quiver with relations.

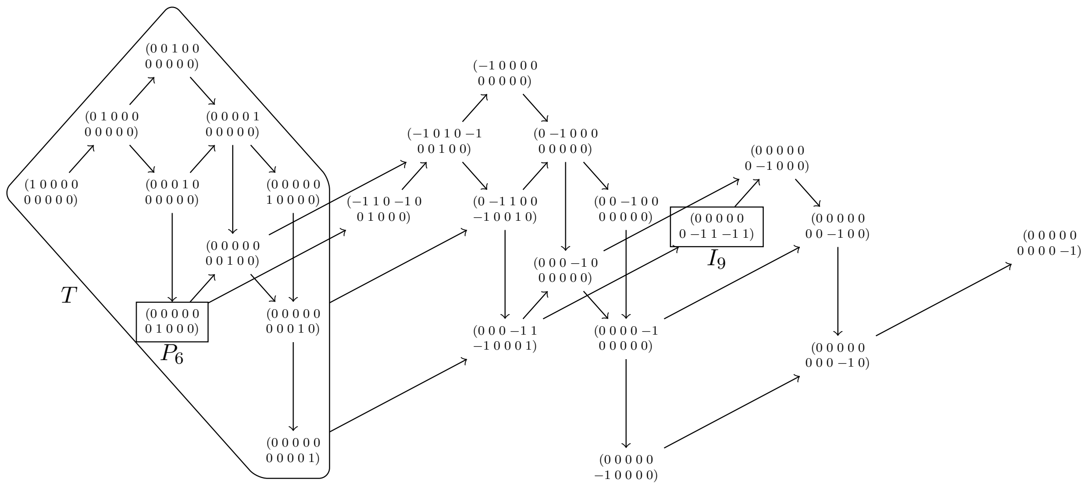


$$(ca, db-ec, fe, gc, he-jg, hd, if-kh, kj, lk)$$

This algebra is 3-representation finite – indeed it is a smaller version of [7, Example 6.13].

We will work in the associated 5-angulated cluster category $\mathscr{C}$, as in Definition 3.18, and illustrate it by drawing our fundamental domain as in Observation 3.20.

We initially consider the cluster tilting object T which is the image of the algebra in the cluster category. Then the indices of all indecomposable objects are given as in the following picture. (We write the 10-dimensional vectors in two lines purely for space reasons.)



The projective P_6 is mutable, its replacement being the injective I_9 — indeed the left mutation 5-angle is given as the image of the exact sequence

$$0 \longrightarrow P_6 \longrightarrow P_7 \longrightarrow P_8 \longrightarrow P_9 \longrightarrow I_9 \longrightarrow 0.$$

We denote by $T^\star$ the cluster tilting object obtained by replacing P_6 by I_9.

Now we can apply the formula of Corollary 5.9 to calculate the index of all indecomposables with respect to $T^\star$.

For those vectors with a positive entry in the 6th coordinate we use the fact that

$$\mathrm{index}_T(\Sigma I_9) = (-1, 1, 0, -1, 0, 0, 1, 0, 0, 0).$$

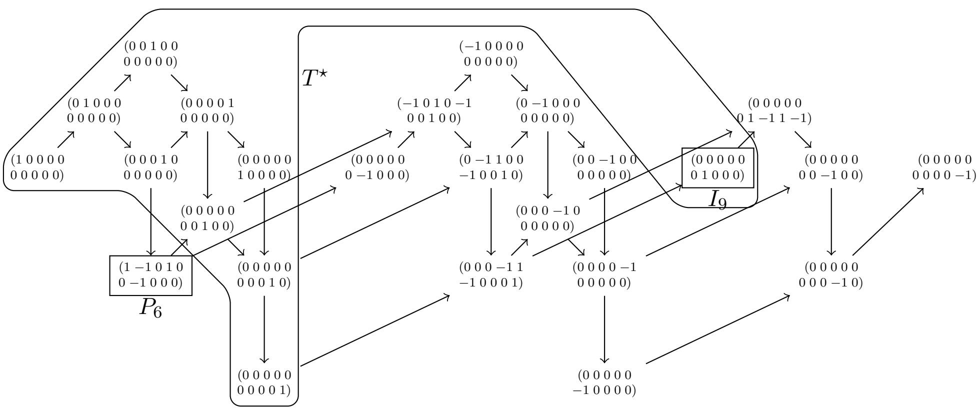
Thus the formula of Corollary 5.9 tells us that if the sixth coordinate is $c > 0$, we should replace it by $-c$ (which we will also write in the 6th position, but which now represents a coefficient of $[I_9]$), and also to add $(c, -c, 0, c, 0, 0, 0, 0, 0, 0)$ to the index.

For a vector with a negative entry in the 6th coordinate we recall that

$$\mathrm{index}_T(I_9) = (0, 0, 0, 0, 0, 0, -1, 1, -1, 1).$$

Thus the formula of Corollary 5.9 tells us that if the sixth coordinate is $c < 0$, we should to change the sign in the 6th coordinate and interpret it as a coefficient of a different basis vector, and also add $(0, 0, 0, 0, 0, 0, 0, c, -c, c)$ to the index.

With these two rules, it is straightforward to calculate all indices:



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