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ABSTRACT. Let K be an algebraically closed field of characteristic 2, G be the algebraic group SL_2 over K, and V be the natural representation of G. Let $b_k^{G,V}$ denote the number of G-indecomposable factors of $V^{\otimes k}$, counted with multiplicity, and let $\delta = \frac{3}{2} - \frac{\log 3}{2\log 2}$. Then there exists a smooth multiplicatively periodic function $\omega(x)$ such that $b_{2k}^{G,V} = b_{2k+1}^{G,V}$ is asymptotic to $\omega(k)k^{-\delta}4^k$. We also prove a lower bound of the form $c_W k^{-\delta}(\dim W)^k$ for $b_k^{G,W}$ for any tilting representation W of G.

1. INTRODUCTION

Let $G = \mathrm{SL}_2$ over an algebraically closed field K of characteristic 2, and let V be the 2-dimensional natural representation of G. Let $b_k^{G,V}$ denote the number of G-indecomposable factors of $V^{\otimes k}$, counted with multiplicity. Coulembier, Ostrik, and Tubbenhauer ask [2, Question 6.1] if there exist $c_1, c_2, \delta > 0$ such that

$$c_1 k^{-\delta} 2^k \leq b_k^{G,V} \leq c_2 k^{-\delta} 2^k \tag{1.1}$$

They give a heuristic argument, due to Etingof, predicting that $\delta = \frac{3}{2} - \frac{\log 3}{2\log 2}$. In this paper, we prove that their prediction is right. Indeed, something stronger is true.

Theorem 1.1. *Defining δ as above, there exists a smooth function $\omega\colon \mathbb{R} \to (0,\infty)$ such that $\omega(4^x)$ is periodic (mod 1) and*

$$b_{2k}^{G,V} = b_{2k+1}^{G,V} \sim \omega(k)k^{-\delta}4^k.$$

We construct the function $\psi(x) = \omega(x)x^{-\delta}$ explicitly as an infinite convolution of distributions of the form $\delta_0 + \varphi_s$, where δ_0 is the delta function concentrated at 0, and the φ_s are rescalings of the theta series associated to the (mod 4) primitive Dirichlet character χ. This theta series is positive and has rapid decay at both 0 and ∞. These facts follow

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almost immediately from (4.8), which is essentially the functional equation of the Dirichlet L-function $L(s,\chi)$.

Our proof interprets $b_{2k}^{G,V}$ as the number of paths of length k, counted with multiplicity, of the *fusion graph* of $V^{\otimes 2}$, the directed graph giving the decomposition into indecomposable factors of the tensor product of a given indecomposable tilting module with $V^{\otimes 2}$. In the characteristic zero case, the vertex set of the graph would be $\mathbb{N}$, and the arrows would connect pairs of consecutive non-negative integers, so we would end up counting left factors of Dyck paths of length k. In our characteristic 2 case, the graph reflects the dyadic nature of the representation category of G. See Figure 2.1 below (which shows only the component of the graph corresponding to even highest weights.)

Classifying paths of length k according to their final endpoint n, we observe two striking departure from the familiar characteristic zero behavior. Firstly, the number of paths with given k is roughly inversely proportional to 2 to the power of the number of 1's in the binary expansion of n. Second, for fixed k, the number of paths terminating in n falls off sharply when n is significantly smaller than $\sqrt{k}$ as well when n is significantly larger than $\sqrt{k}$ (this latter case being in line with characteristic zero behavior). This can be regarded as the discrete analogue of the rapid decay of our theta function at ∞. In proving these claims, we are helped greatly by the fact that the generating function $X_{2^s}(t)$ of paths of length k terminating in 2^s satisfies the recursive formula

$$X_{2^{s+1}}(t) = \frac{X_{2^s}(t)^2}{1-2X_{2^s}(t)^2}. \tag{1.2}$$

I would like to acknowledge a very helpful correspondence with the authors of [1]. Shortly after answering [2, Question 6.1], I learned that they had independently done so and had, indeed, extended their result to all positive characteristics. Our methods were different enough that we agreed it would make sense to write separate papers rather than combine forces.

An early version of [1] asked whether there exists a continuous function ω as in Theorem 1.1. I realized that my approach would give a direct construction of ω. Their most recent draft gives a non-constructive answer to the same question.

Their paper also asks for asymptotic formulas for tensor powers of tilting modules other than V. In the case $p=2$, this paper gives a lower bound of the form $c_W k^{-\delta}(\dim W)^k$, but we do not yet have an upper bound of the same form, let alone an asymptotic formula as in Theorem 1.1.

I would like to thank the referee for a very careful reading of the manuscript and many useful corrections and suggestions.

2. Tilting representations and the fusion graph

For every non-negative integer n, let $T(n)$ denote the (unique) indecomposable tilting module of G with highest weight n. Thus V is isomorphic to $T(1)$. Every tensor product of tilting modules is again a tilting module [3] and is therefore determined by its formal character, which we express as an element of $\mathbb{Z}[t,t^{-1}]$.

The formal characters of the $T(n)$ are well known. Following [7, Proposition 2.6], if

$$n+1 = 2^j + a_{j-1}2^{j-1} + \cdots + a_0,$$

with $a_i \in \{0,1\}$, we define the *support* of n to consist of all integers m in the set

$$\mathsf{supp}(n) = \left\{2^j \pm a_{j-1}2^{j-1} \pm \cdots \pm a_0\right\}.$$

From [7, Proposition 5.4], the formal character of $T(n)$ is

$$\sum_{m\,\in\,\mathsf{supp}(n)} \frac{t^m - t^{-m}}{t - t^{-1}}, \tag{2.1}$$

where m ranges over the support of n. This can be expressed as

$$\chi_n(t) = \frac{\left(t^{2^j} - t^{-2^j}\right)\prod_{\{i|a_i=1\}}\left(t^{2^i} + t^{-2^i}\right)}{t - t^{-1}} = \frac{t^{2^j} - t^{-2^j}}{t - t^{-1}} \prod_{\{i|a_i=1\}} \left(t^{2^i} + t^{-2^i}\right). \tag{2.2}$$

From this, it follows immediately that if $n+1$ is even, $\chi_1(t)\chi_n(t) = \chi_{n+1}(t)$, so

$$V \otimes T(n) \cong T(n+1).$$

By induction on $r \geq 1$,

$$\left(t + t^{-1}\right)\prod_{i=0}^{r-1}\left(t^{2^i} + t^{-2^i}\right) = \left(t^{2^r} + t^{-2^r}\right) + 2\sum_{i=1}^{r}\prod_{j=1}^{i-1}\left(t^{2^j} + t^{-2^j}\right)$$

From this it follows that if 2^r is the highest power of 2 dividing $n+1$,

$$V^{\otimes 2} \otimes T(2n) \cong T(2n+2) \oplus \bigoplus_{i=1}^{r+1} T\left(2n+2-2^i\right)^{\oplus 2},$$

unless $n = 2^r - 1$, in which case we omit the $T(0)$ terms from the above sum:

$$V^{\otimes 2} \otimes T(2n) \cong T(2n+2) \oplus \bigoplus_{i=1}^{r} T\left(2n+2-2^i\right)^{\oplus 2}.$$

Consider the labelled directed graph on non-negative integers n, where there is an arrow from n to $n+1$ labelled 1 and arrows labelled 2 from n to $n+1-2^i$ for $0 \leq i \leq r$, where r is the number of factors of 2 in $n+1$ except that we omit all arrows leading to 0. The multiplicity $x_{n,k}$ of $T(2n)$ as an indecomposable factor in $V^{\otimes 2k}$ is therefore the sum over all directed paths of length k from 0 to n of the product of labels.

Let $X_n = X_n(t) = \sum_k x_{n,k}t^k$ denote the generating function of this sum, so $b_k^{G,V}$ is the sum over n of the t^k coefficient of $X_n(t)$. As there are no edges from any vertex to 0, we have $X_0(t) = 1$. For $m \geq 1$,

$$X_n = t\left(X_{n-1} + 2\sum_{i=0}^{r} X_{2^i+n-1}\right),$$

where 2^r is now the highest power of 2 dividing n. We rewrite this equation

$$X_n = tL_n(X_0, X_1, X_2, \ldots), \tag{2.3}$$

where we define

$$L_n(y_0, y_1, y_2, \ldots) = y_{n-1} + 2\sum_{i=0}^{r} y_{2^i+n-1}.$$

For $n < 2^s$, L_n is a linear combination of $y_0, y_1, \ldots, y_{2^s-1}$, so the system of equations

$$y_i = tL_i(y_0, \ldots, y_{2^s-1}),\ \ 1 \leq i < 2^s \tag{2.4}$$

consists of $2^s - 1$ equations in 2^s variables. The matrix of this system of homogeneous linear equations has rank $2^s - 1$ over $\mathbb{Q}((t))$ because its (mod t) reduction has rank $2^s - 1$. Therefore, the solution set of (2.4) over $\mathbb{Q}((t))$ is 1-dimensional.

For $0 < i < 2^s$,

$$L_{2^s+i}(y_0, y_1, \ldots) = L_i(y_{2^s}, y_{2^s+1}, \ldots)$$

so $(X_{2^s}, X_{2^s+1}, \ldots, X_{2^{s+1}-1})$ is the scalar multiple of $(X_0, X_1, \ldots, X_{2^s-1})$ by an element of $\mathbb{Q}((t))$ (which is, in fact, X_{2^s}, since $X_0 = 1$.) Therefore,

$$X_{2^s+i} = X_{2^s} X_i \tag{2.5}$$

for $0 \le i < 2^s$, which implies the general formula

$$X_{2^{s_1}+\cdots+2^{s_q}} = \prod_{i=1}^{q} X_{2^{s_i}} \tag{2.6}$$

if $s_1 > s_2 > \cdots > s_q$. The sequence X_n is therefore determined by its subsequence as n ranges over powers of 2.

Proposition 2.1. *Let $F(x) = x^2 - 2$, and let $F^{\circ s}(x)$ denote the s-fold iterate of F. Then*

$$(F^{\circ s}(1/t - 2))X_{2^s}(t) = 1.$$

We remark that the fact that the sequence $X_{2^s}^{-1}$ is obtained by iterating F is equivalent to (1.2).

Proof. From Figure 2.1,

$$X_1(t) = \sum_{i=1}^{\infty} 2^{i-1} t^i = \frac{t}{1-2t}, \tag{2.7}$$

so

$$X_1(t)\left(\frac{1}{t} - 2\right) = 1,$$

and the proposition holds for $s = 0$. By (2.3) and (2.5),

$$X_{2^s} = tX_{2^s-1} + 2t\sum_{i=0}^{s} X_{2^s+2^i-1} = tX_{2^s-1} + 2tX_{2^s}\sum_{i=0}^{s} X_{2^i-1}.$$

so rearranging terms and dividing by $tX_{2^s}X_{2^s-1}$, we obtain

$$\frac{1}{X_{2^s}} = \frac{1}{t\prod_{i=0}^{s-1} X_{2^i}} - 2\sum_{i=0}^{s} \frac{1}{\prod_{j=i}^{s-1} X_{2^j}}. \tag{2.8}$$

Dividing both sides by X_{2^s} and subtracting 2,

$$\frac{1}{X_{2^s}^2} - 2 = \frac{1}{t\prod_{i=0}^{s} X_{2^i}} - 2\sum_{i=0}^{s} \frac{1}{\prod_{j=i}^{s} X_{2^j}} - 2 = \frac{1}{X_{2^{s+1}}},$$

where the last equality comes from substituting $s+1$ for s in (2.8). The proposition follows by induction on s. □

The proposition justifies identifying the power series $X_{2^s}(t)$ with the rational function $(F^{\circ s}(1/t-2))^{-1}$.

Proposition 2.2. *For all $s \ge 0$, we have*

$$X_{2^s}(t) = \frac{t^{2^s}}{\prod_{i=1}^{2^s}\left(1 - \left(2 + 2\cos\frac{(2i-1)\pi}{2^{s+1}}\right)t\right)}. \tag{2.9}$$

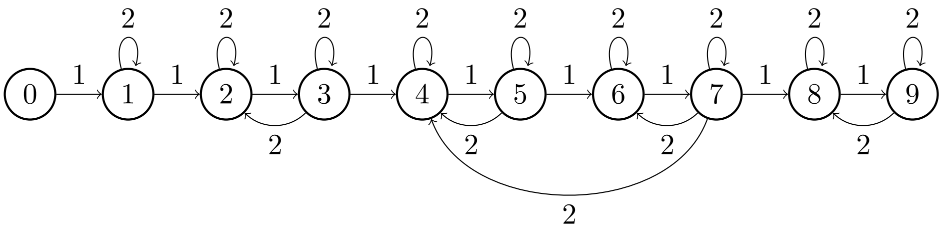

FIGURE 2.1. The fusion graph of $V^{\otimes 2}$

Proof. By induction on s, $F^{\circ s}(2\cos\theta) = 2\cos 2^s\theta$, so $F^{\circ s}(y-2) = 0$ if y is of the form $2 + 2\cos\frac{(2i-1)\pi}{2^{s+1}}$, $i = 0, 1, \ldots, 2^s - 1$. As $\cos\frac{(2i-1)\pi}{2^{s+1}}$ is strictly decreasing as i ranges from 1 to 2^s, this gives 2^s distinct values. It must include all roots of $F^{\circ s}(y-2) = 0$, since this is a polynomial equation of degree 2^s.

Thus,

$$\begin{aligned}\frac{1}{X_{2^s}\left(\frac{1}{t}\right)} &= \prod_{i=1}^{2^s}\left(\frac{1}{t} - \left(2 + 2\cos\frac{(2i-1)\pi}{2^{s+1}}\right)\right) \\ &= \frac{\prod_{i=1}^{2^s}\left(1 - \left(2 + 2\cos\frac{(2i-1)\pi}{2^{s+1}}\right)t\right)}{t^{2^s}},\end{aligned}$$

which implies the proposition. □

Lemma 2.3. *If $P(t)$ is a monic polynomial with distinct roots $r_1, \ldots, r_n$, then*

$$\frac{1}{\prod_{j=1}^{n}(1 - r_j t)} = \sum_{j=1}^{n} \frac{r_j^{n-1}}{P'(r_j)(1 - r_j t)}.$$

Proof. Both are rational functions with simple poles at $1/r_1, \ldots, 1/r_n$ and no other poles, so it suffices to check that the residues are the same. Indeed, the residue of the left hand side at the pole $1/r_i$ is

$$\frac{1/r_i}{\prod_{j\neq i}(1 - r_j/r_i)} = \frac{r_i^{n-2}}{P'(r_i)},$$

which is the same as that on the right hand side. □

Let

$$P_s(x) = \prod_{i=1}^{2^s}\left(x - \left(2 + 2\cos\frac{(2i-1)\pi}{2^{s+1}}\right)\right). \tag{2.10}$$

Lemma 2.4. *For any integer $j \in [1, 2^s]$,*

$$P_s'\left(2 + 2\cos\frac{(2j-1)\pi}{2^{s+1}}\right) = \frac{(-1)^{j+1}2^s}{\sin\frac{(2j-1)\pi}{2^{s+1}}}.$$

Proof. Let $\zeta = \zeta_{2^{s+2}} = e^{\pi i/2^{s+1}}$. Then the roots of $P_s(x)$ are $\beta_{2j-1} = 2 + \zeta^{2j-1} + \zeta^{1-2j}$ as j ranges from 1 to 2^s. For $j \geq 2$,

$$\beta_1 - \beta_{2j-1} = \zeta\left(1 - \zeta^{2j-2}\right)\left(1 - \zeta^{-2j}\right).$$

Since

$$\prod_{j=1}^{2^{s+1}-1}\left(1-\zeta^{2j}\right)=2^{s+1},$$

we have

$$\prod_{j=2}^{2^s}(\beta_1-\beta_{2j-1})=\zeta^{2^s-1}\frac{2^{s+1}}{1-\zeta^{-2}}=\frac{2^{s+1}i}{\zeta-\zeta^{-1}}=\frac{2^s}{\sin\pi/2^{s+1}},$$

which proves the lemma in the $j=1$ case. The remaining cases follow by Galois conjugation. □

Proposition 2.5. *For $k\geq 2^s$, the t^k coefficient of $X_{2^s}(t)$ is given by*

$$2^{-s}\sum_{j=1}^{2^s}(-1)^{j+1}\sin\frac{(2j-1)\pi}{2^{s+1}}\left(2+2\cos\frac{(2j-1)\pi}{2^{s+1}}\right)^{k-1}. \tag{2.11}$$

Proof. Together Proposition 2.2 and Lemmas 2.3 and 2.4 imply that for $k\geq 2^s$, the t^k coefficient of $X_{2^s}(t)$ is given by

$$\sum_{j=1}^{2^s}a_j\left(2+2\cos\frac{(2j-1)\pi}{2^{s+1}}\right)^{k-2^s},$$

where

$$a_j=\frac{(-1)^{j+1}\sin\frac{(2j-1)\pi}{2^{s+1}}\left(2+2\cos\frac{(2j-1)\pi}{2^{s+1}}\right)^{2^s-1}}{2^s}.$$

The proposition follows immediately. □

3. Discrete convolutions

The formula (2.6) can be understood as expressing the sequence of coefficients $x_{n,k}$ of a general $X_n(t)$ as the convolution of the sequences $x_{2^s,k}$ as s ranges over the s_i. As $4^{-k}b^{G,V}_{2k}$ is the sum over all n of $4^{-k}x_{n,k}$, we would like to understand the sum over all finite sets of s-values of the convolutions of the functions $A_s\colon\mathbb{Z}\to\mathbb{R}$ such that $A_s(k)=0$ for $k<0$ and $A_s(k)=4^{-k}x_{2^s,k}$ for $k\geq 0$.

In this section, we analyze such sums more generally. We assume each A_s is non-negative, supported on the natural numbers, with sum $1/2$ and with small differences between consecutive terms and finally that each A_s is concentrated at values of k such that $\log_4 k$ is close to s. It is not difficult to show that our particular functions A_s satisfy these conditions, but we defer this to a later section and work more generally in this section.

If $A\colon\mathbb{Z}\to\mathbb{R}$ is any function and d is a positive integer, we denote by $\|A\|_{a,\infty}$ (resp. $\|A\|_{a,1}$) the ℓ^∞ norm (resp. ℓ^1 norm) of the restriction of A to $[a,\infty)\cap\mathbb{Z}$. If d is a positive integer, we denote by $A^d(x)$ the function $A(x+d)-A(x)$.

Lemma 3.1. *Let $A,B\colon\mathbb{Z}\to[0,\infty)$ be summable and supported on $\mathbb{N}$. We have*

$$\|A*B\|_1=\|A\|_1\|B\|_1$$

and

$$\|A*B\|_\infty\leq\min(\|A\|_1\|B\|_\infty,\|A\|_\infty\|B\|_1).$$

Proof. For the first claim,

$$\|A * B\|_1 = \sum_{n=-\infty}^{\infty} A * B(n) = \sum_{n=-\infty}^{\infty} \sum_{p+q=n} A(p)B(q)$$
$$= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} A(p)B(q) = \|A\|_1 \|B\|_1.$$

Moreover, for all n,

$$A * B(n) = \sum_{p=-\infty}^{\infty} A(p)B(n-p) \leq \sum_{p=-\infty}^{\infty} A(p) \sup_q B(q) = \|A\|_1 \|B\|_\infty.$$

By the symmetry of convolution, this implies the second claim. □

Lemma 3.2. *Let* $A_1, \ldots, A_r$ *be functions* $\mathbb{Z} \to [0, \infty)$ *supported on* $\mathbb{N}$*,* $a_1, \ldots, a_r$ *be non-negative integers, and* a *be an integer greater than or equal to* $a_1 + \cdots + a_r$*. If* $\|A_i\|_1 = \frac{1}{2}$ *for* $i = 1, \ldots, r$*, then*

$$\|A_1 * \cdots * A_r\|_{a,\infty} \leq 2^{1-r} \sum_{i=1}^{r} \|A_i\|_{a_i,\infty}.$$

Proof. If $x_0 \geq a \geq a_1 + \cdots + a_r$, then in any representation of x_0 as a sum $x_1 + \cdots + x_r$ of integers, the condition $x_i \geq a_i$ must be satisfied for at least one value of i. It suffices to prove that

$$\sum_{x_i=a_i}^{\infty} \sum_{x_1+\cdots+x_{i-1}+x_{i+1}+\cdots+x_r=x_0-x_i} A_1(x_1) \cdots A_r(x_r) \leq 2^{1-r} \|A_i\|_{a_i,\infty}.$$

Since $x_i \geq a_i$, this sum is bounded above by

$$\|A_i\|_{a_i,\infty} \sum_{x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_r} A_1(x_1) \cdots A_{i-1}(x_{i-1}) A_{i+1}(x_{i+1}) \cdots A_r(x_r), \tag{3.1}$$

whose ith summand is

$$\|A_i\|_{a_i,\infty} \|A_1\|_1 \cdots \|A_{i-1}\|_1 \|A_{i+1}\|_1 \cdots \|A_r\|_1 = 2^{1-r} \|A_i\|_{a_i,\infty}.$$ □

The following lemma gives explicit form to the principle that the convolution of a rapidly decaying sequence with a slowly varying sequence is well approximated by the termwise product of the second sequence with the sum of the first.

Lemma 3.3. *If* $A, B \colon \mathbb{Z} \to [0, \infty)$ *are supported on* $\mathbb{N}$ *and* a *and* b *are non-negative integers, then*

$$\|A * B - \|A\|_1 B\|_{a+b,\infty} \leq \|A\|_1 \sup_{0 \leq d \leq a} \|B^d\|_{b,\infty} + 2\|A\|_{a,1} \|B\|_\infty.$$

Proof. If $x_0 \geq a + b$, then

$$A * B(x_0) = \sum_{i=0}^{a} A(i)\big(-B(x_0) + B(x_0 - i)\big) + B(x_0) \sum_{i=0}^{a} A(i) + \sum_{i=a+1}^{\infty} A(i)B(x_0 - i)$$
$$= -\sum_{i=0}^{a} A(i)B^i(x_0 - i) + B(x_0) \left(\|A\|_1 - \sum_{i=a+1}^{\infty} A(i) \right) + \sum_{i=a+1}^{\infty} A(i)B(x_0 - i).$$
$$= -\sum_{i=0}^{a} A(i)B^i(x_0 - i) + \|A\|_1 B(x_0) + \sum_{i=a+1}^{\infty} A(i)\big(B(x_0 - i) - B(x_0)\big).$$

We have $\sum_{i=0}^{a} A(i) \leq \|A\|_1$, so the lemma follows. □

Henceforth, we suppose $A_0, A_1, \dots$ are functions $\mathbb{Z} \to [0, \infty)$ which are supported on $\mathbb{N}$. For each finite set $S \subset \mathbb{N}$, we denote by A_S the discrete convolution of A_s over all $s \in S$. For T a subset of $\mathbb{R}$, we define $B_T : \mathbb{Z} \to [0, \infty]$ as the sum of A_S over all non-empty finite subsets S of $T \cap \mathbb{N}$, where a divergent sum gives the value ∞.

We make the following assumptions.

(I) For all s, $\sum_k A_s(k) = 1/2$.
(II) For all $r \geq 0$ there exists C_r such that $A_s(k) < C_r 4^{-s}(4^s/k)^r$ for all $k > 0$.
(III) There exists C such that for all s, k_1, k_2,

$$|A_s(k_1) - A_s(k_2)| < C(|k_1 - k_2|16^{-s} + 8^{-s}).$$

Lemma 3.4. *For $d, m, s \geq 0$ and $S \subset \mathbb{N}$ a finite, non-empty set, the above assumptions imply:*

(1) $\|A_s\|_\infty = O(4^{-s})$.
(2) $\|A_s\|_{2^m,\infty} = O(2^{6s-4m})$.
(3) $\|A_s\|_{2^m,1} = O(2^{6s-3m})$.
(4) $\|A_S\|_\infty = O(2^{-|S|}2^{-2\max S})$.
(5) $\|A_S\|_{2^m,\infty} = O(2^{-|S|}2^{6\max S-4m})$.
(6) $\|A_S^d\|_\infty = O(2^{-|S|}(2^{-3\max S} + 2^{-4\max S}d))$.

Proof. Parts (1) and (2) follow from assumption (II) in the $r = 0$ and $r = 4$ cases respectively.

Part (3) follows from assumption (I) if $m \leq 2s$. Otherwise, (2) implies that for $r \geq 0$,

$$\sum_{k=2^{m+r}}^{2^{m+r+1}-1} A_s(k) = O\left(2^{6s-3m-3r}\right),$$

and summing over r, we get (3).

For the remaining parts, let $S = \{s_1, \dots, s_r\}$ with $s_1 > s_2 > \cdots > s_r$. By (1), $\|A_{s_1}\|_\infty = O(4^{-s_1})$. Thus,

$$\|A_S\|_\infty \leq \|A_{s_1}\|_\infty \|A_{s_2}\|_1 \cdots \|A_{s_r}\|_1 = 2^{1-r}\|A_{s_1}\|_\infty,$$

implying (4).

On the other hand,

$$\sum_{i=1}^{r} 2^{m-i} < 2^m,$$

so by Lemma 3.2, (2), and the fact that $s_i \leq 1 + s_1 - i$, we have

$$\|A_S\|_{2^m,\infty} \leq 2^{1-r} \sum_{i=1}^{r} \|A_{s_i}\|_{2^{m-i},\infty} = O\left(2^{-|S|}2^{6s_1-4m}\right),$$

implying (5).

For (6),

$$A_S^d = A_{s_1}^d * A_{s_2} * \cdots * A_{s_r}.$$

By assumption (III),

$$\left\|A_{s_1}^d\right\|_\infty = O(2^{-3s_1} + 2^{-4s_1}d).$$

Therefore,

$$\left\|A_S^d\right\|_\infty \le O\left(2^{1-r}(8^{-s_1}+16^{-s_1}d)\right) = O\left(2^{-|S|}\left(2^{-3\max S}+2^{-4\max S}d\right)\right).$$

□

Lemma 3.5. *If* $s\in\mathbb{N}$*,* S *is a finite subset of* $(s,\infty)\cap\mathbb{Z}$*,* $n\ge s$*, and* $k\in[4^n,4^{n+1})$*, then*

$$A_{\{s\}\cup S}(k)-\frac{1}{2}A_S(k) = O\left(2^{-|S|}2^{-\left(\max(\max S+n,2n)+\frac{3}{5}(n-s)\right)}\right).$$

Proof. We apply Lemma 3.3 with $A=A_s$, $B=A_S$, and $a=2^{n+s}$ and $b=2^{2n-1}$ to obtain

$$A_{\{s\}\cup S}(k)-\frac{1}{2}A_S(k) = O\left(\sup_{d\le 2^{n+s}}\left\|A_S^d\right\|_{2^{2n-1},\infty}+\left\|2^{n+s}\right\|_{A_s,1}\|A_S\|_\infty\right). \tag{3.2}$$

Suppose $\max S\ge n$. Then

$$3\max S\ge \max S+n+\frac{3}{5}(n-s)$$

and

$$4\max S-n-s\ge \max S+n+\frac{3}{5}(n-s).$$

Using $\|A_S^d\|_{2^{n+s},\infty}\le\|A_S^d\|_\infty$ and applying part (6) of Lemma 3.4,

$$\left\|A_S^d\right\|_{2^{n+s},\infty} = O\left(2^{-|S|}2^{-\left(\max(\max S+n,2n)+\frac{3}{5}(n-s)\right)}\right).$$

As

$$2\max S+3(n-s)\ge \max S+n+\frac{3}{5}(n-s),$$

by parts (3) and (4) of Lemma 3.4,

$$\|A_s\|_{2^{n+s},1}\ \|A_S\|_\infty = O\left(2^{-|S|}2^{-\left(\max(\max S+n,2n)+\frac{3}{5}(n-s)\right)}\right),$$

so the lemma follows from (3.2).

Likewise, if $n-\frac{n-s}{10}\le\max S\le n$, then we have

$$3\max S,\ 4\max S-n-s,\ 2\max S+3(n-s)\ge 2n+\frac{3}{5}(n-s),$$

and the lemma follows as before.

If, on the other hand, $\max S\le n-\frac{n-s}{10}$, then since

$$\left\|A_S^d\right\|_{2^{2n-1},\infty}\le 2\|A_S\|_{2^{2n-1},\infty},$$

by part (5) of Lemma 3.4, we have

$$\|A_S^d\|_{2^{2n-1},\infty} = O\left(2^{-|S|}2^{-\frac{3}{5}(n-s)}2^{-2n}\right).$$

As $\max S\ge s$,

$$2\max S+3n-3s\ge 2n+\frac{3}{5}(n-s),$$

so by parts (3) and (4) of Lemma 3.4,

$$\|A_s\|_{2^{n+s},1}\|A_S\|_\infty = O\left(2^{-|S|}2^{-\frac{3}{5}(n-s)}2^{-2n}\right),$$

and again the lemma follows from (3.2). □

Proposition 3.6. *For all $\epsilon > 0$, there exists $r > 0$ such that if $n \geq r$ is an integer and $k \in [4^n, 4^{n+1})$, then*

$$\left|\left(\frac{3}{2}\right)^{-n} B_{\mathbb{R}}(k) - \left(\frac{3}{2}\right)^{-r} B_{[n-r,n+r]}(k)\right| < \epsilon 4^{-n}.$$

Proof. By part (4) of Lemma 3.4, for $S \subset [0, s)$,

$$A_{\{s\}\,\cup\, S}(k) = O\left(2^{-|S|}4^{-s}\right),$$

so

$$\sum_{s=n+r+1}^{\infty} \sum_{S\,\subset\,[0,s)} A_{\{s\}\,\cup\, S}(k) = O\left(\sum_{s=n+r+1}^{\infty} 4^{-s} \sum_{S\,\subset\,[0,s)} 2^{-|S|}\right)$$
$$= O\left(\sum_{s=n+r+1}^{\infty} 4^{-s}\left(\frac{3}{2}\right)^{s}\right) = O\left(\left(\frac{3}{8}\right)^{n+r}\right).$$

It follows that

$$\left(\frac{3}{2}\right)^{-n}\left(B_{\mathbb{R}}(k) - B_{[0,n+r]}(k)\right) = O\left(4^{-n}\left(\frac{3}{8}\right)^{r}\right). \tag{3.3}$$

By Lemma 3.5 and induction on q, if $S \subset [n-r, n+r]$ and

$$n - r > s'_1 > \cdots > s'_q,$$

then

$$A_{S\,\cup\{s'_1,\dots,s'_q\}}(k) - 2^{-q}A_S(k) = \begin{cases} O\left(2^{-|S|-q}2^{-\frac{3}{5}r}4^{-n}\right) & \text{if } \max S \leq n,\\ O\left(2^{-|S|-q}2^{-\frac{3}{5}r}2^{\max S-n}4^{-n}\right) & \text{if } \max S > n\end{cases}$$

If $s \in [n-r, n]$ and $S \subset [n-r, s)$, then

$$\sum_{S'\,\subset\,[0,n-r)} \left(A_{\{s\}\,\cup\, S\,\cup\, S'}(k) - 2^{-|S'|}A_{\{s\}\,\cup\, S}(k)\right) = O\left(2^{-|S|}\left(\frac{3}{2}\right)^{n-r} 2^{-\frac{3}{5}r}4^{-n}\right),$$

so

$$\sum_{S\,\subset\,[n-r,s)}\ \sum_{S'\,\subset\,[0,n-r)} \left(A_{\{s\}\,\cup\, S\,\cup\, S'}(k) - 2^{-|S'|}A_{\{s\}\,\cup\, S}(k)\right) = O\left(\left(\frac{3}{2}\right)^{s} 2^{-\frac{3}{5}r}4^{-n}\right).$$

Therefore,

$$\left(\frac{3}{2}\right)^{-n} \sum_{s=n-r}^{n}\ \sum_{S\,\subset\,[n-r,s)}\ \sum_{S'\,\subset\,[0,n-r)} \left(A_{\{s\}\,\cup\, S\,\cup\, S'}(k) - 2^{-|S'|}A_{\{s\}\cup S}(k)\right)$$
$$= O\left(2^{-\frac{3}{5}r}4^{-n}\right). \tag{3.4}$$

If $s \in (n, n+r]$ and $S \subset [n-r, s)$, then

$$\sum_{S'\,\subset\,[0,n-r)} \left(A_{\{s\}\,\cup\, S\,\cup\, S'}(k) - 2^{-|S'|}A_{\{s\}\,\cup\, S}(k)\right) = O\left(2^{-|S|}\left(\frac{3}{2}\right)^{n-r} 2^{-\frac{3}{5}r}2^{n-s}4^{-n}\right),$$

so

$$\sum_{S\,\subset\,[n-r,s)}\ \sum_{S'\,\subset\,[0,n-r)} \left(A_{\{s\}\,\cup\, S\,\cup\, S'}(k) - 2^{-|S'|}A_{\{s\}\,\cup\, S}(k)\right) = O\left(\left(\frac{3}{2}\right)^{s} 2^{-\frac{3}{5}r}2^{n-s}4^{-n}\right).$$

Therefore,

$$\left(\frac{3}{2}\right)^{-n}\sum_{s=n-r}^{n}\sum_{S\subset[n-r,s)}\sum_{S'\subset[0,n-r)}\left(A_{\{s\}\cup S\cup S'}(k)-2^{-|S'|}A_{\{s\}\cup S}(k)\right) = O\left(2^{-\frac{3}{5}r}4^{-n}\right). \quad (3.5)$$

By (3.3), (3.4), and (3.5), the sum of all terms in $(3/2)^{-n}B_{\mathbb{R}}(k)$ which do not occur in $(3/2)^{-n}B_{[n-r,n+r]}(k)$ can be taken to be an arbitrarily small multiple of 4^{-n} by making r sufficiently large. □

4. Real convolutions

Let χ denote the unique primitive (mod 4) Dirichlet character, so for every integer r, $\chi(4r+1)=1$ $\chi(4r-1)=-1$, and $\chi(2r)=0$. Let

$$\varphi(x)=\begin{cases}\frac{\pi}{8}\sum_{n=1}^{\infty}\chi(n)ne^{-\frac{\pi^2n^2}{16}x} & \text{if } x>0,\\ 0 & \text{if } x\le 0.\end{cases} \quad (4.1)$$

As (2.11) suggests, the sequence $x_{2^s,k}$ determines a step function which, after suitable rescaling, converges as $s\to\infty$ to φ. See Proposition 5.5 below for the precise statement and proof. We prove in this section a continuous analogue of Proposition 3.6 which enables us to define the function $\psi(x)$ of Theorem 1.1 as a limit of sums of convolutions. There are significant differences between Proposition 3.6 and Proposition 4.6, however. For one thing, because we want to prove the limit function is essentially multiplicatively periodic, the index set for the convolutions must be $\mathbb{Z}$ rather than $\mathbb{N}$. For another, since we want to prove the limit is smooth, we must bound derivatives of all orders. Nevertheless, the proofs are formally very similar.

We begin with some basic facts about convolutions of Schwartz functions over $\mathbb{R}$. The convolution of any two such functions σ and τ is again a Schwartz function, and the derivative of $\sigma*\tau$ is $\sigma'*\tau=\sigma*\tau'$ ([6, V, Proposition 1.11]). If σ and τ are non-negative and supported on $[0,\infty)$, the same will be true of $\sigma*\tau$. We define $\|f\|_{a,\infty}$ (resp. $\|f\|_{a,1}$) to be the L^∞ norm (resp. L^1 norm) of the restriction of f to $[a,\infty)$.

Exactly as in Section 3, we have the following lemma:

Lemma 4.1. *Let* $\sigma,\tau\colon\mathbb{R}\to[0,\infty)$ *be Schwartz functions supported on* $[0,\infty)$. *Then the Schwartz function* $\sigma*\tau$ *satisfies*

$$\|\sigma*\tau\|_1=\|\sigma\|_1\|\tau\|_1$$

and

$$\|\sigma*\tau\|_\infty\le\min(\|\sigma\|_1\|\tau\|_\infty,\|\sigma\|_\infty\|\tau\|_1).$$

Lemma 4.2. *Let* $\sigma_1,\dots,\sigma_r\colon\mathbb{R}\to[0,\infty)$ *be Schwartz functions whose support is contained in* $[0,\infty)$, *let* $a_1,\dots,a_r$ *be non-negative numbers and* $a\ge a_1+\dots+a_r$. *If* $\|\sigma_i\|_1=\frac{1}{2}$ *for* $i=1,2,\dots,r$, *then*

$$\|\sigma_1*\dots*\sigma_r\|_{a,\infty}\le 2^{1-r}\sum_{i=1}^{r}\|\sigma_i\|_{a_i,\infty}.$$

Proof. We proceed by induction on r, the base case being $r = 2$. In this case, for $x_0 \geq a$, we have

$$\sigma_1 * \sigma_2(x_0) = \int_0^{a_1} \sigma_1(x)\sigma_2(x_0 - x)dx + \int_{a_1}^{x_0} \sigma_1(x)\sigma_2(x_0 - x)dx.$$

Since $\sigma_2(x_0 - x) \leq \|\sigma_2\|_{a_2,\infty}$ in the first integral and $\sigma_1(x) \leq \|\sigma_1\|_{a_1,\infty}$ in the second integral, we have

$$\|\sigma_1 * \sigma_2\|_{a,\infty} \leq \|\sigma_1\|_1 \|\sigma_2\|_{a_2,\infty} + \|\sigma_1\|_{a_1,\infty}\|\sigma_2\|_1 = \frac{1}{2}\|\sigma_2\|_{a_2,\infty} + \frac{1}{2}\|\sigma_1\|_{a_1,\infty}. \tag{4.2}$$

The general case now follows by induction. □

Lemma 4.3. *If σ and τ are non-negative Schwartz functions supported on $[0, \infty)$ and $a, b \geq 0$, then*

$$\|\sigma * \tau - \|\sigma\|_1 \tau\|_{a+b,\infty} \leq a\|\sigma\|_1\|\tau'\|_{b,\infty} + 2\|\sigma\|_{a,1}\|\tau\|_\infty.$$

Proof. If $x_0 \geq a + b$, then

$$\begin{aligned}\left|\sigma * \tau(x_0) - \tau(x_0)\int_0^\infty \sigma(x)dx\right| &= \left|\int_0^\infty \sigma(x)(-\tau(x_0) + \tau(x_0 - x))dx\right| \\ &= \left|\int_0^a \sigma(x)(-\tau(x_0) + \tau(x_0 - x))dx + \int_a^\infty \sigma(x)(-\tau(x_0) + \tau(x_0 - x))dx\right| \\ &\leq \|\sigma\|_1 a\|\tau'\|_{b,\infty} + \|\sigma\|_{a,1}(2\|\tau\|_\infty)\end{aligned}$$

since for $x \leq a$, $|\tau(x_0 - x) - \tau(x_0)| \leq a\|\tau'\|_{b,\infty}$ by the mean value theorem. □

Let ϕ be a non-negative Schwartz function supported on $[0, \infty)$ with

$$\|\phi\|_1 = \frac{1}{2}. \tag{4.3}$$

For all $s \in \mathbb{Z}$, we define

$$\phi_s(x) = 4^{-s}\phi(4^{-s}x),$$

so $\|\phi_s\|_1 = \frac{1}{2}$. If p is a non-negative integer, then

$$\left\|\phi_s^{(p)}\right\|_\infty = 4^{-(p+1)s}\left\|\phi^{(p)}\right\|_\infty. \tag{4.4}$$

For any finite subset $S \subset \mathbb{Z}$, we define ϕ_S to be the convolution of ϕ_s over all $s \in S$. If $S = \{s_1, \ldots, s_r\}$, then

$$\phi_S^{(p)} = \phi_{s_1}^{(p)} * \phi_{\{s_2,\ldots,s_r\}}.$$

Lemma 4.4. *If $p \geq 0$, $j > 2p + 2$, $m \in \mathbb{Z}$, and $S \subset \mathbb{Z}$ is finite and non-empty, the above assumptions imply:*

(1) $\left\|\phi_s^{(p)}\right\|_\infty = O\left(2^{-(2p+2)s}\right)$.
(2) $\|\phi_s^{(p)}\|_{2^m,\infty} = O\left(2^{(2j-2p-2)s-jm}\right)$.
(3) $\|\phi_s^{(p)}\|_{2^m,1} = O\left(2^{(2j-2p-2)s-(j-1)m}\right)$.
(4) $\|\phi_S^{(p)}\|_\infty = O\left(2^{-|S|}2^{-(2p+2)\max S}\right)$.
(5) $\|\phi_S^{(p)}\|_{2^m,\infty} = O\left(2^{-|S|}\left(2^{(2j-2p-2)\max S-jm} + 2^{(4j-4p-4)\max S-(2j-1)m}\right)\right)$.

Proof. As $\phi^{(p)}(x)$ is bounded, $|4^{(p+1)s}\phi_s^{(p)}(x)| = |\phi^{(p)}(4^{-s}x)| \le C$ for some constant C, which gives $\|\phi_s^{(p)}\| \le C2^{-(2p+2)s}$, implying part (1).

As $\phi^{(p)}(x)x^j$ is bounded, there exists C such that $\phi^{(p)}(x) \le Cx^{-j}$ for all x, so

$$\phi_s^{(p)}(x) = 4^{-(p+1)s}\phi(4^{-s}x) \le C4^{-(p+1)s}4^{js}x^{-j} \le C2^{(2j-2p-2)s-jm}$$

if $x \ge 2^m$. This gives part (2). Furthermore,

$$\int_{2^m}^{\infty} \phi_s^{(p)}(x)dx \le C2^{(2j-2p-2)s}\int_{2^m}^{\infty} x^{-j}dx = O\left(2^{(2j-2p-2)s-(j-1)m}\right),$$

giving part (3).

Let $S = \{s_1, \dots, s_r\}$, $s_1 > \dots > s_r$. Applying (1) for $s = s_1$, part (4) follows from

$$\left\|\phi_S^{(p)}\right\|_\infty \le \left\|\phi_{s_1}^{(p)}\right\|_\infty \left\|\phi_{\{s_2,\dots,s_r\}}\right\|_1 = 2^{1-r}\left\|\phi_{s_1}^{(p)}\right\|_\infty.$$

Finally, by (4.2),

$$\left\|\phi_S^{(p)}\right\|_{2^m,\infty} \le 2^{1-r}\left\|\phi_{s_1}^{(p)}\right\|_{2^{m-1},\infty} + \left\|\phi_{s_1}^{(p)}\right\|_{2^{m-1},1}\left\|\phi_{\{s_2,\dots,s_r\}}\right\|_{2^{m-1},\infty}.$$

By (2), the first summand on the right hand side is $O(2^{-|S|}2^{(2j-2p-2)s_1-jm+j})$. Applying Lemma 4.2 with $\sigma_i = \phi_{s_{i+1}}$ and $a_i = 2^{m-1-i}$ we get that

$$\left\|\phi_{\{s_2,\dots,s_r\}}\right\|_{2^{m-1},\infty} = O\left(2^{2-r}\sum_{i=1}^{r} 2^{(2j-2p-2)s_{i+1}-j(m-1-i)}\right),$$

and as $(2j-2p-2)s_{i+1} - j(m-i)$ is strictly decreasing as i increases, this is

$$O\left(2^{-r}2^{(2j-2p-2)s_2-jm}\right).$$

By (3),

$$\left\|\phi_{s_1}^{(p)}\right\|_{2^{m-1},1} = O\left(2^{(2j-2p-2)s_1-(j-1)m}\right).$$

Together, these estimates give (5). □

Proposition 4.5. *If p is a non-negative integer, s is a negative integer, and S is a finite subset of $[s,\infty)$, then on $[1,4]$,*

$$\left\|\phi_{\{s\}\cup S}^{(p)}(x) - \frac{1}{2}\phi_S^{(p)}(x)\right\|_{1,1} = O\left(2^{-|S|}2^s\right).$$

Proof. We apply Lemma 4.3 with $\sigma = \phi_s$, $\tau = \phi_S^{(p)}$, $a = 2^s$ and $b = \frac{1}{2}$ to obtain

$$\left\|\phi_{\{s\}\cup S}(x) - \frac{1}{2}\phi_S(x)\right\|_{1,1} = O\left(2^s\left\|\phi_S^{(p+1)}\right\|_{\frac{1}{2},\infty} + \|\phi_s\|_{2^s,1}\left\|\phi_S^{(p)}\right\|_\infty\right). \tag{4.5}$$

Applying part (3) of Lemma 4.4 with $j = 4p+4$, we obtain

$$\|\phi_s\|_{2^s,1} = O\left(2^{(2p+3)s}\right),$$

so part (4) implies

$$\|\phi_s\|_{2^s,1}\left\|\phi_S^{(p)}\right\|_\infty = O\left(2^{(2p+3)s}2^{-|S|}2^{-(2p+2)\max S}\right).$$

If $\max S \geq 0$, we again use (4) to bound $\|\phi_S^{(p+1)}\|_{\frac{1}{2},\infty} \leq \|\phi_S^{(p+1)}\|_\infty$, and (4.5) gives

$$\left\|\phi_{\{s\}\cup S}(x) - \frac{1}{2}\phi_S(x)\right\|_{1,1} = O\left(2^{-|S|}\left(2^s 2^{-(2p+4)\max S} + 2^{(2p+3)s}2^{-(2p+2)\max S}\right)\right)$$
$$= O\left(2^{-|S|}2^s 2^{-(2p+2)\max S}\right),$$

implying the proposition. If $\max S \leq 0$, we apply (5) with $j = 2p+3$ to bound $\|\phi_S^{(p+1)}\|_{\frac{1}{2},\infty}$, so using the fact that $\max S \geq s$, (4.5) implies

$$O\left(2^{-|S|}2^s 2^{5\max S} + 2^{(2p+3)s}2^{-|S|}2^{-(2p+2)\max S}\right) = O\left(2^{-|S|}2^s\right),$$

and the proposition again follows. □

For every subset $T \subset \mathbb{R}$, we define ψ_T to be the sum of ϕ_S over all non-empty finite subsets $S \subset T \cap \mathbb{Z}$. When $T \cap \mathbb{Z}$ is finite, $\psi_T + \delta_0$ is the convolution of the distributions $\phi_s + \delta_0$, $t \in T \cap \mathbb{Z}$, where δ_0 is the delta function concentrated at 0. For r a non-negative integer, we define

$$\psi_r = (3/2)^{-r}\psi_{[-r,r]}. \tag{4.6}$$

Proposition 4.6. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a non-negative Schwartz function supported on $[0,\infty)$ which satisfies (4.3). There exists a unique smooth function $\psi\colon (0,\infty) \to [0,\infty)$ such that the sequence $\psi_1, \psi_2, \ldots$ converges uniformly to ψ on every compact subset of $(0,\infty)$. Moreover,*

$$\psi(4x) = \frac{3}{8}\psi(x).$$

Proof. It suffices to prove the existence of ψ. In fact, we prove slightly more, namely for each fixed $p \in \mathbb{N}$, $(3/2)^{-r_1}\psi^{(p)}_{[-r_1,r_2]}$ converges uniformly on compact subsets of $(0,\infty)$ for any sequence of pairs (r_1, r_2) for which both r_1 and r_2 go to ∞. As

$$\psi_{[-r_1-1,r_2-1]}(x) = 4\psi_{[-r_1,r_2]}(4x),$$

it suffices to prove convergence on the interval $[1,4]$. Moreover,

$$\begin{aligned}\psi(4x) &= \lim_{r_1,r_2\to\infty}\left(\frac{3}{2}\right)^{-r_1}\psi_{[-r_1,r_2]}(4x)\\ &= \lim_{r_1,r_2\to\infty}\frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^{-r_1-1}\psi_{[-r_1-1,r_2-1]}(x)}{4}\\ &= \frac{3}{8}\lim_{r_1,r_2\to\infty}\left(\frac{3}{2}\right)^{-r_1-1}\psi_{[-r_1-1,r_2-1]}(x)\\ &= \frac{3}{8}\psi(x).\end{aligned} \tag{4.7}$$

If $p \in \mathbb{N}$ and $x \in [1,4]$, by part (4) of Lemma 4.4, we have

$$\left(\frac{3}{2}\right)^{-r_1}\psi^{(p)}_{[-r_1,r_2+1]}(x) - \left(\frac{3}{2}\right)^{-r_1}\psi^{(p)}_{[-r_1,r_2]}(x)$$
$$= \left(\frac{3}{2}\right)^{-r_1}\sum_{S\subset[-r_1,r_2]}\phi^{(p)}_{\{r_2+1\}\cup S}(x)$$

$$= O\left(2^{-(2p+2)(r_2+1)}\left(\frac{3}{2}\right)^{-r_1}\sum_{S\subset[-r_1,r_2]}2^{-|S|}\right)$$
$$= O\left(2^{-2(r_2+1)}\left(\frac{3}{2}\right)^{-r_1}\left(\frac{3}{2}\right)^{r_1+r_2+1}\right)$$
$$= O\left(\left(\frac{8}{3}\right)^{-r_2}\right).$$

By Proposition 4.5,

$$\left(\frac{3}{2}\right)^{-r_1-1}\psi^{(p)}_{[-r_1-1,r_2]}(x)-\left(\frac{3}{2}\right)^{-r_1}\psi^{(p)}_{[-r_1,r_2]}(x)$$
$$=\left(\frac{3}{2}\right)^{-r_1-1}\sum_{S\subset[-r_1,r_2]}\left(\phi^{(p)}_{\{-r_1-1\}\cup S}(x)-\frac{1}{2}\phi^{(p)}_S(x)\right)$$
$$= O\left(\left(\frac{3}{2}\right)^{-r_1-1}2^{-r_1-1}\sum_{S\subset[-r_1,r_2]}2^{-|S|}\right)$$
$$= O\left(\left(\frac{3}{2}\right)^{r_2}2^{-r_1-1}\right).$$

Applying these together, we conclude that $|\psi^{(p)}_{r+1}(x)-\psi^{(p)}_r(x)|$ is bounded above on $[1,4]$ by an exponentially decaying function of r, and the sequence converges uniformly. ☐

Proposition 4.7. *The function* $\varphi(x)$ *in* (4.1) *satisfies the hypotheses of Proposition 4.6.*

Proof. The k^{th} derivative of $\varphi(x)$ for $x>0$ is

$$\frac{\pi}{8}\sum_{n=1}^{\infty}\chi(n)\left(-\frac{\pi^2n^2}{16}\right)^k ne^{-\frac{\pi^2n^2}{16}x},$$

which is asymptotic to $\frac{(-1)^k\pi^{2k+1}e^{-\frac{\pi^2}{16}x}}{2^{4k+3}}$ at $+\infty$. Therefore, to prove that φ is a Schwartz function, it suffices to show that it has a kth derivative at 0 for all k, i.e., that

$$\lim_{x\to0^+}\sum_{n=1}^{\infty}\chi(n)n^{2k+1}e^{-\frac{\pi^2n^2}{16}x}=0.$$

By a theorem of de la Vallée Poussin [5, Theorem 10.6], we have

$$\varphi(x)=8(\pi x)^{-3/2}\varphi\left(\frac{16}{\pi^2x}\right) \tag{4.8}$$

for $x>0$. Repeatedly differentiating this identity, we can express $\varphi^{(n)}(x)$ as a linear combination of terms of the form $x^{-j}\varphi^{(k)}(16/\pi^2x)$, where $j\in\{k+3/2,k+5/2,\dots,2k+3/2\}$ and $k\in\{0,1,\dots,n\}$. Each such term has exponential decay at $x=0$ since $\varphi^{(k)}(y)$ has exponential decay at $y=\infty$.

The integral of $\varphi(x)$ over $\mathbb{R}$ is the limit as $a\to0^+$ of $\int_a^\infty\varphi(x)dx$, which can be integrated termwise. Thus,

$$\int_{-\infty}^{\infty}\varphi(x)dx=\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{16n\chi(n)}{\pi^2n^2}=\frac{2L(1,\chi)}{\pi}=\frac{1}{2}.$$

Finally, for the positivity of φ, it suffices by (4.8) to verify it for $x \geq 4/\pi$. This, in turn, follows from the fact that for $m \geq 0$ and $x \geq 4/\pi$,

$$\frac{(4m+3)e^{-\pi^2(4m+3)^2x}}{(4m+1)e^{-\pi^2(4m+1)^2x}} \leq 9e^{-8\pi^2 x} \leq e^{-2\pi} < 1. \qquad \square$$

The function $\omega(x)$ in Theorem 1.1 is defined to be $\psi(x)x^{\delta}$. By (4.7), $\omega(4x) = \omega(x)$.

5. Convergence to ψ

In this section, we prove the main theorem. We follow the notation of Section 4. In particular, $\varphi(x)$ will be defined by (4.1), ψ_r will be defined as in (4.6) where ϕ is taken to be φ, and $\psi(x)$ will be the limit of the ψ_n, as in Proposition 4.6. The key point in the argument is that $A_s = 4^{-k}x_{2^s,k}$ is well approximated by φ_s, and the same thing remains true when we compare convolutions of the sequences A_s indexed by a finite set S and the corresponding functions ϕ_S.

For any function $\sigma\colon \mathbb{R} \to \mathbb{R}$, we write $[\sigma]$ for the sequence obtained by restricting σ to $\mathbb{Z}$. We use the same notation $\|\ \|_1$ and $\|\ \|_\infty$ for norms on $\mathbb{R}$ and $\mathbb{Z}$; which norm is meant in each case should be clear from context.

Lemma 5.1. *For any Schwartz function $\sigma\colon \mathbb{R} \to \mathbb{R}$,*

$$\|[\sigma]\|_1 \leq \|\sigma\|_1 + \|\sigma'\|_1.$$

Proof. By the mean value theorem,

$$|\sigma(x) - \sigma(\lfloor x \rfloor)| \leq \|\sigma'\|_\infty.$$

Therefore, for every integer n,

$$\int_n^{n+1} |\sigma(x) - \sigma(\lfloor x \rfloor)| dx \leq \int_n^{n+1} |\sigma'(x)| dx,$$

so summing over n,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \sigma(x)dx - \sum_{-\infty}^{\infty} \sigma(\lfloor x \rfloor)dx\right| &\leq \sum_{n=-\infty}^{\infty} \int_n^{n+1} |\sigma(x) - \sigma(\lfloor x \rfloor)| dx \\ &\leq \sum_{n=-\infty}^{\infty} \int_n^{n+1} \left|\sigma'(x)\right| dx = \left\|\sigma'\right\|_1. \qquad \square\end{aligned}$$

Lemma 5.2. *For any non-negative Schwartz functions σ and τ,*

$$\big\|[\sigma * \tau] - [\sigma] * [\tau]\big\|_\infty \leq (\|\sigma\|_1 + \|\sigma'\|_1)\|\tau'\|_\infty + \|\sigma'\|_\infty(\|\tau\|_1 + \|\tau'\|_1).$$

Proof. If m and n are integers and $x \in [0,1]$, then

$$\begin{aligned}&\big|\sigma(n)\tau(m-n) - \sigma(n+x)\tau(m-n-x)\big| \\ &\qquad \leq \sigma(n)\big|\tau(m-n) - \tau(m-n+x)\big| + \tau(m-n+x)|\sigma(n) - \sigma(n-x)| \\ &\qquad \leq \sigma(n)\int_{m-n}^{m-n+1} \left|\tau'(t)\right| dt + \tau(m-n)\int_{n-1}^{n} \left|\sigma'(t)\right| dt.\end{aligned}$$

Integrating x over $[0,1]$ and then summing n over $\mathbb{Z}$, we get

$$\left|\sum_{n=-\infty}^{\infty} \sigma(n)\tau(m-n) - \sigma * \tau(m)\right| \leq \sum_{n=-\infty}^{\infty} \sigma(n)\left\|\tau'\right\|_\infty + \sum_{n=-\infty}^{\infty} \tau(m-n)\left\|\sigma'\right\|_\infty,$$

from which the desired inequality follows by Lemma 5.1. $\square$

Lemma 5.3. *If $\sigma_1,\dots,\sigma_r$ are non-negative Schwartz functions with $\|\sigma_i\|_1+\|\sigma_i'\|_1\le\frac{3}{4}$, and $\|\sigma_i'\|_\infty\le\epsilon$, then*

$$\big\|[\sigma_1*\cdots*\sigma_r]-[\sigma_1]*\cdots*[\sigma_r]\big\|_\infty<3\epsilon.$$

Proof. We prove the upper bound $(2r-2)(3/4)^{r-1}\epsilon$ for $r\ge 2$ by induction on r, and that implies the claim. The case $r=2$ follows from Lemma 5.2. If $r\ge 3$, then

$$\begin{aligned}\big\|[\sigma_1*\cdots*\sigma_r]-[\sigma_1]*\cdots*[\sigma_r]\big\|_\infty &\le \big\|[\sigma_1*\cdots*\sigma_r]-[\sigma_1]*[\sigma_2*\cdots*\sigma_r]\big\|_\infty\\ &\quad+\big\|[\sigma_1]*([\sigma_2*\cdots*\sigma_r]-[\sigma_2]*\cdots*[\sigma_r])\big\|_\infty\\ &\le \big\|[\sigma_1*\cdots*\sigma_r]-[\sigma_1]*[\sigma_2*\cdots*\sigma_r]\big\|_\infty\\ &\quad+\big\|[\sigma_1]\big\|_1\big\|([\sigma_2*\cdots*\sigma_r]-[\sigma_2]*\cdots*[\sigma_r])\big\|_\infty.\end{aligned}$$

We have

$$\|\sigma_2*\cdots*\sigma_r\|_1+\big\|(\sigma_2*\cdots*\sigma_r)'\big\|_1\le\big\|(\sigma_2+\sigma_2')\big\|_1\,\|\sigma_3\|_1\cdots\|\sigma_r\|_1\le\left(\frac{3}{4}\right)^{r-1}$$

and

$$\big\|(\sigma_2*\cdots*\sigma_r)'\big\|_\infty\le\|\sigma_2'\|_\infty\|\sigma_3\|_1\cdots\|\sigma_r\|_1\le\left(\frac{3}{4}\right)^{r-2}\epsilon.$$

Applying Lemma 5.2 to σ_1 and $\sigma_2*\cdots*\sigma_r$ and using the induction hypothesis for $\sigma_2,\dots,\sigma_r$,

$$\big\|[\sigma_1*\cdots*\sigma_r]-[\sigma_1]*\cdots*[\sigma_r]\big\|_\infty\le 2\left(\frac{3}{4}\right)^{r-1}\epsilon+(2r-4)\left(\frac{3}{4}\right)^{r-1}\epsilon.\qquad\square$$

Lemma 5.4. *Let $A_1,\dots,A_r$ and $A_1',\dots,A_r'$ are summable functions $\mathbb{Z}\to\mathbb{R}$, then*

$$\big\|A_1*\cdots*A_r-A_1'*\cdots*A_r'\big\|_\infty\le\sum_{i=1}^r\|A_i-A_i'\|_\infty\prod_{j\ne i}\max(\|A_j\|_1,\|A_j'\|_1).$$

Proof. By the triangle inequality,

$$\begin{aligned}\big\|A_1*\cdots*A_r-A_1'*\cdots*A_r'\big\|_\infty&\le\sum_{i=1}^r\big\|A_1*\cdots*A_{i-1}*(A_i-A_i')*A_{i+1}'*\cdots*A_r'\big\|_\infty\\ &\le\sum_{i=1}^r\big\|A_1*\cdots*A_{i-1}*A_{i+1}'*\cdots*A_r'\big\|_1\|A_i-A_i'\|_\infty.\end{aligned}$$

For each i,

$$\begin{aligned}\big\|A_1*\cdots*A_{i-1}*A_{i+1}'*\cdots*A_r'\big\|_1&\le\prod_{j=1}^{i-1}\|A_j\|_1\prod_{j=i+1}^{r}\|A_j'\|_1\\ &\le\prod_{j\ne i}\max(\|A_j\|_1,\|A_j'\|_1).\qquad\square\end{aligned}$$

Proposition 5.5. *For all $k\ge 0$, we have*

$$4^{-k}x_{2^s,k}=\varphi_s(k-1)+O(8^{-s}).$$

Proof. Suppose $k\ge 2^{3s/2}$. For $a>b>0$, the maximum of $e^{-bx}-e^{-ax}$ on $[0,\infty)$ depends only on a/b, so we may consider the case $b=1$. The maximum value is achieved at $\frac{\log a}{a-1}$, and by l'Hôpital's rule, this value divided by $a-1$ approaches $1/e$ as $a\to 1$. As

$$\log\left(\frac{1+\cos x}{2}\right)=-\frac{x^2}{4}+O(x^4),$$

for $j \le 2^{s/4}$,

$$\left(\frac{1+\cos\frac{(2j-1)\pi}{2^{s+1}}}{2}\right)^{k-1} - \exp\left(-\frac{(2j-1)^2\pi^2}{4\cdot 4^{s+1}}(k-1)\right) = O(8^{-s}).$$

Moreover,

$$2^{-s}\sin\frac{(2j-1)\pi}{2^{s+1}} = (2j-1)2^{-2s-1}\pi + O\left(j^3 2^{-4s}\right).$$

Therefore,

$$2^{-s}\sum_{j\le 2^{m/2}}\left|\sin\frac{(2j-1)\pi}{2^{s+1}}\left(\frac{1+\cos\frac{(2j-1)\pi}{2^{s+1}}}{2}\right)^{k-1} - \frac{(2j-1)\pi}{2^{s+1}}\exp\left(-\frac{(2j-1)^2\pi^2}{4\cdot 4^{s+1}}(k-1)\right)\right| = O(8^{-s}).$$

Moreover, for m sufficiently large, the sums

$$2^{-s}\sum_{2^{m/2}<j\le 2^s}\sin\frac{(2j-1)\pi}{2^{s+1}}\left(\frac{1+\cos\frac{(2j-1)\pi}{2^{s+1}}}{2}\right)^{k-1}$$

and

$$\sum_{j>2^{m/2}}\frac{(2j-1)\pi}{2^{s+1}}\exp\left(-\frac{(2j-1)^2\pi^2}{4\cdot 4^{s+1}}(k-1)\right)$$

both have the properties that each term is less than half of the previous term, and the initial term is $o(8^{-s})$. The proposition follows for $k \ge 2^{3s/2}$.

We may therefore assume $3 \le 2^s \le k < 2^{3s/2}$. As $\varphi(x)$ is Schwartz and identically 0 for $x<0$, it follows that $\varphi(x) = o(x^6)$ as $x\to 0$, so $\varphi((k-1)4^{-s-2})$ is $o(8^{-s})$.

To prove that $4^{-k}x_{2^s,k}$ is also $o(8^{-s})$, it suffices to show that $X_{2^s}(1/4-1/4k)$ is $o(8^{-s})$. For this, we observe

$$X_1\left(\frac14-\frac{1}{4k}\right)^{-1} = 2+\frac{4}{k-1}.$$

By induction on i,

$$X_{2^{i+1}}\left(\frac14-\frac{1}{4k}\right)^{-1} = X_{2^i}\left(\frac14-\frac{1}{4k}\right)^{-2}-2 \ge \left(2+\frac{4^i}{k-1}\right)^2-2 > 2+\frac{4^{i+1}}{k-1}$$

for $0\le i\le s-1$. If r is the smallest integer such that $4^r \ge 4(k-1)$, then $r = m/4+O(1)$, and

$$X_{2^r}\left(\frac14-\frac{1}{4k}\right)^{-1} > 6.$$

By induction on i, we have

$$X_{2^i}\left(\frac14-\frac{1}{4k}\right)^{-1} \ge 4^{2^{i-r}}+2$$

for all $i\ge r$, since it holds for $i=r$, and

$$X_{2^{i+1}}\left(\frac14-\frac{1}{4k}\right)^{-1} = X_{2^i}\left(\frac14-\frac{1}{4k}\right)^{-2}-2 \ge \left(4^{2^{i-r}}+2\right)^2-2 > 4^{2^{i+1-r}}+2.$$

for $i \geq r$. Applying this for $i = s$, we get a much stronger upper bound than is needed. □

We now define $A_s(k) = 4^{-k} x_{2^s,k}$ and $A'_s = [\varphi_s]$. As usual, we define $B_T(k)$ to be the sum of $A_S(k)$ over all finite subsets $S \subset T \cap \mathbb{N}$. Thus,

$$B_{\mathbb{R}}(k) = 4^{-k} b_{2k}^{G,V}.$$

Proposition 5.6. *Defining* $A_s(k) = 4^{-k} x_{2^s,k}$*, the sequence* $A_0, A_1, A_2, \ldots$ *satisfies properties* (I)–(III) *of Section 2.*

Proof. We have

$$\sum_{k=0}^{\infty} A_s(k) t^k = X_{2^s}(t/4).$$

By (2.7), $X_1(1/4) = 1/2$. Using (1.2) and induction on n, we deduce that $X_{2^s}(1/4) = 1/2$ for all $s \geq 0$. This implies (I).

By Proposition 5.5, (II) follows from the fact that $\varphi(x) = O(x^{-r})$ as $x \to \infty$, and (III) follows from the fact that $|\varphi'(x)|$ is bounded on $\mathbb{R}$. □

We can now prove Theorem 1.1.

Proof. As $\varphi_s(k-1) - \varphi_s(k) = O(16^{-s})$. By Proposition 5.5, $|A_s(k) - A'_s(k)| = O(8^{-s})$, so Lemma 5.4 implies that for fixed r, for $s \geq 4r$, and for $S = \{s_1, s_2, \ldots, s_m\} \subset [s-r, r+s]$,

$$\left\| A_S - [\varphi_{s_1}] * [\varphi_2] * \cdots * [\varphi_{s_m}] \right\|_\infty = O(8^{-s}).$$

Applying Lemma 5.3 in the case $m = r$ and $\sigma_i = \varphi_{s_i}$, we obtain

$$\|A_S - [\varphi_S]\|_\infty = O(8^{-s}),$$

and summing over $S \subset [s-r, s+r]$, we get

$$\left\| B_{[s-r,s+r]} - [\psi_{[s-r,s+r]}] \right\| \infty = O\left(2^{-5s/2}\right).$$

By Proposition 3.6, for all $\epsilon > 0$, if r is large enough in terms of ϵ, then for all $k \in [4^s, 4^{s+1})$,

$$\left| \left(\frac{3}{2}\right)^{-s} B_{\mathbb{R}}(k) - \left(\frac{3}{2}\right)^{-r} \psi_{[s-r,s+r]}(k) \right| \leq \frac{\epsilon}{2} 4^{-s}.$$

As ψ_r converges to ψ on $[1,4]$, for r sufficiently large

$$\left|\psi_r\left(4^{-s}k\right) - \psi\left(4^{-s}k\right)\right| \leq \frac{\epsilon}{2},$$

so by (4.7),

$$\left| \left(\frac{3}{2}\right)^{-r} 4^s \psi_{[s-r,s+r]}(k) - \left(\frac{8}{3}\right)^{s} \psi(k) \right| \leq \frac{\epsilon}{2}.$$

Therefore,

$$\left| \left(\frac{3}{2}\right)^{-r} \psi_{[s-r,s+r]}(k) - \left(\frac{2}{3}\right)^{s} \psi(k) \right| \leq \frac{\epsilon}{2} 4^{-s},$$

which implies

$$|B_{\mathbb{R}}(k) - \psi(k)| \leq \epsilon \left(\frac{8}{3}\right)^{-s}.$$

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As $k < 4^{s+1}$, and $\omega(x) = \psi(x)x^\delta = \psi(x)x^{\log_4(8/3)}$ is bounded away from zero,

$$\psi(k)\left(\frac{8}{3}\right)^{-s} \geq \frac{3}{8}\omega(k)$$

implies

$$\psi(k)^{-1}B_{\mathbb{R}}(k) \to 1$$

as $k \to \infty$. □

6. A lower bound for powers of any tilting representation

We conclude by proving that a lower bound of the type predicted in [2] exists for all tilting representations of SL_2 in characteristic 2.

Theorem 6.1. *If $G = \mathrm{SL}_2$ over an algebraically closed field of characteristic 2 and W is any tilting representation of G, then there exists $c_W > 0$ such that for all $k \geq 1$,*

$$b_k^{G,W} \geq c_W k^{-\delta}(\dim W)^k.$$

The rest of the section is devoted to a proof of this result. We begin by observing that the map $Q(x) \mapsto Q(V)$ defines an isomorphism from $\mathbb{Z}[x]$ to $\mathsf{Tilt}(G)$, the ring of virtual tilting representations of G. Indeed, the formal character identifies $\mathsf{Tilt}(G)$ with $\mathbb{Z}$-linear combinations of $\chi_n(t)$ as computed in (2.2). The $\mathbb{Z}$-linear combinations of the $\chi_n(\mathbb{Z})$ comprise the ring of $\mathbb{Z}/2\mathbb{Z}$-invariant Laurent polynomials in t with coefficients in $\mathbb{Z}$, where the non-trivial element of the Weyl group $\mathbb{Z}/2\mathbb{Z}$ of SL_2 maps $t \mapsto t^{-1}$. It is clear that

$$Q(x) \mapsto Q(\chi_1(t)) = Q\left(t + t^{-1}\right)$$

gives an isomorphism $\mathbb{Z}[x] \to \mathbb{Z}[t, t^{-1}]^{\mathbb{Z}/2\mathbb{Z}}$.

Lemma 6.2. *If $Q(x) \in \mathbb{Z}[x]$ is such that $W = Q(V)$ is a non-trivial effective representation, then*

(1) $Q(2) = \dim W$*,*
(2) $Q'(2) > 0$
(3) $|Q(x)| < \dim W$ *for all* $x \in (-2, 2)$
(4) $|Q(-2)| = \dim W$ *if and only if* W *purely even or purely odd, i.e., is a direct sum of tilting representations whose highest weights are all even or are all odd.*

Proof. The dimension of W is obtained by substituting $t = 1$ in the formal character $Q(t + t^{-1})$ of W, so it is $Q(2)$.

By the chain rule and l'Hôpital's rule,

$$Q'(2) = \lim_{\theta \to 0} \frac{\frac{d}{d\theta} Q\left(e^{i\theta} + e^{-i\theta}\right)}{\frac{d}{d\theta}\left(e^{i\theta} + e^{-i\theta}\right)} = \left.\frac{\frac{d^2}{d\theta^2} Q\left(e^{i\theta} + e^{-i\theta}\right)}{-2\cos\theta}\right|_{\theta=0} = \frac{I_2(W)}{2},$$

where $I_2(W)$ denotes Dynkin's representation index, which, for an SL_2 representation with formal character $\sum a_n t^n$ is $\sum_n a_n^2$ [4, (2.4)]. This implies (2).

Since W is non-trivial, $Q(t + t^{-1}) - \chi_n(t)$ has non-negative coefficients for some $n \geq 1$, so by (2.1), either the 1 and t^2 coefficients of $Q(t + t^{-1})$ are both positive, or the t and t^{-1} coefficients are both positive. Either way,

$$\left|Q\left(e^{i\theta} + e^{-i\theta}\right)\right| < Q(2)$$

for $0 < \theta < \pi$, implying (3).

Finally, when $\theta = \pi$, $|Q(-2)| \leq Q(2)$ with equality if and only if all the m for which the t^m-coefficient of $Q(t+t^{-1})$ is positive have the same parity. By (2.1), this occurs if and only if all the highest weights have the same parity. □

For each k, the multiplicity of $T(2n)$ as a (virtual) factor of $P(V)$ determines an additive map $\mu_n\colon \mathbb{Z}[x] \to \mathbb{Z}$. By definition, $\mu_n(x^{2k}) = x_{n,k}$ and $\mu_n(x^{2k+1}) = 0$.

Let $n = 2^{s_1} + \cdots + 2^{s_r}$ with $s_1 > s_2 > \cdots > s_r$. For each integer $s \geq -2$ and integer j, let

$$\beta_{s,j} = \zeta_{2^{s+2}}^{j} + 2 + \zeta_{2^{s+2}}^{-j},$$

with the convention that $\beta_{s,j} = 4$ for $s < -2$ so that

$$(\beta_{s,j} - 2)^2 = \beta_{s-1,j}$$

for all $s, j \in \mathbb{Z}$. By Proposition 2.1 and (2.9), it follows that

$$P_{s'}(\beta_{s,j}) = \beta_{s-s',j} - 2, \tag{6.1}$$

where $P_{s'}$ is defined as in (2.10).

Let

$$R_s = \left\{\beta_{s,1}, \beta_{s,3}, \beta_{s,5}, \ldots, \beta_{s,2^{s+1}-1}\right\},$$

so R_s is the set of roots of $P_s(x)$.

By (2.6) and Proposition 2.2, $x_{n,k}$ is the t^k-coefficient of

$$\frac{t^n}{\prod_{i=1}^{r}\prod_{\beta \in R_{s_i}}(1-\beta t)}.$$

If $P^{(n)}$ denotes the product of the P_{s_i}, then its roots are contained in the set $\{\beta_{s_1,j} \mid j \in \mathbb{Z}\}$. By Lemma 2.3,

$$X_n(t) = \sum_{i=0}^{\infty} \sum_{\{\beta \,|\, P^{(n)}(\beta)=0\}} \frac{\beta^{i+n-1}t^{i+n}}{(P^{(n)})'(\beta)}.$$

Therefore,

$$\mu_n\left(x^{2k}\right) = \sum_{\{\beta \,|\, P^{(n)}(\beta)=0\}} \frac{\beta^{k-1}}{(P^{(n)})'(\beta)}.$$

It follows that if $Q(x)$ is a linear combination of even powers of x, then

$$\mu_n(Q(x)) = \sum_{\{\beta \,|\, P^{(n)}(\beta)=0\}} \frac{Q(\beta^{1/2})}{\beta(P^{(n)})'(\beta)}. \tag{6.2}$$

Since μ_n vanishes on odd powers of x, for general Q, we have

$$\mu_n(Q(x)) = \frac{1}{2}\sum_{\{\beta \,|\, P^{(n)}(\beta)=0\}} \frac{Q\left(\beta^{1/2}\right) + Q\left(-\beta^{1/2}\right)}{\beta\left(P^{(n)}\right)'(\beta)}. \tag{6.3}$$

Lemma 6.3. *If* $n = 2^{s_1} + \cdots + 2^{s_r}$ *and* $\beta_{s_1,j}$ *is a root of* $P^{(n)}$*, then*

$$\left|\left(P^{(n)}\right)'(\beta_{s_1,j})\right| > \frac{2^{-(\log_2 j)^2}}{4j^2}\left|\left(P^{(n)}\right)'(\beta_{s_1,1})\right|.$$

Proof. If $j \le 2^{s-s'}$, then by (6.1),

$$P_{s'}(\beta_{s,j}) = 2\cos\frac{2\pi j}{2^{s+2-s'}} = 2\sin\frac{2\pi\left(2^{s-s'}-j\right)}{2^{s+2-s'}} \ge 2\left(1-j2^{s'-s}\right)$$

as $\sin x \ge \frac{2x}{\pi}$ on $[0,\pi/2]$. For all $s' \ne s$ and j odd, $P_{s'}(\beta_{s,j})$ is not zero and is twice the sine of an integer multiple of $\frac{2\pi}{2^{s+2-s'}}$, so it is at least $2^{1+s'-s}$ in absolute value. For a given j, there are $\lfloor \log_2 j\rfloor$ choices of $s' < s$ for which $j \ge 2^{s-s'}$, and the lower bounds they give are $2^{-0}, 2^{-1}, \dots, 2^{1-\lfloor \log_2 j\rfloor}$. From these observations, we see that

$$2^{r-1} \ge \prod_{\left\{i \,\middle|\, P_{s_i}(\beta_{s_1,j})\ne 0\right\}} |P_{s_i}(\beta_{s_1,j})| \ge 2^{r-1}2^{-(\log_2 j)^2}\prod_{l=1}^{\infty}\left(1-2^{-l}\right) \ge 2^{r-3}2^{-(\log_2 j)^2}.$$

On the other hand, by Lemma 2.4, if $\beta_{s,j}$ is a root of $P_{s'}(x)$, then

$$\left|P'_{s'}(\beta_{s,j})\right| = \frac{2^{s'}}{\sin\frac{j\pi}{2^{s+1}}}.$$

Combining these facts, we see that if $\beta_{s_1,j}$ is a root of $P_{s_i}(x)$, then

$$\frac{2^{s_i}}{\sin\frac{j\pi}{2^{s_1+1}}}2^{r-1} \ge \left|(P^{(n)})'(\beta_{s_1,j})\right| \ge \frac{2^{s_i}}{\sin\frac{j\pi}{2^{s_1+1}}}2^{r-3}2^{-(\log_2 j)^2}. \tag{6.4}$$

If $\beta_{s_1,j}$ is a root of P_{s_i}, then j is divisible by $2^{s_1-s_i}$, so $2^{s_i-s_1} \ge j^{-1}$, and

$$\frac{\left|\left(P^{(n)}\right)'(\beta_{s_1,j})\right|}{\left|\left(P^{(n)}\right)'(\beta_{s_1,1})\right|} \ge \frac{2^{-(\log_2 j)^2}}{4j^2}. \tag{6.5}$$

□

We can now prove Theorem 6.1.

Proof of Theorem 6.1. Let $W = Q(V)$. We assume first that W is neither purely even nor purely odd. In this case, by Lemma 6.2, there exist $\epsilon, c_1 > 0$ such that for all $v \in [4-\epsilon, 4]$ and all $u < v$, we have

$$Q(\sqrt{v}) - Q(\sqrt{u}) \ge c_1(v-u).$$

We claim that W determines a positive integer h such that for all sufficiently large s, all $n \in [2^s, 2^{s+1})$, all $k \in [4^{s+h}, 4^{s+h+1})$, and all $j \in [1, 2^s)$,

$$\frac{Q\left(\beta_{s,j}^{1/2}\right)^k}{\left|\beta_{s,j}(P^{(n)})'(\beta_{s,j})\right|} \le 4^{1-j}\frac{Q\left(\beta_{s,1}^{1/2}\right)^k}{\left|\beta_{s,1}(P^{(n)})'(\beta_{s,1})\right|}. \tag{6.6}$$

Indeed for s sufficiently large, $\beta_{s,1} \in [4-\epsilon, 4]$, so

$$\begin{aligned}\frac{Q\left(\beta_{s,1}^{1/2}\right)^k}{Q\left(\beta_{s,j}^{1/2}\right)^k} &\geq \left(1+\frac{c_1(\beta_{s,1}-\beta_{s,j})}{Q(2)}\right)^k \\ &\geq (1+c_2(\beta_{s,1}-\beta_{s,j}))^{4^{s+h}} \\ &= \left(1+c_2\left(4\sin\frac{(j+1)\pi}{2^{s+2}}\sin\frac{(j-1)\pi}{2^{s+2}}\right)\right)^{4^{s+h}} \\ &\geq \left(1+c_2\frac{j^2-1}{4^s}\right)^{4^{s+h}} \geq \exp\left(c_3\left(j^2-1\right)4^h\right)\end{aligned}$$

for some $c_2 > c_3 > 0$ which do not depend on j or h. For $j < 2^s$, also $\beta_{s,j} \geq 2$, so

$$\frac{\beta s, j}{\beta_{s,1}} \geq \frac{1}{2}.$$

Thus (6.6) follows easily from (6.5) when h is sufficiently large. It implies that the absolute value of the sum of all terms

$$\frac{Q\left(\beta_{s,j}^{1/2}\right)^k}{\left|\beta_{s,j}\left(P^{(n)}\right)'(\beta_{s,j})\right|},\ 2 \leq j < 2^s$$

is less than $\frac{1}{4}+\frac{1}{16}+\cdots = \frac{1}{3}$ times

$$\frac{Q\left(\beta_{s,1}^{1/2}\right)^k}{\left|\beta_{s,1}\left(P^{(n)}\right)'(\beta_{s,1})\right|}. \tag{6.7}$$

By part (3) of Lemma 6.2, the terms $Q(\beta_{s,j}^{1/2})^k$ for $j \geq 2^s$ and $Q(-\beta_{s,j}^{1/2})^k$ for any j are bounded above by $((1-c_4)\dim W)^k$ for some $c_4 > 0$ which does not depend on s, j, or k. Therefore,

$$\frac{Q\left(\pm\beta_{s,j}^{1/2}\right)^k\left|\beta_{s,1}\left(P^{(n)}\right)'(\beta_{s,1})\right|}{Q\left(\beta_{s,1}^{1/2}\right)^k\left|\beta_{s,j}\left(P^{(n)}\right)'(\beta_{s,j})\right|}$$

is bounded above by a term of the form $(1-c_5)^{4^{s+h}}$ for some $c_5 > 0$.

When s and h are sufficiently large, therefore, the expression for $\mu_n(Q(x)^k)$ in (6.3) consists of a dominant term which is half the value (6.7), and a sum of other terms, whose total absolute values is less than half as large as the dominant term. Therefore,

$$\mu_n(Q^k) \geq \frac{1}{4}\frac{Q\left(\beta_{s,1}^{1/2}\right)^k}{\beta_{s,1}\left(P^{(n)}\right)'(\beta_{s,1})}.$$

As $\beta_{s,1} < 4$, (6.4) implies

$$\begin{aligned}\mu_n\left(Q^k\right) > \frac{1}{16}\frac{Q\left(\beta_{s,1}^{1/2}\right)^k}{\left(P^{(n)}\right)'(\beta_{s,1})} &\geq 2^{1-r-s}\sin\frac{\pi}{2^{s+1}}Q\left(\beta_{s,1}^{1/2}\right)^k \\ &\geq 2^{-3-r-2s}Q\left(\beta_{s,1}^{1/2}\right)^k\end{aligned}$$

for $k \in [4^{s+h}, 4^{s+h+1})$. Summing over all $n \in [2^s, 2^{s+1})$, we get

$$b_k^{G,W} \geq \sum_{n=2^s}^{2^{s+1}-1} \mu_n(Q^k) \geq \left(\frac{3}{2}\right)^s 2^{-2s-4} Q\left(\beta_{s,1}^{1/2}\right)^k = \frac{1}{16}\left(\frac{3}{8}\right)^s Q\left(\beta_{s,1}^{1/2}\right)^k.$$

As $Q'(2) > 0$, there exists $c_6 > 0$ such that $Q(\beta_{s,1}^{1/2}) > \dim W - c_6(4^{-s})$, so for fixed h and $k < 4^{s+h}$, we can bound $4^{-k}Q(\beta_{s,1}^{1/2})^k$ away from 0.

Finally we consider the cases that W is purely even or purely odd. In the purely even case (i.e., when $Q(x)$ is an even function) we use (6.2) instead of (6.3), so we sum only over the non-negative square roots of the $\beta_{s,j}$. Since part (3) of Lemma 6.2 still holds, it remains true that the terms $Q(\beta_{s,j}^{1/2})^k$ for $j \geq 2^s$ are bounded above by $((1-c_4)\dim W)^k$ for some $c_4 > 0$. For W purely odd and for even k, Q^k is an even function, and we proceed as in the purely even case. Finally, for W odd and k odd, we use the fact, true for all groups and all representations, that $b_{k+1}^{G,W} \geq b_k^{G,W}$. □

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