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Luke Kershaw & Jeremy Rickard

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A finite dimensional algebra with infinite delooping level

Luke Kershaw and Jeremy Rickard *

Abstract. We give an example of a finite dimensional algebra with infinite delooping level, based on an example of a semi-Gorenstein-projective module due to Ringel and Zhang.

1. Introduction

One of the most celebrated open problems in the representation theory of finite dimensional algebras is the finitistic dimension conjecture, publicized by Bass [2] in 1960, and subsequently proven to imply a host of other homological conjectures.

In recent years other, potentially stronger, conditions that would imply the finitistic dimension conjecture have been considered. One is the notion of "injective generation" [8], that we consider briefly in Section 5. But our main focus is the finiteness of delooping level, introduced by Gélinas [5]. He works in greater generality, but since the main example in this paper is a finite dimensional algebra, we shall describe the delooping level in that context.

Given a finite dimensional algebra A and a finitely generated right A-module M, the delooping level of M is the smallest nonnegative integer n such that $\Omega^n M$ is a direct summand (in the stable module category) of $\Omega^{n+1}N$ for some other finitely generated module N, or is infinite if no such n exists. The delooping level is denoted by $\operatorname{dell} M$.

Then the delooping level $\operatorname{dell} A$ is defined to be the largest delooping level of a simple A-module.

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* Corresponding author.

One of the main properties of the delooping level is that $\operatorname{dell} A$ is an upper bound for the (big) finitistic dimension of the opposite algebra A^{op} and so if $\operatorname{dell} A < \infty$ then the big finitistic dimension conjecture holds for A^{op} [5, Proposition 1.3].

In this paper we shall give what we believe to be the first known example of a finite dimensional algebra with infinite delooping level. This does not give a counterexample to the finitistic dimension conjecture.

Although the delooping level of an algebra A is defined in terms of the delooping levels of *simple* modules, we show in Section 2 that if there is any finitely generated A-module M with $\operatorname{dell} M = \infty$, then there is another finite dimensional algebra B with $\operatorname{dell} B = \infty$.

Our main result, Proposition 4.1, is that a particular "semi-Gorenstein-projective" module studied by Ringel and Zhang [9] has infinite delooping level. Gorenstein projective modules are easily seen to have delooping level equal to zero, but the same argument does not apply to semi-Gorenstein-projective modules. What inspired us to look at this example is that Gélinas showed in [5, Theorem 1.10] that the module N in the definition of delooping level can always be taken to be $\Sigma^{n+1}\Omega^n M$ (where Σ denotes the left adjoint of Ω), which in the case of a semi-Gorenstein-projective module simplifies to ΣM, so that proving $\operatorname{dell} M = \infty$ reduces to proving that $\Omega^n M$ is never a summand of $\Omega^n(\Omega\Sigma)M$.

There are other examples known of modules that are semi-Gorenstein-projective, but not Gorenstein projective. The first was given by Jorgensen and Şega [6], and Marczinzik [7] gave others. It might be interesting to study the delooping level of these examples.

2. Delooping levels of one point extensions

The proof of the following proposition uses the construction of a one-point extension algebra, a particular kind of triangular matrix algebra. More details about this construction can be found in, for example, [1, Section III.2].

Proposition 2.1. *If M is a finitely generated module with infinite delooping level for a finite dimensional k-algebra A, then there is a finite dimensional k-algebra B with infinite delooping level.*

Proof. Let B be the one point extension algebra

$$B = A[M] = \begin{pmatrix} k & M \\ 0 & A \end{pmatrix},$$

and let S be the simple B-module $\begin{pmatrix} k & M \end{pmatrix} / \begin{pmatrix} 0 & M \end{pmatrix}$

Then for $n \geq 1$, $\Omega^n S = \begin{pmatrix} 0 & \Omega^{n-1}M \end{pmatrix}$. If S has finite delooping level, then for all sufficiently large n, $\Omega^n S$ is a summand of $\Omega^{n+1}X$ for some B-module X, and $\Omega X = \begin{pmatrix} 0 & N \end{pmatrix}$ for some A-module N, and $\Omega^{n+1}X = \begin{pmatrix} 0 & \Omega^n N \end{pmatrix}$.

Therefore $\Omega^{n-1}M$ is a summand of $\Omega^n N$, and so M has finite delooping level. □

3. Ringel and Zhang's example

In [9], Ringel and Zhang exhibited an example of a three-dimensional module M for a six-dimensional local finite dimensional algebra Λ that is semi-Gorenstein-projective (meaning that $\operatorname{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$), but is not torsionless (i.e., is not a submodule of a projective Λ-module), and hence not Gorenstein-projective.

In this section, we gather the information that we will need about this example. All of the results in this section are taken from Ringel and Zhang's paper, except for the remarks about the subalgebra Γ. Note that they work with left modules, whereas we are using right modules, so the algebra we define is the opposite of theirs.

3.1. **The algebra $\Lambda = \Lambda(q)$ and the subalgebra Γ [9, §6.1].** Let k be a field, and $q \in k$ an element with infinite multiplicative order. Then $\Lambda = \Lambda(q)$ is the algebra

$$k\langle x, y, z\rangle / \left(x^2, y^2, z^2, zy, yx + qxy, zx - xz, yz - xz\right),$$

which is easily seen to be six-dimensional, with basis $1, x, y, z, yx, zx$.

Let Γ be the two-dimensional subalgebra of Λ generated by x. Note that Λ is a free right Γ-module.

3.2. **The modules $M(\alpha)$ [9, §6.1, §6.3].** For each $\alpha \in k$, $M(\alpha)$ is the three-dimensional Λ-module with basis v, v', v'', where $vx = \alpha v'$, $vy = v'$, $vz = v''$, with v' and v'' annihilated by x, y and z.

Next, we gather the results that we need regarding the modules $M(\alpha)$. We shall state them in terms of the effect of Ω and Σ on the modules, but apart from the brief remark about the subalgebra Γ, everything in the following proposition is a restatement of results of Ringel and Zhang [9].

Proposition 3.1. *Let $\alpha \in k$.*

(1) If $\alpha \neq 1$ then $\Omega M(\alpha) = M(q\alpha)$.
(2) If $\alpha \neq q$ then $\Sigma M(\alpha) = M(q^{-1}\alpha)$.
(3) $\Omega\Sigma M(q)$ is a two-dimensional module that is free as a Γ-module.

Proof. Both (1) and (2) follow immediately from [9, Lemma 6.4 and Lemma 3.2]. Note that Ringel and Zhang use the notation $\mho M(\alpha)$ where we use $\Sigma M(\alpha)$.

For any module M, $\Omega\Sigma M$ is the maximal torsionless quotient of M. For $M = M(q)$, this is shown in [9, Lemma 6.2] to be $M(q)/M(q)z$, which is freely generated as a Γ-module by v. ☐

4. The delooping level of $M(q)$

Proposition 4.1. *The Λ-module $M(q)$ has infinite delooping level.*

Proof. Gélinas showed that if a module M has delooping level $n < \infty$, then $\Omega^n M$ is a stable direct summand of $\Omega^{n+1}\Sigma^{n+1}\Omega^n M$ [5, Theorem 1.10]. We shall show that this is not the case for $M = M(q)$.

By Proposition 3.1 (1), $\Omega^n M(q) \cong M(q^{n+1})$ is three-dimensional.

By Proposition 3.1 (2), $\Sigma^n\Omega^n M(q) \cong M(q)$, so

$$\Omega^{n+1}\Sigma^{n+1}\Omega^n M \cong \Omega^{n+1}\Sigma M(q).$$

But since $\Omega\Sigma M(q)$ and Λ are both free as Γ-modules,

$$\Omega^{n+1}\Sigma M(q) = \Omega^n \left(\Omega\Sigma M(q)\right)$$

is also free as a Γ-module and therefore every direct summand has even dimension. ☐

Corollary 4.2. *The one-point extension algebra $\Lambda[M(q)]$ has infinite delooping level.*

Proof. Proposition 3.1 and Proposition 4.1. ☐

5. Other properties of $\Lambda[M(q)]$

In this final section, we consider some other related properties of the algebra $\Lambda[M(q)]$.

One of the main reasons that the delooping level was introduced was its connection with the finitistic dimension conjecture: for Artinian rings A, Gélinas proved that the delooping level $\operatorname{dell} A$ of A is an upper bound for the big finitistic dimension $\operatorname{Findim} A^{op}$ of its opposite algebra, so that finiteness of the delooping level of A implies the big finitistic dimension conjecture for A^{op} [5, Proposition 1.3].

It is therefore natural to wonder whether $\Lambda[M(q)]^{op}$ might have infinite finitistic dimension. This is not the case.

Proposition 5.1. $\operatorname{Findim} \Lambda[M(q)]^{op} = 1$

Proof. Since

$$\Lambda[M(q)]^{op} = \begin{pmatrix} k & 0 \\ M(q) & \Lambda^{op} \end{pmatrix}$$

there is an obvious module with projective dimension equal to one – namely, $\begin{pmatrix} 0 & \Lambda^{op} \end{pmatrix} = \begin{pmatrix} M(q) & \Lambda^{op} \end{pmatrix} / \begin{pmatrix} M(q) & 0 \end{pmatrix}$, and so $1 \leq \operatorname{Findim} \Lambda[M(q)]^{op}$.

But since $\Lambda[M(q)]^{op}$ is a triangular matrix ring, it is well-known [4, Corollary 4.21] that

$$\operatorname{Findim} \Lambda[M(q)]^{op} \leq 1 + \operatorname{Findim} k + \operatorname{Findim} \Lambda^{op},$$

and since Λ^{op} is a local finite dimensional algebra, $\operatorname{Findim} \Lambda^{op} = 0$. So $\operatorname{Findim} \Lambda[M(q)]^{op} \leq 1$. □

Another property that has recently been shown to imply the big finitistic conjecture for a finite dimensional algebra A is "injective generation". If the unbounded derived category $\mathcal{D}(A)$ is equal to $\operatorname{Loc}(\operatorname{Inj} A)$, the localizing subcategory of $\mathcal{D}(A)$ generated by injective modules, then the big finitistic dimension conjecture holds for A [8, Theorem 4.3].

We shall end by showing that the algebra $\Lambda[M(q)]^{op}$ is not a counterexample to injective generation.

Proposition 5.2. *Injectives generate for* $\Lambda[M(q)]^{op}$*.*

Proof. Since $\Lambda[M(q)]^{op}$ is a triangular matrix ring, by a result of Cummings it is sufficient to show that injectives generate for Λ^{op} [3, Example 6.11].

Since $M(0) = \Omega M(0)$, $M(0)$ has finite syzygy type as a Λ-module, and so, taking the vector space dual, $DM(0)$ has finite cosyzygy type as a Λ^{op}-module, which implies that $DM(0)$ is in $\operatorname{Loc}(\operatorname{Inj} \Lambda^{op})$ [8, Proposition 7.2].

Since $DM(0)$ has radical length two, there is a short exact sequence

$$0 \to S_1 \to DM(0) \to S_2 \to 0$$

where S_1 and S_2 are nonzero semisimple Λ^{op}-modules. Taking the coproduct of infinitely many copies of this sequence, we get a short exact sequence

$$0 \to S \to \bigoplus DM(0) \to S \to 0$$

where S is an infinitely generated semisimple module. Splicing together copies of this sequence gives a resolution

$$\cdots \to \bigoplus DM(0) \to \bigoplus DM(0) \to S \to 0$$

by coproducts of copies of $DM(0)$, and so S is in $\operatorname{Loc}(\operatorname{Inj} \Lambda^{op})$ [8, Proposition 2.1(h)]. Therefore the unique simple Λ^{op}-module is in $\operatorname{Loc}(\operatorname{Inj} \Lambda^{op})$ and so $\mathcal{D}(\Lambda^{op}) = \operatorname{Loc}(\operatorname{Inj} \Lambda^{op})$ [8, Proposition 2.1(e) and Lemma 6.1]. □

5.1. **The opposite algebra.** The reader might wonder about corresponding properties of the opposite algebra. In that case two things are easier, because $\left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & U \end{smallmatrix}\right)$, where U is any one-dimensional subspace of $\operatorname{rad}^2(\Lambda)$, are simple left submodules of $\Lambda[M(q)]$, so that every simple module for $\Lambda[M(q)]^{op}$ is isomorphic to a submodule of a projective module, a property which is well known ([2, Lemma 6.2]) to be equivalent to the condition

$$\operatorname{Findim} \Lambda[M(q)] = 0,$$

and is trivially equivalent to the condition

$$\operatorname{dell} \Lambda[M(q)]^{op} = 0.$$

It is also the case that injectives generate for $\Lambda[M(q)]$. It follows from the results of [9, Section 6.5] that $M(0)^* = \operatorname{Hom}_\Lambda(M(0), \Lambda)$ is a three-dimensional Λ^{op}-module isomorphic to its syzygy, and from there the proof follows that of Proposition 5.2: $D(M(0)^*)$ is in $\operatorname{Loc}(\operatorname{Inj} \Lambda)$, and so the unique simple Λ-module is in $\operatorname{Loc}(\operatorname{Inj} \Lambda)$, and so injectives generate for Λ, and so injectives generate for $\Lambda[M(q)]$.

References

[1] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, 1995.

[2] Hyman Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Am. Math. Soc. **95** (1960), 466–488.

[3] Charley Cummings, *Ring constructions and generation of the unbounded derived module category*, Algebr. Represent. Theory **26** (2023), no. 1, 281–315.

[4] Robert M. Fossum, Phillip A. Griffith, and Idun Reiten, *Trivial extensions of abelian categories. Homological algebra of trivial extensions of abelian categories with applications to ring theory*, Lecture Notes in Mathematics, vol. 456, Springer, 1975.

[5] Vincent Gélinas, *The depth, the delooping level and the finitistic dimension*, Adv. Math. **394** (2022), Paper no. 108052.

[6] Liana M. Jorgensen, David A. andŞega, *Independence of the total reflexivity conditions for modules*, Algebr. Represent. Theory **9** (2006), no. 2, 217–226.

[7] René Marczinzik, *Gendo-symmetric algebras, dominant dimensions and Gorenstein homological algebra*, 2021, `https://arxiv.org/abs/1608.04212v2`.

[8] Jeremy Rickard, *Unbounded derived categories and the finitistic dimension conjecture*, Adv. Math. **354** (2019), Paper no. 106735 (21 pages).

[9] Claus Michael Ringel and Pu Zhang, *Gorenstein-projective and semi-Gorenstein-projective modules*, Algebra Number Theory **14** (2020), no. 1, 1–36.

— Luke Kershaw —
School of Mathematics, University of Bristol, Fry Building, Woodland Road, Bristol, BS8 1UG, United Kingdom
E-mail address: `l.kershaw@bristol.ac.uk`

— Jeremy Rickard —
School of Mathematics, University of Bristol, Fry Building, Woodland Road, Bristol, BS8 1UG, United Kingdom
E-mail address: `j.rickard@bristol.ac.uk`

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