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Abstract. This paper gives a Schur–Weyl duality approach to the representation theory of the affine Hecke algebras of type C with unequal parameters. The first step is to realize the affine braid group of type C_k as the group of braids on k strands with two poles. Generalizing familiar methods from the one pole (type A) case, this provides commuting actions of the quantum group $U_q\mathfrak{g}$ and the affine braid group of type C_k on a tensor space $M \otimes N \otimes V^{\otimes k}$. Special cases provide Schur–Weyl pairings between the affine Hecke algebra of type C_k and the quantum group of type $\mathfrak{gl}_n$, resulting in natural labelings of many representations of the affine Hecke algebras of type C by partitions. Following an analysis of the structure of weights of affine Hecke algebra representations (extending the one parameter case to the three parameter case necessary for affine Hecke algebras of type C), we provide an explicit identification of the affine Hecke algebra representations that appear in tensor space.

1. Introduction

This paper explores a Schur–Weyl duality approach to the representations of the affine Hecke algebras of type C with unequal parameters. Following Kazhdan–Lusztig [13], the irreducible representations of the affine Hecke algebra are usually constructed via the K-theory of generalized Springer fibers. This method works well when an algebraic group is available, which is only for special cases of the three parameters t, t_0, t_k of the affine Hecke algebras of type C.

G. Lusztig gave a general approach to the unequal parameter case using Kazhdan–Lusztig bases and cells. In [16], the challenges for pushing this method through in type C are outlined in a set of conjectures, many of which have now been settled in work of Geck, Bonnafé, and others (see [2, 9, 11] and references there). Another analytic approach, closer to the original classification and construction of Kazhdan–Lusztig, is given by Opdam and Solleveld (see [20, 26] and the references there). In the type C case, Kato [12] explained

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that the "exotic nilpotent cone" can be used to replace the Kazhdan–Lusztig geometry and obtain a complete geometric classification of the irreducible representations of affine Hecke algebras (with mild restrictions on parameters).

In the type A case, there is a powerful alternative to the geometric method via Schur–Weyl duality (see for example [1, 21, 28]). In this paper we provide an analogue of this Schur–Weyl duality approach for the type C case, with unequal parameters. This is a generalization of the degenerate case studied by Daugherty [4].

The method is the following: Let $U_q\mathfrak{gl}_n$ be the Drinfeld–Jimbo quantum group corresponding to the general linear Lie algebra, and let $V = \mathbb{C}^n$ be the standard representation of $U_q\mathfrak{gl}_n$. Write $L(\lambda)$ for the irreducible polynomial representation of $U_q\mathfrak{gl}_n$ indexed by the partition λ, let $M = L((a^c))$ and $N = L((b^d))$ be irreducible representations of $U_q\mathfrak{gl}_n$ indexed by $a \times c$ and $b \times d$ rectangles. There is an action of an extension of the affine Hecke algebra of type C_k, denoted H_k^{ext}, with parameters

$$t^{\frac{1}{2}} = q, \qquad t_0^{\frac{1}{2}} = -iq^{b+d}, \qquad \text{and} \qquad t_k^{\frac{1}{2}} = -iq^{a+c} \qquad \left(\text{where } i = \sqrt{-1}\right),$$

such that

$$M \otimes N \otimes V^{\otimes k} \qquad \text{is a } \left(U_q\mathfrak{gl}_n, H_k^{\mathrm{ext}}\right)\text{-bimodule.}$$

We show that the commuting actions of $U_q\mathfrak{gl}_n$ and H_k^{ext} provide a Schur–Weyl duality, which can be used to derive the representation theory of H_k^{ext} from the quantum group $U_q\mathfrak{gl}_n$. We work out the combinatorics of this correspondence, relating the natural indexing of H_k^{ext}-modules coming from the Schur–Weyl duality to the other indexings, by describing the weights for the action of the polynomial part (generated by Bernstein generators) on each irreducible module.

A significant portion of the work in identifying the centralizer of the $U_q\mathfrak{gl}_n$ action on $M \otimes N \otimes V^{\otimes k}$ as an extended affine Hecke algebra of type C is in relating Coxeter and Bernstein presentations, and putting the parameter conversions into focus. The relationships between these presentations are given in Theorem 2.1 for the affine braid group of type C, and in Theorem 2.2 for the affine Hecke algebra of type C. Sections 3, 4 and 5 could, perhaps have stood as papers on their own. In Section 3, we give the combinatorics of local regions and standard tableaux for the case of type C with unequal parameters (following the equal parameter case done in [22]). The main result of Section 3, Theorem 3.5, provides a classification and a construction of all irreducible calibrated H_k^{ext}-modules. As in [22], this classification is via *skew local regions*, whose precise definition of skew local regions depends on the careful analysis of the structure of the irreducible representations of rank two affine Hecke algebras. This analysis was done in the single parameter case in [23]. Since the corresponding analysis for *three distinct parameters in the type C_2 case* is, to our knowledge, not available in the literature, we have provided it in Section 4. This will ensure that our classification of calibrated irreducible representations for H_k^{ext} with distinct parameters, as given in Theorem 3.5, is on firm footing. The construction of the action of H_k^{ext} on tensor space is completed in Theorems 5.1 and 5.4. Finally, armed with these tools we prove the main result, Theorem 5.5, which determines exactly which representations of H_k^{ext} appear in tensor space, comparing the natural indexing from the highest weight theory for $\mathfrak{gl}_n$ to the combinatorics of the weights of the action of the polynomial part of H_k^{ext}.

Following the schematic from [21], one would like to generalize the analysis in this paper by replacing finite-dimensional M and N with, for example, other modules from category $\mathcal{O}$. In the finite-dimensional case, the key is that R-matrices for $M \otimes V$ and $N \otimes V$

have only two eigenvalues. This strongly restricts the choices for M and N. Non-finite-dimensional choices of modules M and N that satisfy these conditions exist in category $\mathcal{O}$, but additional work toward understanding the combinatorics of $M \otimes N \otimes V^{\otimes k}$ in these cases is needed. See [3] for important work in this direction.

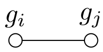
The seeds of the idea for this paper were sown during conversations of A. Ram with P. Pyatov and V. Rittenberg in Bonn in 2005. They suggested that one should analyze two boundary spin chains by R-matrices, thus implying the possibility for Schur–Weyl duality approach to representations of affine braid groups of type C. This idea was completed in the degenerate case in [4], and significant information was obtained in the Temperley–Lieb case in [6] (see also references there). In [5] we shall complete the connection to the statistical mechanics by using the results of this paper to identify the representations of the two boundary Temperley–Lieb algebra given, in a diagrammatic form, by de Gier and Nichols in [6].

2. The two boundary Hecke algebra

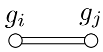
In this section we define the two boundary braid group and Hecke algebras and establish multiple presentations of each. The conversion between presentations is important for matching the algebraic approach to the representation theory with the Schur–Weyl duality approach that we give in Section 5.

For generators g_i, g_j, encode relations graphically by

means $g_i g_j = g_j g_i$,



means $g_i g_j g_i = g_j g_i g_j$, and (2.1)

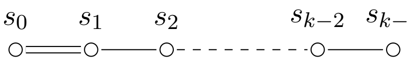


means $g_i g_j g_i g_j = g_j g_i g_j g_i$.

For example, the group of signed permutations,

$$\mathcal{W}_0 = \left\{ \begin{matrix} \text{bijections } w\colon \{-k, \dots -1, 1, \dots k\} \to \{-k, \dots -1, 1, \dots k\} \\ \text{such that } w(-i) = -w(i) \quad \text{for} \quad i = 1, \dots k \end{matrix} \right\}, \tag{2.2}$$

has a presentation by generators $s_0, s_1, \dots s_{k-1}$, with relations

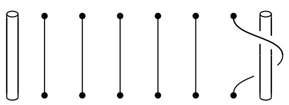
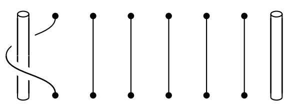


and $s_i^2 = 1$ for $i = 0, 1, 2, \dots k-1$. (2.3)

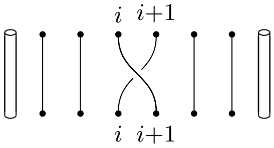
2.1. **The two boundary braid groups $\mathcal{B}_k$ and $\mathcal{B}_k^{\mathrm{ext}}$.** The *two boundary braid group* is the group $\mathcal{B}_k$ generated by $\bar{T}_0, \bar{T}_1, \dots \bar{T}_k$, with relations

. (2.4)

Pictorially, the generators of $\mathcal{B}_k$ are identified with the braid diagrams

$\bar{T}_k =$ , $\bar{T}_0 =$ ,

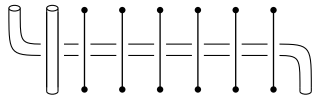
and

$$\bar{T}_i =$$ 

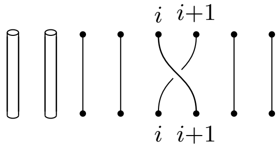
for $i = 1, \ldots, k-1$, (2.5)

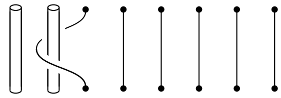
and the multiplication of braid diagrams is given by placing one diagram on top of another. These pictures represent an embedding of $\mathcal{B}_k$ into the braid group on $k+2$ strands. The fixed strands can be viewed as poles.

To make explicit the Schur–Weyl duality approach to representations of $\mathcal{B}_k$ appearing in Section 5, it is useful to move the rightmost pole to the left by conjugating by the diagram

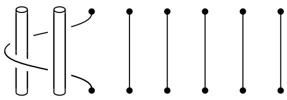
$$\sigma =$$  , (2.6)

where σ is an element of the braid group on $k+2$ strands. This gives a different embedding of $\mathcal{B}_k$ into the braid group on $k+2$ with generators of $\mathcal{B}_k$ having the form

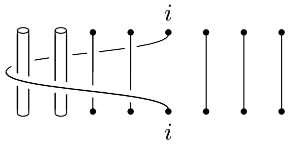
$$T_i = \sigma \bar{T}_i \sigma^{-1} =$$ 

, $$Y_1 = \sigma \bar{T}_0 \sigma^{-1} =$$  , (2.7)

and

$$X_1 = T_1^{-1} T_2^{-1} \cdots T_{k-1}^{-1} \sigma \bar{T}_k \sigma^{-1} T_{k-1} \cdots T_1 =$$  . (2.8)

Define

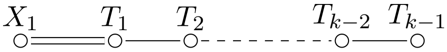
$$Z_1 = X_1 Y_1 \quad \text{and} \quad Z_i = T_{i-1} T_{i-2} \cdots T_1 X_1 Y_1 T_1 \cdots T_{i-1} =$$ 

, (2.9)

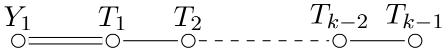
for $i = 2, \ldots k$.

Theorem 2.1. *The two boundary braid group $\mathcal{B}_k$ is presented in the following three ways, using the notation defined in* (2.1)*.*

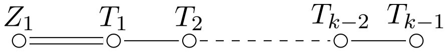
(a) $\mathcal{B}_k$ is presented by generators $X_1, Y_1, Z_1, T_1, \ldots, T_{k-1}$ and relations



(a1)



(a2)



(a3)

and

$$Z_1 = X_1 Y_1. \tag{a4}$$

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(b) $\mathcal{B}_k$ *is presented by generators* $X_1, Y_1, T_1, \ldots T_{k-1}$ *and relations* (a1), (a2), *and*

$$\left(T_1 X_1 T_1^{-1}\right) Y_1 = Y_1 \left(T_1 X_1 T_1^{-1}\right). \tag{b3}$$

(c) $\mathcal{B}_k$ *is presented by generators* $Z_1, \ldots, Z_k, Y_1, T_1, \ldots, T_{k-1}$, *and relations* (a2),

$$Z_i Z_j = Z_j Z_i \quad \textit{for} \quad i, j = 1, \ldots k, \tag{c1}$$

$$Y_1 Z_i = Z_i Y_1 \quad \textit{for} \quad i = 2, \ldots k, \ \textit{and} \tag{c2}$$

$$\begin{aligned} T_i Z_j = Z_j T_i \quad \textit{for} \quad j \neq i, i+1, \\ \textit{with} \quad i = 1, \ldots k-1, \quad \textit{and} \quad j = 1, \ldots k, \end{aligned} \tag{c3}$$

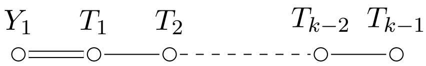
and

$$Z_{i+1} = T_i Z_i T_i \quad \textit{for} \quad i = 1, \ldots k-1. \tag{c4}$$

Proof. With σ as in (2.6) let $T_k = \sigma \bar{T}_k \sigma^{-1}$, so that the original generators are the σ-conjugates of

$$T_0, T_1, \ldots T_k. \tag{o}$$

Conjugate the relations in (2.4) by σ to rewrite them in the form



$$, \quad T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k, \tag{o1}$$

$$T_k Y_1 = Y_1 T_k, \quad \text{and} \quad T_k T_i = T_i T_k, \quad \text{for } i = 1, \ldots k-2. \tag{o2}$$

The conversions between the generators in presentations (a), (b), and (c) are given in (2.7), (2.8), and (2.9). For generators (a) and (b) in terms of generators (o), the key relations are

$$Y_1 = T_0, \quad X_1 = T_1^{-1} \cdots T_{k-1}^{-1} T_k T_{k-1} \cdots T_1 \quad \text{and} \quad T_k = T_{k-1} \cdots T_1 X_1 T_1^{-1} \cdots T_{k-1}^{-1}.$$

Relations (a) *from relations* (b). Relation (a4) is the conversion from generators (b) to generators (a). The relations in (a3) then follow from

$$T_i Z_1 = T_i X_1 Y_1 = X_1 T_i Y_1 = X_1 Y_1 T_i = Z_1 T_i, \qquad \text{for } i = 2, \ldots k-1,$$

and

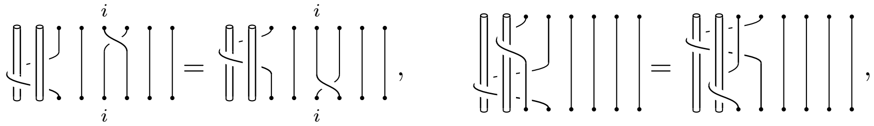
$$\begin{aligned} T_1 Z_1 T_1 Z_1 &= T_1 X_1 Y_1 T_1 X_1 Y_1 = T_1 X_1 \left(Y_1 T_1 X_1 T_1^{-1}\right) T_1 Y_1 = T_1 X_1 \left(T_1 X_1 T_1^{-1} Y_1\right) T_1 Y_1 \\ &= X_1 T_1 X_1 T_1 T_1^{-1} Y_1 T_1 Y_1 = X_1 T_1 X_1 T_1^{-1} T_1 Y_1 T_1 Y_1 = X_1 T_1 X_1 T_1^{-1} Y_1 T_1 Y_1 T_1 \\ &= X_1 Y_1 T_1 X_1 T_1^{-1} T_1 Y_1 T_1 = Z_1 T_1 Z_1 T_1. \end{aligned}$$

Relations (b) *from Relations* (a). Multiplying

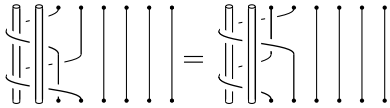
$$\begin{aligned} T_1 X_1 \left(T_1 X_1 T_1^{-1} Y_1\right) T_1 Y_1 &= X_1 T_1 X_1 T_1 T_1^{-1} Y_1 T_1 Y_1 = X_1 T_1 X_1 T_1^{-1} T_1 Y_1 T_1 Y_1 \\ &= X_1 T_1 X_1 T_1^{-1} Y_1 T_1 Y_1 T_1 = X_1 Y_1 T_1 X_1 T_1^{-1} T_1 Y_1 T_1 = Z_1 T_1 Z_1 T_1 \\ &= T_1 Z_1 T_1 Z_1 = T_1 X_1 Y_1 T_1 X_1 Y_1 = T_1 X_1 \left(Y_1 T_1 X_1 T_1^{-1}\right) T_1 Y_1 \end{aligned}$$

on the left by $(T_1 X_1)^{-1}$ and on the right by $(T_1 Y_1)^{-1}$ gives $T_1 X_1 T_1^{-1} Y_1 = Y_1 T_1 X_1 T_1^{-1}$, establishing (b3).

Relations (b) *from relations* (o). The pictorial computations

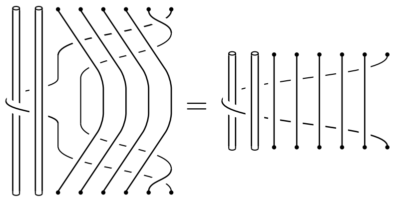


and

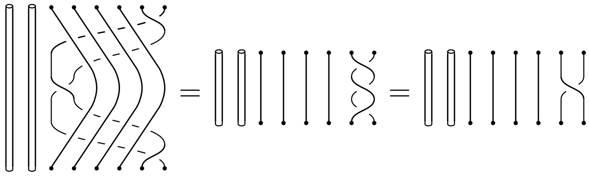


show that $X_1T_i = T_iX_1$ for $i = 1, 2, \dots k-1$, $Y_1T_1X_1T_1^{-1} = T_1X_1T_1^{-1}Y_1$, and $X_1T_1X_1T_1 = T_1X_1T_1X_1$. Hence the relations (a1) and (a2) follow from the relations in (o1) and (o2).

Relations (o) *from relations* (b). The first set of relations in (o1) are the same as the relations in (a2). Let $A = T_{k-1} \cdots T_1$ and $B = T_{k-1} \cdots T_2$. Since X_1 commutes with T_i for $i = 2, \dots k-1$, then $BX_1B^{-1} = X_1$ so that

$ABX_1B^{-1}A^{-1} =$  $= T_k,$

and

$ABT_1B^{-1}A^{-1} =$  $= T_{k-1}.$

Thus, by conjugation by AB, the relation $X_1T_1X_1T_1 = T_1X_1T_1X_1$ becomes $T_kT_{k-1}T_kT_{k-1} = T_{k-1}T_kT_{k-1}T_k$, establishing the second relation in (o1). For $i = 1, \dots k-2$,

$$\begin{aligned}
T_iT_k &= T_iT_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_k^{-1} \\
&= T_{k-1} \cdots T_{i+2}T_iT_{i+1}T_i \cdots T_1X_1T_1^{-1} \cdots T_k^{-1} \\
&= T_{k-1} \cdots T_{i+2}T_{i+1}T_iT_{i+1}T_{i-1} \cdots T_1X_1T_1^{-1} \cdots T_k^{-1} \\
&= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_{i-1}^{-1}T_{i+1}T_i^{-1}T_{i+1}^{-1} \cdots T_k^{-1} \\
&= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_{i-1}^{-1}T_i^{-1}T_{i+1}^{-1}T_iT_{i+2}^{-1} \cdots T_k^{-1} \\
&= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots \cdots T_k^{-1}T_i = T_kT_i.
\end{aligned}$$

Similarly, (b3) gives

$$\begin{aligned}
Y_1T_k &= Y_1T_{k-1} \cdots T_2T_1X_1T_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1} \\
&= T_{k-1} \cdots T_2 \left(Y_1T_1X_1T_1^{-1}\right) T_2^{-1} \cdots T_{k-1}^{-1} \\
&= T_{k-1} \cdots T_2 \left(T_1X_1T_1^{-1}Y_1\right) T_2^{-1} \cdots T_{k-1}^{-1} \\
&= T_{k-1} \cdots T_2T_1X_1T_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1}Y_1 = T_kY_1,
\end{aligned}$$

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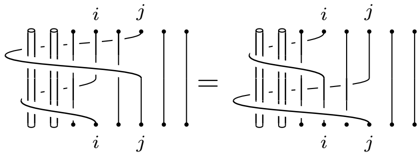
giving the relations in (o2).

Relations (c) *from relations* (o). The first set of relations in (o1) are the same as the relations in (a2). Relations (c4) are exactly the definitions in the second part of (2.9). The pictorial computation

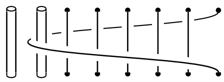
$$Z_jZ_i =$$  $$= Z_iZ_j$$

give relations (c1). Similarly, pictorial computations readily show that $Y_1Z_i = Z_iY_1$ for $i > 1$ and $T_iZ_j = Z_jT_i$ for $i \neq j, j+1$, proving relations (c2) and (c3).

Generators (o) *from generators* (c). The key formula for the generator T_k is

$$\begin{aligned} T_k &= T_{k-1}\cdots T_1\left(T_1^{-1}\cdots T_{k-1}^{-1}T_kT_{k-1}\cdots T_1\right)\\ &\qquad Y_1(T_1\cdots T_{k-1})\left(T_{k-1}^{-1}\cdots T_1^{-1}\right)Y_1^{-1}\left(T_1^{-1}\cdots T_{k-1}^{-1}\right)\\ &= (T_{k-1}\cdots T_1)X_1Y_1(T_1\cdots T_{k-1})T_{s_\varphi} = Z_kT_{s_\varphi}^{-1}, \end{aligned}$$

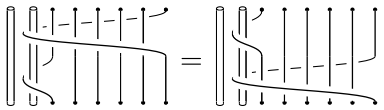
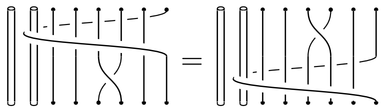
where

$$T_{s_\varphi} = T_{k-1}T_{k-2}\cdots T_1Y_1T_1\cdots T_{k-2}T_{k-1} =$$  .

Relations (o) *from relations* (c). The first set of relations in (o1) are the same as the relations in (a2). The relations

$$T_{s_\varphi}Y_1 = Y_1T_{s_\varphi} \qquad \text{and} \qquad T_{s_\varphi}T_i = T_iT_{s_\varphi}, \qquad \text{for } i = 1, \ldots, k-2, \tag{2.10}$$

are verified pictorially by

 and  .

or by direct computation using the relations in (a2).

By (2.8) and (2.9), $Z_k = T_kT_{s_\varphi}$ and, by (c3) and (c2) respectively,

$$\begin{aligned} T_kT_i = Z_kT_{s_\varphi}^{-1}T_i = Z_kT_iT_{s_\varphi}^{-1} = T_iZ_kT_{s_\varphi}^{-1} = T_iT_k, \quad &\text{for } i = 1, \ldots, k-2,\\ \text{and} \quad T_kY_1 = Z_kT_{s_\varphi}^{-1}Y_1 = Z_kY_1T_{s_\varphi}^{-1} = Y_1Z_kT_{s_\varphi}^{-1} = Y_1T_k, & \end{aligned} \tag{2.11}$$

which proves the relations in (o2).

By the relations in (2.11) and the second set of relations in (2.10),

$$\left(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\right)T_k = T_k\left(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\right)$$

and

$$\left(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\right)T_{s_\varphi} = T_{s_\varphi}\left(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\right),$$

so that $(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1})(T_kT_{s_\varphi}) = (T_kT_{s_\varphi})(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1})$. Using these and the equality

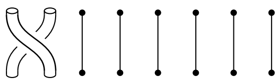
$$T_{k-1}Z_kZ_{k-1} = T_{k-1}Z_{k-1}Z_k = Z_kT_{k-1}^{-1}Z_k = Z_kZ_{k-1}T_{k-1},$$

we have

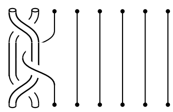
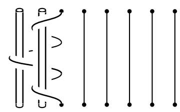
$$\begin{aligned}
T_{k-1}Z_kZ_{k-1} &= T_{k-1}Z_k\left(T_{k-1}^{-1}Z_kT_{k-1}^{-1}\right) = T_{k-1}(T_kT_{s_\varphi})T_{k-1}^{-1}(T_kT_{s_\varphi})T_{k-1}^{-1}\\
&= T_{k-1}T_kT_{k-1}\left(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\right)T_kT_{s_\varphi}T_{k-1}^{-1}\\
&= \Big(T_{k-1}T_kT_{k-1}T_k)(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\Big)\\
&= Z_kZ_{k-1}T_{k-1} = Z_k\left(T_{k-1}^{-1}Z_kT_{k-1}^{-1}\right)T_{k-1} = (T_kT_{s_\varphi})T_{k-1}^{-1}(T_kT_{s_\varphi})\\
&= T_kT_{k-1}\left(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\right)(T_kT_{s_\varphi})\\
&= T_kT_{k-1}(T_kT_{s_\varphi})\left(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\right)\\
&= (T_kT_{k-1}T_kT_{k-1})\left(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}\right).
\end{aligned}$$

Then multiplying on the right by $(T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1}T_{s_\varphi}T_{k-1}^{-1})^{-1}$ establishes the last relation in (o1). □

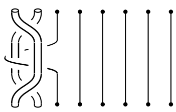
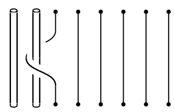
In preparation for the definition of the group $\mathcal{B}_k^{\mathrm{ext}}$ introduce a new (diagrammatic) generator

$P^{1/2} =$  . (2.12)

Then

$P^{1/2}Y_1P^{-1/2} =$  $=$  $= Y_1^{-1}X_1Y_1$ (2.13)

and

$P^{1/2}X_1P^{-1/2} =$  $=$  $= Y_1$ (2.14)

Following these pictorial computations, the *extended affine braid group* is the group $\mathcal{B}_k^{\mathrm{ext}}$ generated by $\mathcal{B}_k$ and P with the additional relations

$$PX_1P^{-1} = Z_1^{-1}X_1Z_1, \qquad PY_1P^{-1} = Z_1^{-1}Y_1Z_1, \tag{2.15}$$

$$PZ_1P^{-1} = Z_1, \qquad \text{and} \qquad PT_iP^{-1} = T_i \text{ for } i = 1,\dots,k-1. \tag{2.16}$$

Note that the element

$$Z_0 = PZ_1\cdots Z_k \quad \text{is central in } \mathcal{B}_k^{\mathrm{ext}} \tag{c0}$$

since the group $\mathcal{B}_k^{\mathrm{ext}}$ is a subgroup of the braid group on $k+2$ strands, and Z_0 is the generator of the center of the braid group on $k+2$ strands (see [10, Theorem 4.2]). So

$$\text{if } \mathcal{D} = \left\{Z_0^j \,\middle|\, j\in\mathbb{Z}\right\} \qquad \text{then} \qquad \mathcal{B}_k^{\mathrm{ext}} = \mathcal{D}\times\mathcal{B}_k, \quad \text{with } \mathcal{D}\cong\mathbb{Z}.$$

2.2. **The two boundary Hecke algebra H_k^{ext}.** In this subsection we define the two boundary Hecke algebras and relate it to the presentation of the affine Hecke algebra of type C that is found, for example, in [16, Proposition 3.6] and [18, (4.2.4)].

Fix $a_1, a_2, b_1, b_2, t^{\frac12} \in \mathbb{C}^\times$. The *extended two boundary Hecke algebra* H_k^{ext} is the quotient of $\mathcal{B}_k^{\mathrm{ext}}$ by the relations

$$(X_1-a_1)(X_1-a_2) = 0, \quad (Y_1-b_1)(Y_1-b_2) = 0, \quad \text{and} \quad \left(T_i - t^{\frac12}\right)\left(T_i + t^{-\frac12}\right) = 0, \tag{h}$$

for $i = 1, \dots k-1$. Let

$$t_k^{\frac{1}{2}} = a_1^{\frac{1}{2}}(-a_2)^{-\frac{1}{2}} \quad \text{and} \quad t_0^{\frac{1}{2}} = b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}}. \tag{2.17}$$

With $Z_i \in H_k^{\mathrm{ext}}$ as in (2.9), define

$$T_0 = b_1^{-\frac{1}{2}}(-b_2)^{-\frac{1}{2}}Y_1, \qquad W_i = -(a_1a_2b_1b_2)^{-\frac{1}{2}}Z_i \quad \text{for } i = 1, \dots, k, \text{ and} \tag{2.18}$$

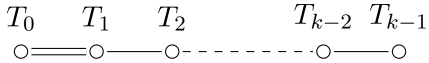
$$W_0 = PW_1 \cdots W_k = (-1)^k(a_1a_2b_1b_2)^{-\frac{k}{2}}PZ_1 \cdots Z_k = (-1)^k(a_1a_2b_1b_2)^{-\frac{k}{2}}Z_0. \tag{2.19}$$

Then

$$X_1 = Z_1Y_1^{-1} = a_1^{\frac{1}{2}}(-a_2)^{\frac{1}{2}}W_1T_0^{-1}. \tag{2.20}$$

Theorem 2.2. *Fix $t_0, t_k, t \in \mathbb{C}^\times$ and use notations for relations as defined in* (2.1)*. The extended affine Hecke algebra H_k^{ext} defined in* (h) *is presented by generators, T_0, T_1, ..., T_{k-1}, W_0, W_1, ..., W_k and relations*

$$W_0 \in Z\left(H_k^{\mathrm{ext}}\right), \tag{B1}$$



;

$$W_iW_j = W_jW_i, \quad \textit{for} \quad i, j = 0, 1, \dots k; \tag{B2}$$

$$T_0W_j = W_jT_0, \quad \textit{for} \quad j \neq 1; \tag{B3}$$

$$\begin{gathered} T_iW_j = W_jT_i, \quad \textit{for } i = 1, \dots k-1 \\ \textit{and} \quad j = 1, \dots k \textit{ with } j \neq i, i+1; \end{gathered} \tag{B4}$$

$$\begin{gathered} \left(T_0 - t_0^{\frac{1}{2}}\right)\left(T_0 + t_0^{-\frac{1}{2}}\right) = 0, \\ \textit{and} \quad \left(T_i - t^{\frac{1}{2}}\right)\left(T_i + t^{-\frac{1}{2}}\right) = 0 \quad \textit{for } i = 1, \dots k-1. \end{gathered} \tag{H}$$

For $i = 1, \dots k-1$,

$$\begin{aligned} T_iW_i &= W_{i+1}T_i + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\frac{W_i - W_{i+1}}{1 - W_iW_{i+1}^{-1}}, \\ T_iW_{i+1} &= W_iT_i + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\frac{W_{i+1} - W_i}{1 - W_iW_{i+1}^{-1}}, \end{aligned} \tag{C1}$$

and

$$T_0W_1 = W_1^{-1}T_0 + \left(\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right) + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right)W_1^{-1}\right)\frac{W_1 - W_1^{-1}}{1 - W_1^{-2}}. \tag{C2}$$

Proof. The conversion between the different sets of generators of H_k^{ext} is provided by (2.18). *Equivalence between (c0–c4) and the second and third relations of* (h) *with the relations (B1–B4) and* (H). Since T_0 and Y_1 differ by a constant, and W_i and Z_i differ by a constant, the relations in (c0–c4) are equivalent to the relations in (B1–B4), respectively. Since

$$\begin{aligned} 0 = (Y_1 - b_1)(Y_1 - b_2) &= b_1^{\frac{1}{2}}(-b_2)^{\frac{1}{2}}\left(T_0 - b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}}\right)b_1^{\frac{1}{2}}(-b_2)^{\frac{1}{2}}\left(T_0 + b_1^{-\frac{1}{2}}(-b_2)^{\frac{1}{2}}\right) \\ &= -b_1b_2\left(T_0 - t_0^{\frac{1}{2}}\right)\left(T_0 + t_0^{-\frac{1}{2}}\right), \end{aligned}$$

the relations (H) are equivalent to the second and third relations in (h).

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Relations (C1–C2) from relations (c0–c4) and (h). From (2.9) and (2.18), $W_{i+1} = T_i W_i T_i$, and by the last relation in (h), $T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. So

$$T_i W_i = W_{i+1} T_i^{-1} = W_{i+1}\left(T_i - \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\right) = W_{i+1}T_i + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\frac{W_i - W_{i+1}}{1 - W_i W_{i+1}^{-1}}$$

and

$$T_i W_{i+1} = T_i^2 W_i T_i = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) W_{i+1} + W_i T_i = W_i T_i + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\frac{W_{i+1} - W_i}{1 - W_i W_{i+1}^{-1}},$$

which establishes the relations in (C1).

By the first relation in (h), $X_1^{-1} = -a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1})$. Since $W_1 = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}X_1T_0$ and $T_0 - T_0^{-1} = t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}$,

$$\begin{aligned} T_0W_1 - W_1^{-1}T_0 &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}\left(T_0X_1T_0 - a_1(-a_2)T_0^{-1}X_1^{-1}T_0\right) \\ &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}\left(T_0X_1T_0 + a_1a_2T_0^{-1}\left(-a_1^{-1}a_2^{-1}X_1 + \left(a_1^{-1} + a_2^{-1}\right)\right)T_0\right) \\ &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}\left(\left(T_0 - T_0^{-1}\right)X_1T_0 + (a_1 - (-a_2))\right) \\ &= \left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right)W_1 + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right), \end{aligned}$$

which establishes (C2).

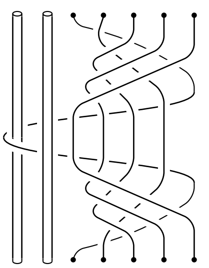
The first relation in (h) *from the relations (B1–B4),* (H) *and (C1–C2).* By (C2),

$$\begin{aligned} a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}\left(T_0X_1T_0 - a_1(-a_2)T_0^{-1}X_1^{-1}T_0\right) &= T_0W_1 - W_1^{-1}T_0 \\ &= \left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right)W_1 + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right) \\ &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}\left(\left(T_0 - T_0^{-1}\right)X_1T_0 + (a_1 - (-a_2))\right) \\ &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}\left(T_0X_1T_0 + a_1a_2T_0^{-1}\left(-a_1^{-1}a_2^{-1}X_1 + \left(a_1^{-1} + a_2^{-1}\right)\right)T_0\right), \end{aligned}$$

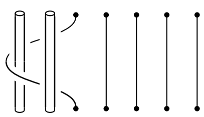
giving $X_1^{-1} = -a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1})$, which establishes the first relation in (h). □

Remark 2.3. Let $w_A = s_1s_2\ldots s_{k-1}s_{k-2}\cdots s_1$ be the longest element of the group $WA_k = \langle s_1, \ldots s_{k-1}\rangle$, and let $T_{w_A} = T_1T_2\ldots T_{k-1}T_{k-2}\cdots T_1$ be the corresponding element of H_k or H_k^{ext}. Let

$$T_{0^\vee} = T_{w_A}^{-1}T_kT_{w_A} = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}$$



$$= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}$$



$$= W_1T_0^{-1},$$

and note that $T_{w_A}^{-1}T_{k-1}T_{w_A} = T_1$. Then

$$\left(T_{0^\vee} - t_k^{\frac{1}{2}}\right)\left(T_{0^\vee} + t_k^{-\frac{1}{2}}\right) = 0 \quad \text{and} \quad T_{0^\vee}T_1T_{0^\vee}T_1 = T_1T_{0^\vee}T_1T_{0^\vee}.$$

As vector spaces,

$$H_k^{\mathrm{ext}} = \mathbb{C}\left[W_0^{\pm1}, W_1^{\pm1}, \dots W_k^{\pm1}\right] \otimes H_k^{\mathrm{fin}}, \tag{2.21}$$

where H_k^{fin} is the subalgebra of H_k^{ext} generated by $T_0, T_1, \dots T_{k-1}$. The algebra H_k^{fin} is the Iwahori-Hecke algebra of finite type C_k. If $s_0, s_1, \dots s_{k-1}$ are the generators of $\mathcal{W}_0$ as given in (2.3), write $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ for a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, so that

$$\{T_w \mid w \in \mathcal{W}_0\} \qquad \text{is a } \mathbb{C}\text{-basis of } H_k^{\mathrm{fin}}.$$

Thus (2.21) means that any element $h \in H_k^{\mathrm{ext}}$ can be written uniquely as

$$h = \sum_{w \in \mathcal{W}_0} h_w T_w, \qquad \text{with} \quad h_w \in \mathbb{C}\left[W_0^{\pm1}, W_1^{\pm1}, \dots W_k^{\pm1}\right].$$

Let

$$W^\lambda = W_0^{\lambda_0}W_1^{\lambda_1}W_2^{\lambda_2}\cdots W_k^{\lambda_k} \qquad \text{for } \lambda = (\lambda_0, \lambda_1, \dots \lambda_k) \in \mathbb{Z}^{k+1}. \tag{2.22}$$

Relations (C1) and (C2) produce an action of $\mathcal{W}_0$ on

$$\mathbb{C}\left[W_0^{\pm1}, W_1^{\pm1}, \dots W_k^{\pm1}\right] = \operatorname{span}_{\mathbb{C}}\left\{W^\lambda \;\middle|\; \lambda = (\lambda_0, \lambda_1, \dots \lambda_k) \in \mathbb{Z}^{k+1}\right\}.$$

Namely, for $w \in \mathcal{W}_0$ and $\lambda \in \mathbb{Z}^{k+1}$,

$$wW^\lambda = W^{w\lambda}, \qquad \text{where} \qquad s_0\lambda = s_0(\lambda_0, \lambda_1, \dots \lambda_k) = (\lambda_0, -\lambda_1, \dots \lambda_k), \quad \text{and}$$
$$s_i\lambda = s_i(\lambda_0, \lambda_1, \dots \lambda_k) = (\lambda_0, \lambda_1, \dots \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \dots \lambda_k), \tag{2.23}$$

for $i = 1, 2, \dots k-1$ (see [22, (1.12)]). With this notation, for $\lambda \in \mathbb{Z}^{k+1}$, the relations (C1) and (C2) give

$$T_iW^\lambda = W^{s_i\lambda}T_i + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\frac{W^\lambda - W^{s_i\lambda}}{1 - W_iW_{i+1}^{-1}} \tag{2.24}$$

and

$$T_0W^\lambda = W^{s_0\lambda}T_0 + \left(\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right) + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right)W_1^{-1}\right)\frac{W^\lambda - W^{s_0\lambda}}{1 - W_1^{-2}}, \tag{2.25}$$

and, replacing $s_i\lambda$ by μ,

$$W^\mu T_i = T_iW^{s_i\mu} + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\frac{W^\mu - W^{s_i\mu}}{1 - W_iW_{i+1}^{-1}} \tag{2.26}$$

and

$$W^\mu T_0 = T_0W^{s_0\mu} + \left(\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right) + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right)W_1^{-1}\right)\frac{W^\mu - W^{s_0\mu}}{1 - W_1^{-2}}, \qquad \text{for } \mu \in \mathbb{Z}^{k+1}. \tag{2.27}$$

The subalgebra $H_k \subseteq H_k^{\mathrm{ext}}$ generated by $W_1, \dots, W_k$ and $T_0, \dots T_{k-1}$ is the affine Hecke algebra of type C considered, for example, in [16]. The following theorem determines the center of H_k^{ext} and shows that, as algebras, H_k^{ext} is a tensor product of H_k by the algebra of Laurent polynomials in one variable. It follows that the irreducible representations of H_k^{ext} are indexed by $\mathbb{C}^\times \times \widehat{H}_k$, where $\widehat{H}_k$ is an indexing set for the irreducible representations of H_k. This indexing set will be used in Theorem 3.5 (a) and Theorem 5.5.

Theorem 2.4. *Let H_k be the subalgebra of H_k^{ext} generated by $W_1, \dots W_k$ and $T_0, \dots, T_{k-1}$. As algebras,*

$$H_k^{\mathrm{ext}} \cong \mathbb{C}\left[W_0^{\pm 1}\right] \otimes H_k, \tag{2.28}$$

The center of H_k^{ext} is

$$Z(H_k^{\mathrm{ext}}) = \mathbb{C}\left[W_0^{\pm 1}\right] \otimes \mathbb{C}\left[W_1^{\pm 1}, \dots W_k^{\pm 1}\right]^{\mathcal{W}_0},$$

and H_k^{ext} is a free module of rank $\mathrm{Card}(\mathcal{W}_0)^2 = 2^{2k}(k!)^2$ over $Z(H_k^{\mathrm{ext}})$.

Proof. As observed in (c0), Z_0 is central in $\mathcal{B}_k^{\mathrm{ext}}$, and so $W_0 = (-1)^k(a_1a_2b_1b_2)^{k/2}Z_0$ is central in H_k^{ext}. Thus

$$H_k^{\mathrm{ext}} = \mathbb{C}\left[W_0^{\pm 1}\right] \otimes H_k.$$

By the formulas (2.23), the Laurent polynomial ring $\mathbb{C}[W_1^{\pm 1}, \dots W_k^{\pm 1}]$ is a $\mathcal{W}_0$-submodule of $\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots W_k^{\pm 1}]$, and

$$\mathbb{C}\left[W_0^{\pm 1}, W_1^{\pm 1}, \dots W_k^{\pm 1}\right]^{\mathcal{W}_0} = \mathbb{C}\left[W_1^{\pm 1}, \dots W_k^{\pm 1}\right]^{\mathcal{W}_0} \otimes \mathbb{C}\left[W_0^{\pm 1}\right]. \tag{2.29}$$

The proof that $Z(H_k^{\mathrm{ext}}) = \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots W_k^{\pm 1}]^{\mathcal{W}_0}$ is exactly as in [24, Theorem 4.12]. The fact that H_k^{ext} is a free module of rank $\mathrm{Card}(\mathcal{W}_0)^2$ over $\mathbb{C}[\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots W_k^{\pm 1}]^{\mathcal{W}_0}]$ follows from (2.21) and [22, Theorem 1.17]. ☐

2.3. **Weights of representations and intertwiners.** Let $t^{\frac{1}{2}} \in \mathbb{C}^\times$ be such that $(t^{\frac{1}{2}})^\ell \neq 1$ for $\ell \in \mathbb{Z}$. (This restriction allows us to avoid the combinatorics of periodic configurations that would be necessary to handle the case when $t^{\frac{1}{2}}$ is a root of unity, see [22, § 7]). All irreducible complex representations γ of the algebra $\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}]$ are one-dimensional. Identify the sets

$$\begin{aligned}
\mathcal{C} = &\left\{\text{irreducible representations } \gamma \text{ of } \mathbb{C}\left[W_0^{\pm 1}, W_1^{\pm 1}, \dots W_k^{\pm 1}\right]\right\} \\
&\leftrightarrow \quad \left\{\text{sequences} \quad (z, \gamma_1, \dots, \gamma_k) \in (\mathbb{C}^\times)^{k+1}\right\} \\
&\rightarrow \quad \left\{\text{sequences} \quad (\zeta, c_1, \dots c_k) \in \mathbb{C}^{k+1}\right\}
\end{aligned} \tag{2.30}$$

via

$$\gamma(W_0) = z = (-1)^k t^\zeta \qquad \text{and} \qquad \gamma(W_i) = \gamma_i = -t^{c_i} \text{ for } i = 1, \dots k \tag{2.31}$$

(the choice of sign in the last equation is an artifact of equations (5.32) and (5.33) and an effort to make the combinatorics of contents of boxes Section 5 optimally helpful). It should be noted that the last arrow in (2.30) involves a choice of branch of the logarithm. The action of $\mathcal{W}_0$ from (2.23) induces an action of $\mathcal{W}_0$ on $\mathcal{C}$ by

$$(w\gamma)(W^\lambda) = \gamma\left(W^{w^{-1}\lambda}\right), \qquad \text{for } w \in \mathcal{W}_0 \text{ and } \lambda \in \mathbb{Z}^{k+1}. \tag{2.32}$$

Letting $c_{-i} = -c_i$, the action of $\mathcal{W}_0$ can be given, equivalently, on sequences $(\zeta, c_1, \dots, c_k)$, by

$$w(\zeta, c_1, \dots c_k) = \left(\zeta, c_{w^{-1}(1)}, \dots c_{w^{-1}(k)}\right), \qquad \text{for } w \in \mathcal{W}_0. \tag{2.33}$$

Let $\widetilde{H}_k^{\mathrm{ext}}$ be the extensions of H_k^{ext} by the rational functions in $W_1, \dots W_k$:

$$\widetilde{H}_k^{\mathrm{ext}} = \mathbb{C}\left[W_0^{\pm 1}\right] \otimes \mathbb{C}(W_1, \dots W_k) \otimes H_k^{\mathrm{fin}},$$

where H_k^{fin} is the subalgebra of H_k^{ext} generated by $T_0, T_1, \dots T_{k-1}$. The *intertwining operators* for $\widetilde{H}_k^{\mathrm{ext}}$ are

$$\tau_0 = T_0 - \frac{\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right) + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right) W_1^{-1}}{1 - W_1^{-2}} \qquad \text{and} \qquad \tau_i = T_i - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - W_i W_{i+1}^{-1}} \tag{2.34}$$

for $i = 1, 2, \dots k-1$. Proposition 2.5 shows that these operators satisfy $\tau_0 W^\lambda = W^{s_0\lambda}\tau_0$ and $\tau_i W^\lambda = W^{s_i\lambda}\tau_i$ so that, for $w \in \mathcal{W}_0$ and $\lambda = (\lambda_0, \dots \lambda_k) \in \mathbb{Z}^{k+1}$,

$$\tau_w W^\lambda = W^{w\lambda}\tau_w, \qquad \text{where } \tau_w = \tau_{i_1} \dots \tau_{i_\ell} \tag{2.35}$$

for a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$.

Each H_k^{ext}-module M can be written as $M = \bigoplus_{\gamma \in \mathcal{C}} M_\gamma^{\mathrm{gen}}$, where for $\gamma = (z, \gamma_1, \dots \gamma_k) \in \mathcal{C}$,

$$M_\gamma^{\mathrm{gen}} = \left\{ m \in M \;\middle|\; \begin{matrix} \text{there exists } N \in \mathbb{Z}_{>0} \text{ such that } (W_0 - z)^N m = 0 \\ \text{and } (W_i - \gamma_i)^N m = 0 \text{ for } i = 1, \dots, k \end{matrix} \right\} \tag{2.36}$$

is the *generalized weight space* associated to γ. The intertwiners (2.34) define vector space homomorphisms

$$\tau_0 \colon M_\gamma^{\mathrm{gen}} \longmapsto M_{s_0\gamma}^{\mathrm{gen}} \qquad \text{and} \qquad \tau_i \colon M_\gamma^{\mathrm{gen}} \longmapsto M_{s_i\gamma}^{\mathrm{gen}} \ \text{ for } i = 1, \dots, k-1, \tag{2.37}$$

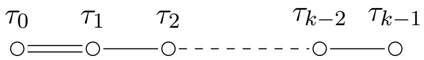
where

τ_0 is defined only when $\gamma_1^2 \neq 1$, so that $\left(1 - W_1^{-2}\right)^{-1}$ is well-defined on M_γ^{gen} and

$\tau_i$is defined only when $\gamma_i \neq \gamma_{i+1}$, so that $\left(1 - W_i W_{i+1}^{-1}\right)^{-1}$ is well-defined on M_γ^{gen}

for $i = 1, \dots k-1$.

Proposition 2.5 (Intertwiner presentation)**.** *The algebra* $\widetilde{H}_k^{\mathrm{ext}}$ *is generated by* W_0 *and* $\tau_0, \dots \tau_k$ *and* $\mathbb{C}(W_1, \dots W_k)$ *with relations*



(2.38)

in the notation of (2.1)*;*

$$\tau_0 W_1 = W_1^{-1}\tau_0 \qquad \textit{and} \qquad \tau_0 W_j = W_j \tau_0 \ \textit{for } j \neq 1; \tag{2.39}$$

for $i = 1, \dots k-1$,

$$\begin{aligned} \tau_i W_i = W_{i+1}\tau_i \quad \textit{and} \quad \tau_i W_{i+1} &= W_i \tau_i \quad \textit{for } i > 0, \\ \textit{and} \quad \tau_i W_j &= W_j \tau_i \quad \textit{for } j \neq i, i+1; \end{aligned} \tag{2.40}$$

$$\tau_0^2 = \frac{\left(1 - t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} W_1^{-1}\right)}{1 - W_1^{-1}} \frac{\left(1 + t_0^{\frac{1}{2}} t_k^{-\frac{1}{2}} W_1^{-1}\right)}{1 + W_1^{-1}} \frac{\left(1 + t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}} W_1^{-1}\right)}{1 + W_1^{-1}} \frac{\left(1 - t_0^{-\frac{1}{2}} t_k^{-\frac{1}{2}} W_1^{-1}\right)}{1 - W_1^{-1}}; \tag{2.41}$$

and

$$\tau_i^2 = \frac{\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}} W_i^{-1} W_{i+1}\right)\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}} W_{i+1}^{-1} W_i\right)}{\left(1 - W_i^{-1} W_{i+1}\right)\left(1 - W_{i+1}^{-1} W_i\right)} \quad \textit{for } i = 1, \dots k-1. \tag{2.42}$$

Proof. The proof of the relations in (2.38) is accomplished exactly as in the proof of [22, Proposition 2.14 (e)]; relation (2.42) is [22, Proposition 2.14 (c)]. Let us check the relations in (2.40) and (2.41).

Using (C1),

$$\begin{aligned}\tau_i W_i &= \left(T_i - \frac{t^{\frac12}-t^{-\frac12}}{1-W_iW_{i+1}^{-1}}\right)W_i\\ &= W_{i+1}T_i + \left(t^{\frac12}-t^{-\frac12}\right)\frac{W_i-W_{i+1}}{1-W_iW_{i+1}^{-1}} - \left(t^{\frac12}-t^{-\frac12}\right)\frac{W_i}{1-W_iW_{i+1}^{-1}}\\ &= W_{i+1}\left(T_i - \frac{t^{\frac12}-t^{-\frac12}}{1-W_iW_{i+1}^{-1}}\right) = W_{i+1}\tau_i.\end{aligned}$$

Similarly, using (C2),

$$\begin{aligned}\tau_0 W_1 &= \left(T_0 - \frac{\left(t_0^{\frac12}-t_0^{-\frac12}\right)+\left(t_k^{\frac12}-t_k^{-\frac12}\right)W_1^{-1}}{1-W_1^{-2}}\right)W_1\\ &= W_1^{-1}T_0 + \left(t_0^{\frac12}-t_0^{-\frac12}\right)W_1 + \left(t_k^{\frac12}-t_k^{-\frac12}\right) - \frac{\left(t_0^{\frac12}-t_0^{-\frac12}\right)W_1+\left(t_k^{\frac12}-t_k^{-\frac12}\right)}{1-W_1^{-2}}\\ &= W_1^{-1}\left(T_0 - \frac{\left(t_0^{\frac12}-t_0^{-\frac12}\right)+\left(t_k^{\frac12}-t_k^{-\frac12}\right)W_1^{-1}}{1-W_1^{-2}}\right) = W_1^{-1}\tau_0.\end{aligned}$$

For $i = 0, \ldots, k-1$ and $j \neq i, i+1$, τ_i and W_j commute by the second set of relations in (C1). These computations establish the relations in (2.39) and (2.40).

By the first relation in (H), $T_0 = T_0^{-1} + (t_0^{\frac12} - t_0^{-\frac12})$, so that

$$\begin{aligned}\tau_0 &= T_0 - \frac{\left(t_0^{\frac12}-t_0^{-\frac12}\right)+\left(t_k^{\frac12}-t_k^{-\frac12}\right)W_1^{-1}}{1-W_1^{-2}}\\ &= T_0^{-1} + \left(t_0^{\frac12}-t_0^{-\frac12}\right) + \frac{\left(t_0^{\frac12}-t_0^{-\frac12}\right)W_1^2+\left(t_k^{\frac12}-t_k^{-\frac12}\right)W_1}{1-W_1^2}\\ &= T_0^{-1} + \frac{\left(t_0^{\frac12}-t_0^{-\frac12}\right)+\left(t_k^{\frac12}-t_k^{-\frac12}\right)W_1}{1-W_1^2}.\end{aligned}$$

Then

$$\begin{aligned}\tau_0^2 &= \tau_0\left(T_0 - \frac{\left(t_0^{\frac12}-t_0^{-\frac12}\right)+\left(t_k^{\frac12}-t_k^{-\frac12}\right)W_1^{-1}}{1-W_1^{-2}}\right)\\ &= \tau_0T_0 - \left(\frac{\left(t_0^{\frac12}-t_0^{-\frac12}\right)+\left(t_k^{\frac12}-t_k^{-\frac12}\right)W_1}{1-W_1^2}\right)\tau_0\end{aligned}$$

$$= \left(T_0^{-1} + \frac{\left(t_0^{\frac12} - t_0^{-\frac12}\right) + \left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1}{1 - W_1^2} \right) T_0$$

$$- \left(\frac{\left(t_0^{\frac12} - t_0^{-\frac12}\right) + \left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1}{1 - W_1^2} \right) \left(T_0 - \frac{\left(t_0^{\frac12} - t_0^{-\frac12}\right) + \left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1^{-1}}{1 - W_1^{-2}} \right)$$

$$= 1 + \left(\frac{\left(t_0^{\frac12} - t_0^{-\frac12}\right) + \left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1}{1 - W_1^2} \right) \left(\frac{\left(t_0^{\frac12} - t_0^{-\frac12}\right) + \left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1^{-1}}{1 - W_1^{-2}} \right)$$

$$= 1 - \left(\frac{\left(t_0^{\frac12} - t_0^{-\frac12}\right) W_1^{-2} + \left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1^{-1}}{1 - W_1^{-2}} \right) \left(\frac{\left(t_0^{\frac12} - t_0^{-\frac12}\right) + \left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1^{-1}}{1 - W_1^{-2}} \right)$$

$$= \frac{\begin{pmatrix} 1 - 2W_1^{-2} + W_1^{-4} - \left(\left(t_0^{\frac12} - t_0^{-\frac12}\right)^2 + \left(t_k^{\frac12} - t_k^{-\frac12}\right)^2 \right) W_1^{-2} \\ - \left(t_0^{\frac12} - t_0^{-\frac12}\right)\left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1^{-1} - \left(t_0^{\frac12} - t_0^{-\frac12}\right)\left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1^{-3} \end{pmatrix}}{\left(1 - W_1^{-2}\right)^2}$$

so that

$$\tau_0^2 = \frac{\left(1 - t_0^{\frac12} t_k^{\frac12} W_1^{-1}\right)}{1 + W_1^{-1}} \frac{\left(1 + t_0^{\frac12} t_k^{-\frac12} W_1^{-1}\right)}{1 - W_1^{-1}} \frac{\left(1 + t_0^{-\frac12} t_k^{\frac12} W_1^{-1}\right)}{1 - W_1^{-1}} \frac{\left(1 - t_0^{-\frac12} t_k^{-\frac12} W_1^{-1}\right)}{1 + W_1^{-1}},$$

establishing (2.41). □

3. Calibrated representations of H_k^{ext}

A *calibrated* H_k^{ext}*-module* is an H_k^{ext}-module M such that $W_0, W_1, \dots W_k$ are simultaneously diagonalizable as operators on M. In the context of (2.36), M is calibrated if

$$M = \bigoplus_{\gamma \in \mathcal{C}} M_\gamma, \quad \text{where}$$

$$M_\gamma = \{m \in M \mid W_0 m = zm \quad \text{and} \quad W_i m = \gamma_i m \text{ for } i = 1, \dots k\} \tag{3.1}$$

for $\gamma = (z, \gamma_1, \dots \gamma_k) \in \mathcal{C}$. Another formulation is that M is calibrated if M has a basis of simultaneous eigenvectors for $W_0, \dots W_k$. This section follows the framework of [22] in developing combinatorial tools for describing the structure and the classification of irreducible calibrated H_k^{ext}-modules. In Section 5 we will use this combinatorics to analyze and classify the H_k^{ext}-modules arising in the Schur–Weyl duality settings.

With notations as in the definition of $\mathcal{W}_0$ in (2.2), the *reflection representation* of $\mathcal{W}_0$ is the action of $\mathcal{W}_0$ on $\mathfrak{h}_{\mathbb{R}} = \mathbb{R}^k$ given by

$$w(c_1, \dots c_k) = \left(c_{w^{-1}(1)}, \dots c_{w^{-1}(k)}\right), \qquad \text{where } c_{-i} = -c_i \text{ for } i = 1, 2, \dots k.$$

The dual space $\mathfrak{h}_\mathbb{R}^*$ has basis $\varepsilon_1, \dots, \varepsilon_k$, where $\varepsilon_i \colon \mathfrak{h}_\mathbb{R} \to \mathbb{R}$ is the $\mathbb{R}$-linear map given by $\varepsilon_i(\gamma_1, \dots \gamma_k) = \gamma_i$. With $\varepsilon_{-i} = -\varepsilon_i$, the action of $\mathcal{W}_0$ on $\mathbb{R}^k$ produces an action on $\mathfrak{h}_\mathbb{R}^*$ given by $w\varepsilon_i = \varepsilon_{w^{-1}(i)}$.

Let

$$\begin{aligned} R^+ &= \{\varepsilon_1, \dots \varepsilon_k\} \sqcup \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \mid 1 \le i < j \le k\} \\ &= \{\varepsilon_1, \dots \varepsilon_k\} \sqcup \{\varepsilon_j - \varepsilon_i \mid 1 \le i < j \le k\} \sqcup \{\varepsilon_j - \varepsilon_{-i} \mid 1 \le i < j \le k\} \\ &= \{\varepsilon_1, \dots \varepsilon_k\} \sqcup \{\varepsilon_j - \varepsilon_i \mid i, j \in \{-k, \dots, -1, 1, \dots k\}, i < j, i \neq -j\}. \end{aligned}$$

If $w \in \mathcal{W}_0$, the *inversion set of* w is

$$R(w) = \left\{\alpha \in R^+ \,\middle|\, w\alpha \notin R^+\right\} \tag{3.2}$$

$$\begin{aligned} &= \{\varepsilon_i \mid \text{if } i > 0 \text{ and } w(i) < 0\} \sqcup \{\varepsilon_j - \varepsilon_i \mid \text{if } 0 < i < j \text{ and } w(i) > w(j)\} \\ &\qquad \sqcup \{\varepsilon_j + \varepsilon_i \mid \text{if } 0 < i < j \text{ and} - w(i) > w(j)\}. \end{aligned} \tag{3.3}$$

The chambers are the connected components of

$$\mathfrak{h}_\mathbb{R} \backslash \bigcup_{\alpha \in R^+} \mathfrak{h}^\alpha, \qquad \text{where} \quad \mathfrak{h}^\alpha = \{\gamma \in \mathfrak{h}_\mathbb{R} \mid \alpha(\gamma) = 0\}.$$

The fundamental chamber in $\mathfrak{h}_\mathbb{R}$ is

$$C = \left\{\mathbf{c} \in \mathfrak{h}_\mathbb{R} \,\middle|\, \alpha(\gamma) \in \mathbb{R}_{>0} \text{ for } \alpha \in R^+\right\} = \left\{(c_1, \dots, c_k) \in \mathbb{R}^k \,\middle|\, 0 < c_1 < c_2 < \cdots < c_k\right\},$$

and the group $\mathcal{W}_0$ can be identified with the set of chambers via the bijection

$$\begin{matrix} \mathcal{W}_0 & \longleftrightarrow & \{\text{chambers}\} \\ w & \longmapsto & w^{-1}C \end{matrix}. \qquad \text{Since} \quad w^{-1}C = \left\{\mathbf{c} \in \mathfrak{h}_\mathbb{R} \,\middle|\, \begin{matrix} \alpha(\mathbf{c}) \in \mathbb{R}_{<0} \text{ if } \alpha \in R(w) \text{ and} \\ \alpha(\mathbf{c}) \in \mathbb{R}_{>0} \text{ if } \alpha \in R^+ \backslash R(w) \end{matrix}\right\},$$

the set $R(w)$ determines w.

3.1. **Local regions.** For $\gamma = (\gamma_1, \dots \gamma_k) \in (\mathbb{C}^\times)^k$, define

$$\begin{aligned} Z(\gamma) &= \{\varepsilon_i \mid \gamma_i = \pm 1\} \sqcup \left\{\varepsilon_j - \varepsilon_i \,\middle|\, 0 < i < j,\, \gamma_i\gamma_j^{-1} = 1\right\} \\ &\qquad \sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j,\, \gamma_i\gamma_j = 1\}, \end{aligned}$$

$$\begin{aligned} P(\gamma) &= \left\{\varepsilon_i \,\middle|\, \gamma_i \in \left\{\left(t_0^{\frac{1}{2}}t_k^{\frac{1}{2}}\right)^{\pm 1}, \left(-t_0^{-\frac{1}{2}}t_k^{\frac{1}{2}}\right)^{\pm 1}\right\}\right\} \\ &\qquad \sqcup \left\{\varepsilon_j - \varepsilon_i \,\middle|\, 0 < i < j, \gamma_i\gamma_j^{-1} = t^{\pm 1}\right\} \\ &\qquad \sqcup \left\{\varepsilon_j + \varepsilon_i \,\middle|\, 0 < i < j, \gamma_i\gamma_j = t^{\pm 1}\right\}. \end{aligned} \tag{3.4}$$

These sets keep track of the zeros and poles of the right hand side of (2.42)and (2.41) when $\tau_0^2, \tau_1^2, \dots, \tau_n^2$ are acting on M_γ^{gen}. Using the conversion from γ_i to c_i as in (2.31), let

$$\gamma_i = -t^{c_i}, \quad \text{and set} \quad -t^{r_1} = -t_k^{\frac{1}{2}}t_0^{-\frac{1}{2}} \quad \text{and} \quad -t^{r_2} = t_k^{\frac{1}{2}}t_0^{\frac{1}{2}}, \tag{3.5}$$

so that $-t^{\pm r_1}$ and $-t^{\pm r_2}$ are the eigenvalues of W_1 that cause τ_0^2 to have a nonzero kernel (see (2.41)). Then, for $\mathbf{c} = (c_1, \dots c_k) \in \mathbb{C}^k$ let $c_{-i} = -c_i$ and define

$$\begin{aligned} Z(\mathbf{c}) &= \{\varepsilon_i \mid c_i = 0\} \sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = 0\} \\ &\qquad \sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } c_j + c_i = 0\}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} P(\mathbf{c}) &= \{\varepsilon_i \mid c_i \in \{\pm r_1, \pm r_2\}\} \sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = \pm 1\} \\ &\qquad \sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } c_j + c_i = \pm 1\}. \end{aligned} \tag{3.7}$$

A *local region* is a pair $(\mathbf{c}, J)$ with $\mathbf{c} \in \mathbb{C}^k$ and $J \subseteq P(\mathbf{c})$. The set of *standard tableaux* of shape $(\mathbf{c}, J)$ is

$$\mathcal{F}^{(\mathbf{c},J)} = \{w \in \mathcal{W}_0 \mid R(w) \cap Z(\mathbf{c}) = \emptyset,\ R(w) \cap P(\mathbf{c}) = J\}. \tag{3.8}$$

As in [22, § 5 and § 8] the local regions $(\mathbf{c}, J)$ and standard tableaux $w \in \mathcal{F}^{(\mathbf{c},J)}$ can be converted to configurations of boxes κ and standard tableaux S of shape κ similar to those that are familiar in the literature on irreducible representations of Weyl groups of classical types. As explained in [22, § 5.11], the definitions of $Z(\mathbf{c})$ and $P(\mathbf{c})$ make it possible to view the general case $\mathbf{c} \in \mathbb{C}^k$ as pieced together from the cases $\mathbf{c} \in (\mathbb{Z} + \beta)^k$ where β runs over a set of representatives of the $\mathbb{Z}$-cosets in $\mathbb{C}$. Below we make the conversion between local regions and configurations of boxes explicit for the cases when $\mathbf{c} \in \mathbb{Z}^k$ and $\mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k$. These are the cases that appear in the Schur–Weyl duality approach to the representations of H_k^{ext} explored in Section 5. As in [22, § 8], it is also true that these cases are sufficient to determine the general $\mathbf{c} \in (\mathbb{Z} + \beta)^k$ setting.

Let $(\mathbf{c}, J)$ be a local region with $\mathbf{c} = (c_1, \dots c_k)$,

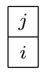
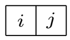
$$\mathbf{c} \in \mathbb{Z}^k \quad \text{or} \quad \mathbf{c} \in \left(\mathbb{Z} + \frac{1}{2}\right)^k, \qquad \text{and} \qquad 0 \le c_1 \le \cdots \le c_k. \tag{3.9}$$

Start with an infinite arrangement of NW to SE diagonals, numbered consecutively from $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, increasing southwest to northeast (see Example 3.1). The *configuration* κ of boxes corresponding to the local region $(\mathbf{c}, J)$ has $2k$ boxes (labeled $\mathrm{box}_{-k}, \dots \mathrm{box}_{-1}$, $\mathrm{box}_1, \dots \mathrm{box}_k$) with the following conditions.

(κ1) Location: box_i is on diagonal c_i, where $c_{-i} = -c_i$ for $i \in \{-k, \dots -1\}$.

(κ2) Same diagonals: box_i is NW of box_j if $i < j$ and box_i and box_j are on the same diagonal.

(κ3) Adjacent diagonals:

- If $\varepsilon_j - \varepsilon_i \in J$, then box_j is NW (strictly north and weakly west) of box_i: 
- If $\varepsilon_j - \varepsilon_i \in P(\mathbf{c}) - J$, then box_j is SE (weakly south and strictly east) of box_i: 

(κ4) Markings: There is a marking on each of the diagonals r_1, $-r_1$, r_2 and $-r_2$.

- If $\varepsilon_i \in J$, box_i is NW of the marking on diagonal c_i: 
- If $\varepsilon_i \in P(\mathbf{c}) - J$, then box_i is SE of the marking in diagonal c_i : 

Condition (κ1) enables the values $(c_{-k}, \dots, c_{-1}, c_1, \dots, c_k)$ to be read off of configuration κ. The sets $Z(\mathbf{c})$, $P(\mathbf{c})$, and J can also be determined from the configuration κ since

$$\begin{aligned} Z(\mathbf{c}) = \{\varepsilon_i \mid 0 < i \text{ and } \mathrm{box}_i \text{ is in diagonal } 0\} \\ \sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } \mathrm{box}_i \text{ and } \mathrm{box}_j \text{ are in the same diagonal}\} \\ \sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } \mathrm{box}_{-i} \text{ and } \mathrm{box}_j \text{ are in the same diagonal}\}, \end{aligned}$$

$$\begin{aligned} P(\mathbf{c}) = \{\varepsilon_i \mid 0 < i \text{ and } \mathrm{box}_i \text{ is in diagonal } r_1 \text{ or } r_2\}, \\ \sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } \mathrm{box}_i \text{ and } \mathrm{box}_j \text{ are in adjacent diagonals}\} \\ \sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } \mathrm{box}_{-i} \text{ and } \mathrm{box}_j \text{ are in adjacent diagonals}\}, \end{aligned}$$

and

$$J = \{\varepsilon_i \in P(\mathbf{c}) \mid \text{box}_i \text{ is NW of the marking}\}$$
$$\sqcup \{\varepsilon_j - \varepsilon_i \in P(\mathbf{c}) \mid \text{box}_j \text{ is northwest of box}_i\}$$
$$\sqcup \{\varepsilon_j + \varepsilon_i \in P(\mathbf{c}) \mid \text{box}_j \text{ is northwest of box}_{-i}\}\,.$$

A *standard filling* of the boxes of κ is a bijective function $S\colon \kappa \to \{-k, \dots -1, 1, \dots k\}$ such that

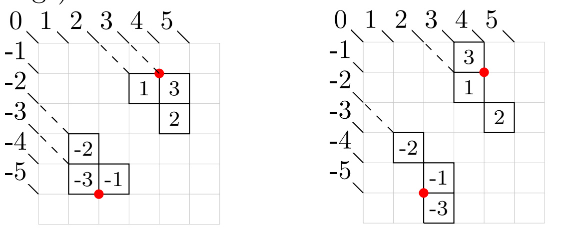
(S1) Symmetry: $S(\text{box}_{-i}) = -S(\text{box}_i)$.
(S2) Same diagonals:
If $0 < i < j$ and box_i and box_j are on the same diagonal then $S(\text{box}_i) < S(\text{box}_j)$.
(S3) Adjacent diagonals:
If $0 < i < j$, box_i and box_j are on adjacent diagonals, and box_j is NW of box_i, then $S(\text{box}_j) < S(\text{box}_i)$.
If $0 < i < j$, box_i and box_j are on adjacent diagonals, and box_j is SE of box_i, then $S(\text{box}_j) > S(\text{box}_i)$.
(S4) Markings:
If box_i is on a marked diagonal and is SE of the marking, then $S(\text{box}_i) > 0$.
If box_i is on a marked diagonal and is NW of the marking, then $S(\text{box}_i) < 0$.

The *identity filling* of a configuration κ is the filling F of the boxes of κ given by $F(\text{box}_i) = i$, for $i = -k, \dots, -1, 1, \dots k$. The identity filling of κ is usually not a standard filling of κ (see Example 3.1).

Example 3.1. Let $k = 4$, $r_1 = 1$, and $r_2 = 3$. Consider $\mathbf{c} = (-3, -2, -2, 2, 2, 3)$. Then

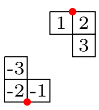
$$Z(\mathbf{c}) = \{\varepsilon_2 - \varepsilon_1\} \qquad \text{and} \qquad P(\mathbf{c}) = \{\varepsilon_3,\ \varepsilon_3 - \varepsilon_1,\ \varepsilon_3 - \varepsilon_2\}\,.$$

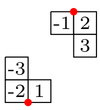
The box configurations corresponding to $J = \{\varepsilon_3 - \varepsilon_2\}$ and $J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$ (filled with their identity fillings) are

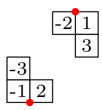


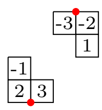
$J = \{\varepsilon_3 - \varepsilon_2\}$ $\qquad$ $J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$

For both configurations, the identity filling is not a standard filling. Examples of standard fillings of the configuration corresponding to $J = \{\varepsilon_3 - \varepsilon_2\}$ include



,


, and


, but not


.

The proof of the following proposition is a straightforward, though slightly tedious, check that the conditions $R(w) \cap Z(\mathbf{c}) = \emptyset$ and $R(w) \cap P(\mathbf{c}) = J$ from (3.8) convert to the conditions (S2), (S3), (S4) on standard fillings of shape κ. The proof is similar to the proof of [22, Theorem 5.9].

Proposition 3.2. *Let κ be a configuration of boxes corresponding to a local region $(\mathbf{c}, J)$ with $\mathbf{c} \in \mathbb{Z}^k$ or $\mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k$. For $w \in \mathcal{W}_0$ let S_w be the filling of the boxes of κ given by*

$$S_w(\text{box}_i) = w(i), \quad \textit{for } i = -k, \dots -1, 1, \dots k.$$

The map

$$\begin{array}{rcl} \mathcal{F}^{(\mathbf{c},J)} & \longrightarrow & \{\textit{standard fillings } S \textit{ of the boxes of } \kappa\} \\ w & \longmapsto & S_w \end{array} \qquad \textit{is a bijection.}$$

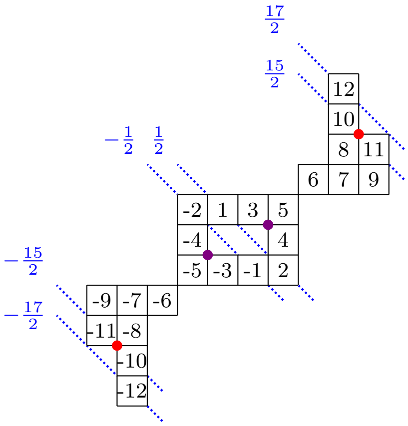
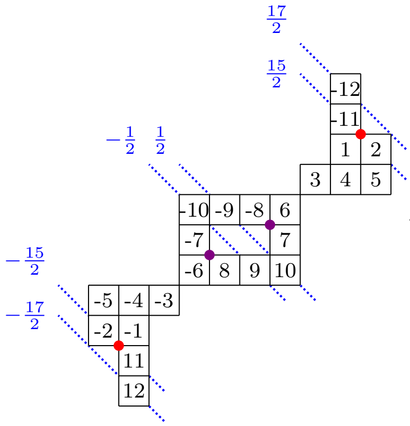
Example 3.3. Let $k = 12$, $r_1 = \frac{3}{2}$, $r_2 = \frac{15}{2}$, $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{13}{2}, \frac{15}{2}, \frac{15}{2}, \frac{17}{2})$ and

$$J = \left\{ \begin{array}{l} \varepsilon_3, \varepsilon_{10}, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \varepsilon_8 - \varepsilon_7, \\ \varepsilon_{10} - \varepsilon_8, \varepsilon_{10} - \varepsilon_9, \varepsilon_{11} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{10}, \varepsilon_{12} - \varepsilon_{11} \end{array} \right\}$$

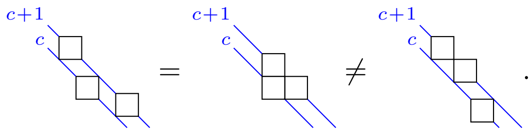
Let

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ -9 & 10 & -8 & 7 & 6 & 3 & 4 & 1 & 5 & -11 & 2 & -12 \end{pmatrix} \in \mathcal{F}^{(\mathbf{c},J)}.$$

Then, for the corresponding configuration of boxes κ, the identity filling F, and the standard filling S_w corresponding to w are

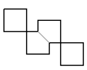
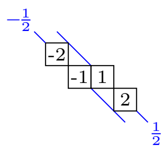
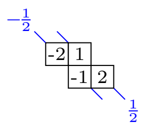
$F =$  and $S_w =$  .

Remark 3.4. Borrowing a physical intuition, configurations are invariant under sliding boxes along diagonals like beads on an abacus, so long as boxes that run into each other are not allowed to exchange places, i.e. for most $c \in \mathbb{Z}$,

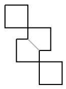
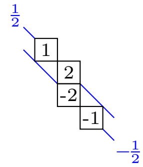


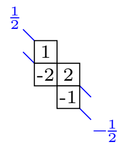
Then by arranging configurations so that the boxes are packed together, standard fillings of configurations are exactly analogous to standard tableaux for partitions.

The only exception to this physical intuition is for boxes on the diagonals $\pm\frac{1}{2}$. Note that if $c_i = \frac{1}{2}$, then box_i and box_{-i} are on adjacent diagonals. However, since $2\varepsilon_i = \varepsilon_i - \varepsilon_{-i} \notin R^+$ and therefore never in $P(\mathbf{c})$, the relative positions of box_i and box_{-i} will never be recorded in the set J. For example, in Figure 4.2, the point where $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$ has two configurations, each with two boxes overlapping in indication that box_i and box_{-i} may "slide past each other". The drawing

 represents the equivalence of  and  ,

(with boxes filled in the identity filling) where box_1 and box_{-1} can move freely past each other, and

 represents the equivalence of 

and 

,

where box_2 and box_{-2} can move freely past each other. In these two examples $\varepsilon_1 - \varepsilon_{-2} \in P(\mathbf{c})$ and $\varepsilon_2 - \varepsilon_{-1} \in P(\mathbf{c})$ and so the relative orientation of box_2 and box_{-1} and the relative orientation of box_1 and box_{-2} are recorded in J. Each configuration has exactly two standard fillings.

3.2. **Classifying and constructing calibrated representations.** Theorem 3.5 below provides an indexing of the calibrated irreducible H_k^{ext}-modules by skew local regions. A *skew local region* is a local region $(\mathbf{c}, J)$, $\mathbf{c} = (c_1, \dots c_k)$, such that if $w \in \mathcal{F}^{(\mathbf{c},J)}$ then $w\mathbf{c} = ((w\mathbf{c})_1, \dots (w\mathbf{c})_n)$ satisfies

$$\begin{aligned} &(w\mathbf{c})_1 \neq 0, \quad (w\mathbf{c})_2 \neq 0, \quad (w\mathbf{c})_1 \neq -(w\mathbf{c})_2, \\ &(w\mathbf{c})_i \neq (w\mathbf{c})_{i+1} \quad \text{for } i = 1, \dots k-1, \\ \text{and} \quad &(w\mathbf{c})_i \neq (w\mathbf{c})_{i+2} \quad \text{for } = 1, \dots k-2. \end{aligned} \tag{3.10}$$

Theorem 3.5 is completely analogous to the same theorem for the case $t^{\frac12} = t_0^{\frac12} = t_k^{\frac12}$ in [22, Theorem 3.5]. As explained in the discussion and remarks before [22, Lemma 3.1] in [22, § 3], getting exactly the right definition of skew local region for the purpose of Theorem 3.5 is accomplished by a detailed computation of the irreducible representations in rank two cases. More specifically, for $I \subseteq \{0, \dots, k\}$, let H_I be the subalgebra of H_k^{ext} generated by $\{T_i\}_{i \in I}$ and $\mathbb{C}[W_1^{\pm1}, \dots W_k^{\pm1}]$. Then the conditions in (3.10) guarantee that for $w \in \mathcal{F}^{(\mathbf{c},J)}$ and $i, j \in \{0, 1, \dots sk-1\}$,

$$\text{there exists a calibrated } H_{\{i,j\}}\text{-module } M \text{ with } M_{w\mathbf{c}}^{\text{gen}} \neq 0.$$

The cases where $H_{\{i,j\}}$ is of type $A \times A_1$ or of type A_2 are checked in [23]. However, when $H_{\{i,j\}}$ is of type C_2 and there are *three distinct parameters*, we do not know a reference for this. So in the effort to provide a more complete presentation, we have done the appropriate analysis in Section 4 for all generic choices of the three parameters $t^{\frac12}$, $t_0^{\frac12}$, and $t_k^{\frac12}$, as given in the following theorem (see also (4.1)).

Theorem 3.5. *Assume $t^{\frac12}$, $t_0^{\frac12}$, and $t_k^{\frac12}$ are invertible, $t^{\frac12}$ is not a root of unity, and*

$$t_0^{\frac12} t_k^{\frac12}, -t_0^{-\frac12} t_k^{\frac12} \notin \left\{1, -1, t^{\pm\frac12}, -t^{\pm\frac12}, t^{\pm1}, -t^{\pm1}\right\} \quad \text{and} \quad t_0^{\frac12} t_k^{\frac12} \neq \left(-t_0^{-\frac12} t_k^{\frac12}\right)^{\pm1}.$$

(a) *Let $(\mathbf{c}, J)$ be a skew local region and let $z \in \mathbb{C}^\times$. Define*

$$H_k^{(z,\mathbf{c},J)} = \operatorname{span}_{\mathbb{C}} \left\{ v_w \;\middle|\; w \in \mathcal{F}^{(\mathbf{c},J)} \right\}, \tag{3.11}$$

so that the symbols v_w are a labeled basis of the vector space $H_k^{(z,\mathbf{c},J)}$. Let

$$\gamma_i = -t^{c_i} \quad \text{for } i = 1, 2, \dots k, \qquad \text{and} \qquad \gamma_0 = z\gamma_{w^{-1}(1)}^{-1} \cdots \gamma_{w^{-1}(k)}^{-1}.$$

Then the following formulas make $H_k^{(z,\mathbf{c},J)}$ *into an irreducible* H_k^{ext}*-module:*

$$PW_1\cdots W_k v_w = z v_w, \qquad P v_w = \gamma_0 v_w, \qquad W_i v_w = \gamma_{w^{-1}(i)} v_w, \tag{3.12}$$

$$\begin{aligned} T_i v_w &= [T_i]_{ww} v_w + \sqrt{-\left([T_i]_{ww} - t^{\frac{1}{2}}\right)\left([T_i]_{ww} + t^{-\frac{1}{2}}\right)}\; v_{s_i w}, \\ &\text{for} \quad i = 1, \ldots, k-1, \end{aligned} \tag{3.13}$$

$$T_0 v_w = [T_0]_{ww} v_w + \sqrt{-\left([T_0]_{ww} - t_0^{\frac{1}{2}}\right)\left([T_0]_{ww} + t_0^{-\frac{1}{2}}\right)}\; v_{s_0 w}, \tag{3.14}$$

where $v_{s_i w} = 0$ *if* $s_i w \notin \mathcal{F}^{(\mathbf{c},J)}$*, and*

$$\begin{aligned} [T_i]_{ww} &= \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)}\gamma^{-1}_{w^{-1}(i+1)}} \\ \text{and} \quad [T_0]_{ww} &= \frac{\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right) + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right)\gamma^{-1}_{w^{-1}(1)}}{1 - \gamma^{-2}_{w^{-1}(1)}}. \end{aligned} \tag{3.15}$$

(b) The map

$$\begin{array}{ccc} \mathbb{C}^\times \times \{\textit{skew local regions } (\mathbf{c}, J)\} & \longleftrightarrow & \{\textit{irreducible calibrated } H_k^{\mathrm{ext}}\textit{-modules}\} \\ (z, \mathbf{c}, J) & \longmapsto & H_k^{(z,\mathbf{c},J)} \end{array}$$

is a bijection.

Proof. This result follows from [22, Theorems 3.2 and 3.5]. It is only necessary to establish that the formulas in (3.12), (3.13), and (3.14) are correct. These are derived in a similar manner to [22, Proposition 3.3] as follows. As in [22, Theorem 3.2], if M is an irreducible calibrated H_k^{ext}-module then

$$M = \bigoplus_{w \in \mathcal{W}_0} M^{\mathrm{gen}}_{w\gamma}, \qquad \text{with } \dim(M^{\mathrm{gen}}_{w\gamma}) = 1 \text{ if } M^{\mathrm{gen}}_{w\gamma} \neq 0.$$

For $w \in \mathcal{W}_0$, if $M^{\mathrm{gen}}_{w\gamma} \neq 0$, let v_w be a nonzero vector in $M^{\mathrm{gen}}_{w\gamma}$; otherwise if $M^{\mathrm{gen}}_{w\gamma} = 0$, let $v_w = 0$. By (2.37), $\tau_i v_w = [T_i]_{s_i w, w} v_{s_i w}$ for some constant $[T_i]_{s_i w, w}$ and the definition of τ_i in (2.34) gives that

$$T_i v_w = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)}\gamma^{-1}_{w^{-1}(i+1)}} v_w + [T_i]_{s_i w, w} v_{s_i w} \qquad \text{for } i = 1, \ldots k, \tag{3.16}$$

and

$$T_0 v_\gamma = \frac{\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right) + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right)\gamma^{-1}_{w^{-1}(1)}}{1 - \gamma^{-2}_{w^{-1}(1)}} v_w + [T_0]_{s_0 w, w} v_{s_0 w}. \tag{3.17}$$

Thus T_0 is an operator on the subspace $\mathrm{span}_{\mathbb{C}}\{v_w, v_{s_0 w}\}$ satisfying $(T_0 - t_0^{\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}) = 0$ by (H). Restricting to the action on $\mathrm{span}_{\mathbb{C}}\{v_w, v_{s_0 w}\}$, the formulas in (3.14) now follow from the following argument about general 2×2 matrices.

If a 2×2 matrix $[T_0]$ has eigenvalues α_1 and α_2,

$$[T_0] = \begin{pmatrix} [T_0]_{ww} & [T_0]_{w, s_0 w} \\ [T_0]_{s_0 w, w} & [T_0]_{s_0 w, s_0 w} \end{pmatrix}, \qquad \text{then} \qquad ([T_0] - \alpha_1)([T_0] - \alpha_2) = 0$$

is the characteristic polynomial for $[T_0]$, and it follows that

$$\mathrm{Tr}([T_0]) = [T_0]_{ww} + [T_0]_{s_0w,s_0w} = \alpha_1 + \alpha_2, \qquad \text{and}$$
$$\det([T_0]) = [T_0]_{ww}[T_0]_{s_0w,s_0w} - [T_0]_{w,s_0w}[T_0]_{s_0w,w} = \alpha_1\alpha_2.$$

Thus

$$\begin{aligned}-[T_0]_{w,s_0w}[T_0]_{s_0w,w} &= \alpha_1\alpha_2 - [T_0]_{ww}[T_0]_{s_0w,s_0w} = \alpha_1\alpha_2 - [T_0]_{ww}((\alpha_1+\alpha_2) - [T_0]_{ww}) \\ &= \alpha_1\alpha_2 - (\alpha_1+\alpha_2)[T_0]_{ww} + ([T_0]_{ww})^2 = ([T_0]_{ww} - \alpha_1)([T_0]_{ww} - \alpha_2).\end{aligned}$$

Choosing a normalization of v_{s_0w} so that the matrix of $[T_0]$ is symmetric, then $[T_0]_{w,s_0w}$ $= [T_0]_{s_0w,w}$ and

$$[T_0]_{s_0w,w} = \sqrt{([T_0]_{s_0w,w})^2} = \sqrt{[T_0]_{w,s_0w}[T_0]_{s_0w,w}} = \sqrt{-([T_0]_{ww} - \alpha_1)([T_0]_{ww} - \alpha_2)}.$$

□

4. Classification of irreducible representations of H_2

In this section we do a *complete* classification of the irreducible representations of the algebra H_2^{ext}. This classification is summarized in the pictures in Figures 4.1 and 4.2. In particular, a corollary of the classification of irreducible H_2^{ext}-modules in this section is the following.

Corollary 4.1. *An irreducible representation M of H_2^{ext} is calibrated if and only if M satisfies*

$$M_\gamma^{\mathrm{gen}} \neq 0 \quad \textit{if and only if} \quad \gamma = (-t^{c_1}, -t^{c_2}),$$
$$\textit{where } (c_1, c_2) \textit{ satisfies the conditions in the first line of } (3.10).$$

This corollary provides a sound basis for the definition of a skew local region (see the remarks immediately after the definition of skew local region in (3.10)). The condition in Corollary 4.1 is equivalent to saying that if $M_\gamma^{\mathrm{gen}} \neq 0$ then $\gamma = (-t^{c_1}, -t^{c_2})$ is regular so that we are in one of the cases shown in Figure 4.1. The classification and construction of calibrated representations of H_k^{ext} in terms of skew local regions in Theorem 3.5 is important for determining the irreducible representations of H_k^{ext} that arise in the Schur–Weyl duality framework (see Theorem 5.5).

We will do the classification of irreducible H_2^{ext} representations under *generality assumptions on the parameters*: $t^{\frac12}$ is not a root of unity and

$$t_0^{\frac12}t_k^{\frac12}, -t_0^{-\frac12}t_k^{\frac12} \notin \left\{1, -1, t^{\pm\frac12}, -t^{\pm\frac12}, t^{\pm1}, -t^{\pm1}\right\} \quad \text{and} \quad t_0^{\frac12}t_k^{\frac12} \neq \left(-t_0^{-\frac12}t_k^{\frac12}\right)^{\pm1}. \tag{4.1}$$

Whilst these restrictions look technical, they are explained by the (rank 2) computation of $\mathcal{W}_0$-orbits given in equation (4.4). Similar methods apply to the non-generic cases but the final classification needs to be stated differently and we will not treat the non-generic cases here. The non-generic case $t_0^{\frac12} = t_k^{\frac12} = t^{\frac12}$ is done in [22, 23] and [25]; the case where $t_0^{\frac12} = t_k^{\frac12} \neq t^{\frac12}$ appears in [8] (see also [14]).

The algebra H_2 is generated by $W_1^{\pm1}, W_2^{\pm1}, T_0$, and T_1, and the Weyl group $\mathcal{W}_0$ is generated by s_0 and s_1 with relations $s_i^2 = 1$ and $s_0s_1s_0s_1 = s_1s_0s_1s_0$. By (2.28),

$$H_2^{\mathrm{ext}} = \mathbb{C}[W_0^{\pm}] \otimes H_2 \quad \text{as algebras,}$$

and therefore it is sufficient to do the classification of irreducible representations of H_2. This is because all irreducible representations of $\mathbb{C}[W_0^{\pm 1}]$ are one dimensional and determined by the image of W_0; and all irreducible representations of H_2^{ext} are the tensor product of an irreducible representation of $\mathbb{C}[W_0^{\pm 1}]$ and an irreducible representation of H_2.

The group $\mathcal{W}_0$ acts on $(\mathbb{C}^\times)^2$ by

$$s_0(\gamma_1,\gamma_2) = \left(\gamma_1^{-1},\gamma_2\right) \qquad \text{and} \qquad s_1(\gamma_1,\gamma_2) = (\gamma_2,\gamma_1). \tag{4.2}$$

By (2.34), the intertwiners are

$$\tau_0 = T_0 - \frac{\left(t_0^{\frac12} - t_0^{-\frac12}\right) + \left(t_k^{\frac12} - t_k^{-\frac12}\right) W_1^{-1}}{1 - W_1^{-2}} \qquad \text{and} \qquad \tau_1 = T_1 - \frac{t^{\frac12} - t^{-\frac12}}{1 - W_1 W_2^{-1}}.$$

4.1. **Classification of central characters.** Following [23, § 5], the classification of irreducible H_k^{ext}-modules begins with a classification of pairs $(Z(\mathbf{c}), P(\mathbf{c})) = (Z(\gamma), P(\gamma))$, where, as in (2.31)), the conversion between $\gamma = (\gamma_1, \gamma_2)$ and $\mathbf{c} = (c_1, c_2)$ is given by (2.31),

$$\gamma_1 = -t^{c_1}, \quad \gamma_2 = -t^{c_2}, \qquad \text{and write } (Z(\mathbf{c}), P(\mathbf{c})) = (Z(\gamma), P(\gamma)).$$

Enumerate the possible pairs $(Z(\mathbf{c}), P(\mathbf{c})) = (Z(\gamma), P(\gamma))$ by taking note of the following:

(0) Since $(Z(w\gamma), P(w\gamma)) = (wZ(\gamma), wP(\gamma))$, it is sufficient to do the analysis for a single representative γ of each $\mathcal{W}_0$-orbit on $(\mathbb{C}^\times)^k$.
(1) The $\mathcal{W}_0$-orbits of roots are $\{\pm\varepsilon_1, \pm\varepsilon_2\}$ and $\{\pm(\varepsilon_2 \pm \varepsilon_1)\}$, and our preferred representative of the $\mathcal{W}_0$-orbit will have ε_1 or $\varepsilon_2 - \varepsilon_1$ in $Z(\gamma)$ if $Z(\gamma) \neq \emptyset$.
(2) If $Z(\gamma) = \emptyset$ and $P(\gamma) \neq \emptyset$ then our preferred representative of the $\mathcal{W}_0$-orbit will have ε_1 or $\varepsilon_2 - \varepsilon_1$ in $P(\gamma)$.

With these preferences, the classification of $(Z(\gamma), P(\gamma))$ is accomplished by noting that

(a) if $\gamma \in \{(1,1), (-1,-1)\}$ then $(Z(\gamma), P(\gamma)) = (\{\varepsilon_1, \varepsilon_2, \varepsilon_2 \pm \varepsilon_1\}, \emptyset)$;
(b) if $\gamma \in \{(1,-1), (-1,1)\}$ then $(Z(\gamma), P(\gamma)) = (\{\varepsilon_1, \varepsilon_2\}, \emptyset)$;
(c) $\varepsilon_2 - \varepsilon_1 \in Z(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_1)$;
(d) $\varepsilon_2 + \varepsilon_1 \in Z(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_1^{-1})$;
(e) $\varepsilon_1 \in Z(\gamma)$ if and only if $\gamma = (1, \gamma_2)$ or $\gamma = (-1, \gamma_2)$;
(f) $\varepsilon_2 \in Z(\gamma)$ if and only if $\gamma = (\gamma_1, 1)$ or $\gamma = (\gamma_1, -1)$;
(g) $\varepsilon_1 \in P(\gamma)$ if and only if

$$\gamma = (\gamma_1, \gamma_2) \quad \text{with} \quad \gamma_1 \in \left\{ t_0^{\frac12} t_k^{\frac12}, -t_0^{-\frac12} t_k^{\frac12}, -t_0^{\frac12} t_k^{-\frac12}, t_0^{-\frac12} t_k^{-\frac12} \right\};$$

(h) $\varepsilon_2 - \varepsilon_1 \in P(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_2 = \gamma_1 t^{\pm 1}$;
(i) $\varepsilon_2 + \varepsilon_1 \in P(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 \gamma_2 = t^{\pm 1}$.

Representatives of the 12 possible $(Z(\mathbf{c}), P(\mathbf{c}))$ with $Z(\mathbf{c}) = \emptyset$ are displayed in Figure 4.1. Representatives of the 9 possible $(Z(\mathbf{c}), P(\mathbf{c}))$ with $Z(\mathbf{c}) \neq \emptyset$ are displayed in Figure 4.2. It works out that, in each case, the pair $(Z(\mathbf{c}), P(\mathbf{c}))$ is attained by an element $\mathbf{c}$ that has real coordinates (the one complex character in the equal parameter case that behaves differently from the real characters, namely the point t_b in [23, Figure 5.1], does not appear in the generic unequal parameter case assumed in (4.1)).

With notation as at the beginning of Section 3, in Figures 4.1 and 4.2, the fundamental region C is the shaded area, the solid lines are the hyperplanes $\mathfrak{h}^\alpha$ for $\alpha \in R^+$, and the dotted hyperplanes are labeled by the equation that defines them. If $\mathbf{c} = (c_1, c_2) \in C$, so that $0 \le c_1 \le c_2$, then

$$Z(\mathbf{c}) = \{\text{solid hyperplanes through } \mathbf{c}\} \quad \text{and} \quad P(\mathbf{c}) = \{\text{dotted hyperplanes through } \mathbf{c}\}.$$

The bijection

$$\begin{matrix} \mathcal{W}_0 & \leftrightarrow & \{\text{chambers}\} \\ w & \mapsto & w^{-1}C \end{matrix} \qquad \text{identifies each } \mathcal{F}^{(\mathbf{c},J)} \text{ with a set of chambers,} \tag{4.3}$$

a *local region* in $\mathfrak{h}^*_{\mathbb{R}}$. As illustrated by the example at the bottom right of Figures 4.1 and 4.2, $\mathcal{F}^{(\mathbf{c},J)}$ is identified with the set of chambers that are on the negative side of the hyperplanes in J and on the positive side of the hyperplanes in $P(\mathbf{c}) - J$. For each $(\mathbf{c}, J)$ the corresponding configuration of boxes κ is displayed in the local region of chambers corresponding to the elements of $\mathcal{F}^{(\mathbf{c},J)}$ by (4.3). In Figure 4.1, only the boxes on positive diagonals are shown, since they determine the entire doubled configuration when $Z(\mathbf{c}) = \emptyset$. The diagram at the bottom right of each figure gives an example of the correspondence between chambers corresponding to $\mathcal{F}^{(\mathbf{c},J)}$, the elements of $\mathcal{F}^{(\mathbf{c},J)}$, and the standard fillings of the corresponding configuration of boxes κ: the point $\mathbf{c} = (r_1 - 1, r_1)$ in the bottom right of Figure 4.1, and the point $\mathbf{c} = (0, 1)$ in the bottom right of Figure 4.2.

In Figure 4.2, the small graphs nearby each marked $\mathbf{c} = (c_1, c_2)$ indicate the structure (generalized weight spaces and intertwiner maps) of the irreducible modules M of central character $\mathbf{c}$. This structure is determined below in Section 4.2. There is a vertex in the chamber $w^{-1}C$ for each element of a basis of $M^{\text{gen}}_{w\mathbf{c}}$ and there is an edge if the matrix of τ_i (or T_i if τ_i is not defined on $M^{\text{gen}}_{w\mathbf{c}}$) is nonzero in the entry corresponding to the two vertices that are connected.

4.2. **Construction of the irreducible H_2-modules.** The group $\mathcal{W}_0$ acts on $(\mathbb{C}^\times)^2$ as in (4.2) and the *central characters* are the $\mathcal{W}_0$-orbits on $(\mathbb{C}^\times)^2$. The *regular central characters* are the $\mathcal{W}_0$-orbits of $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{C}^\times)^2$ that have $Z(\gamma) = \emptyset$, i.e. where the intertwining operators in (2.37) are defined. Let $\mathbb{C}[W] = \mathbb{C}[W_1^{\pm 1}, W_2^{\pm 1}] \subseteq H_2$. By Kato's criterion (see [22, Proposition 2.11b]), for central characters $\gamma = (\gamma_1, \gamma_2)$ with $P(\gamma) = \emptyset$ there is a single irreducible module of dimension eight given by

$$L_{(\gamma_1,\gamma_2)} = \operatorname{Ind}^H_{\mathbb{C}[W]}(\mathbb{C}_{\gamma_1,\gamma_2}), \qquad \text{where } \mathbb{C}_{\gamma_1,\gamma_2} = \mathbb{C}v \text{ with } W_1 v = \gamma_1 v \text{ and } W_2 v = \gamma_2 v.$$

All irreducible modules with $Z(\gamma) = \emptyset$ are calibrated and can be constructed as in Theorem 3.5.

Representatives of the $\mathcal{W}_0$-orbits of $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{C}^\times)^2$ that have $Z(\gamma) \neq \emptyset$ and $P(\gamma) \neq \emptyset$ are as follows:

$\gamma = (\gamma_1, \gamma_2)$	$Z(\gamma)$	$P(\gamma)$
$\left(t^{\frac12}, t^{\frac12}\right), \left(-t^{\frac12}, -t^{\frac12}\right)$	$\{\varepsilon_2 - \varepsilon_1\}$	$\{\varepsilon_2 + \varepsilon_1\}$
$\left(t_0^{\frac12} t_k^{\frac12}, t_0^{\frac12} t_k^{\frac12}\right), \left(-t_0^{-\frac12} t_k^{\frac12}, -t_0^{-\frac12} t_k^{\frac12}\right)$	$\{\varepsilon_2 - \varepsilon_1\}$	$\{\varepsilon_1, \varepsilon_2\}$
$(1, t), (-1, -t)$	$\{\varepsilon_1\}$	$\{\varepsilon_2 - \varepsilon_1, \varepsilon_2 + \varepsilon_1\}$
$\left(\pm 1, t_0^{\frac12} t_k^{\frac12}\right), \left(\pm 1, -t_0^{-\frac12} t_k^{\frac12}\right),$	$\{\varepsilon_1\}$	$\{\varepsilon_2\}$

(4.4)

This classification is valid under the generality assumption on the parameters (4.1), which guarantees that none of these representatives are in the $\mathcal{W}_0$-orbit of another.

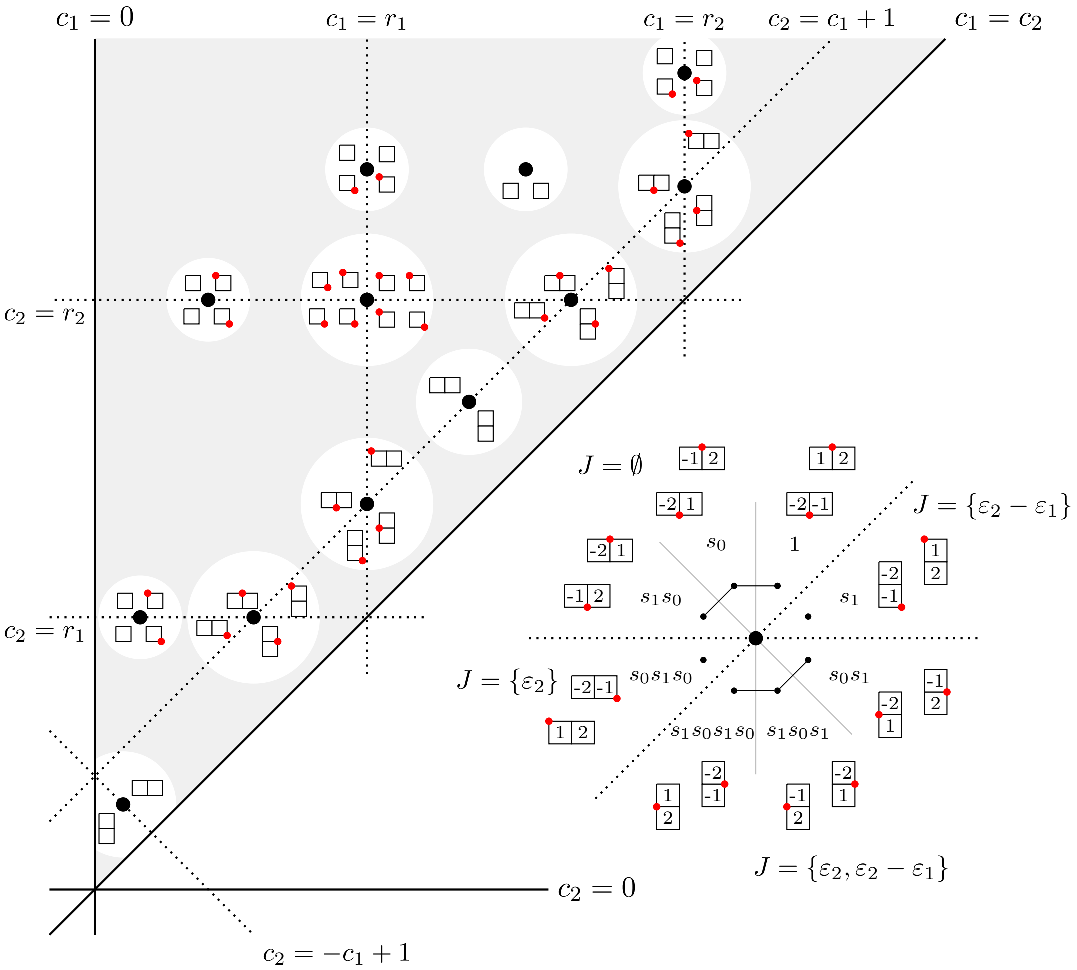


FIGURE 4.1. Regular central characters in rank 2. See the description in Section 4.1.

The following analysis of modules of central character $\gamma = (\gamma_1, \gamma_2)$ in (4.4) shows that no irreducible calibrated H_2-modules appear at these central characters. As in (3.5), the values r_1 and r_2 are defined by

$$-t^{r_1} = -t_k^{\frac{1}{2}} t_0^{-\frac{1}{2}} \quad \text{and} \quad -t^{r_2} = t_k^{\frac{1}{2}} t_0^{\frac{1}{2}}.$$

Case $(\gamma_1, \gamma_2) = (-1, -t^{r_i})$ *for* $i = 1$ *or* 2. Let $H_{\{0\}}$ be the subalgebra of H_2 generated by $T_0, W_1^{\pm 1}, W_2^{\pm 1}$. For each of $i = 1$ and $i = 2$, there are two irreducible modules of central character $\mathbf{c} = (0, r_i)$:

$$L^+_{(0,r_i)} = \mathrm{Ind}_{H_{\{0\}}}^{H_2} \left(\mathbb{C}_{(r_i,0)}\right), \qquad \text{where } \mathbb{C}_{(r_i,0)} = \mathbb{C}v \text{ with } \begin{array}{l} W_1 v = -t^{r_i} v, \\ W_2 v = -v, \\ T_0 v = t_0^{\frac{1}{2}} v, \end{array}$$

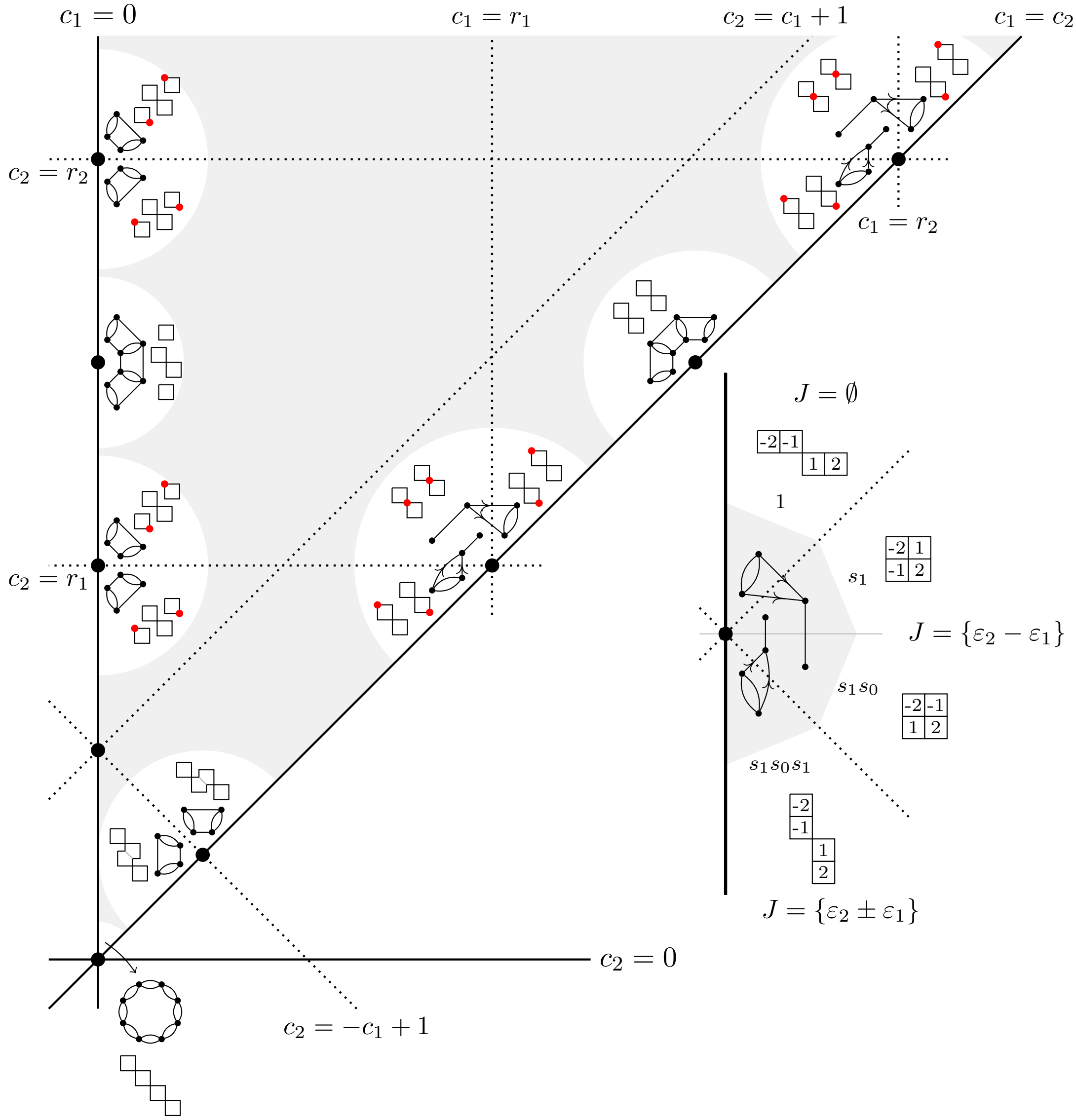


FIGURE 4.2. Non-regular points.

and

$$L^-_{(0,r_i)} = \mathrm{Ind}_{H_{\{0\}}}^{H_2}\left(\mathbb{C}_{(-r_i,0)}\right), \qquad \text{where } \mathbb{C}_{(-r_i,0)} = \mathbb{C}v \text{ with } \begin{aligned} W_1 v &= -t^{-r_i} v, \\ W_2 v &= -v, \\ T_0 v &= -t_0^{-\frac{1}{2}} v. \end{aligned}$$

With $M = L^+_{(0,r_i)}$, the generalized weight space decomposition is

$$M = M^{\mathrm{gen}}_{(r_i,0)} \oplus M^{\mathrm{gen}}_{(0,r_i)}, \qquad \text{with} \quad \dim\left(M^{\mathrm{gen}}_{(r_i,0)}\right) = \dim\left(M^{\mathrm{gen}}_{(0,r_i)}\right) = 2. \tag{4.5}$$

The element $W_1W_2^{-1}$ acts on $M^{\mathrm{gen}}_{(r_i,0)}$ with eigenvalues t^{r_i}. Since the parameters are generic (see (4.1)), $t^{r_i} \neq t^{\pm 1}$ and thus, by (2.42), τ_1^2 has no kernel. Thus the intertwiner $\tau_1\colon M^{\mathrm{gen}}_{(r_i,0)} \to M^{\mathrm{gen}}_{(0,r_i)}$ is invertible and $M = L^+_{(0,r_i)}$ is irreducible. Replacing r_i with $-r_i$ in (4.5) yields the decomposition of $M = L^-_{(0,r_i)}$ analogously.

Case $(\gamma_1,\gamma_2) = (-t^{\frac12}, -t^{\frac12})$. Let $H_{\{1\}}$ be the subalgebra of H_2 generated by $T_1, W_1^{\pm1}$ and $W_2^{\pm1}$. There are two irreducible modules of central character $\mathbf{c} = (\frac12,\frac12)$:

$$L^+_{(\frac12,\frac12)} = \mathrm{Ind}_{H_{\{1\}}}^{H_2}\left(\mathbb{C}_{(-\frac12,\frac12)}\right), \quad \text{where } \mathbb{C}_{(-\frac12,\frac12)} = \mathbb{C}v \text{ with } \begin{array}{r} W_1v = -t^{-\frac12}v, \\ W_2v = -t^{\frac12}v, \\ T_1v = t^{\frac12}v, \end{array}$$

and

$$L^-_{(\frac12,\frac12)} = \mathrm{Ind}_{H_{\{1\}}}^{H_2}\left(\mathbb{C}_{(\frac12,-\frac12)}\right), \quad \text{where } \mathbb{C}_{(\frac12,-\frac12)} = \mathbb{C}v \text{ with } \begin{array}{r} W_1v = -t^{\frac12}v, \\ W_2v = -t^{-\frac12}v, \\ T_1v = -t^{-\frac12}v. \end{array}$$

With $M = L^+_{(\frac12,\frac12)}$, the generalized weight space decomposition is

$$M = M^{\mathrm{gen}}_{(\frac12,\frac12)} \oplus M^{\mathrm{gen}}_{(-\frac12,\frac12)}, \qquad \text{with} \quad \dim\left(M^{\mathrm{gen}}_{(\frac12,\frac12)}\right) = \dim\left(M^{\mathrm{gen}}_{(-\frac12,\frac12)}\right) = 2. \tag{4.6}$$

The element W_1^{-1} acts on $M^{\mathrm{gen}}_{(\frac12,\frac12)}$ with eigenvalues $-t^{\frac12}$. Since the parameters are generic (see (4.1)), $-t^{\frac12} \notin \{-t^{\pm r_1}, -t^{\pm r_2}\}$ and thus, by (2.41), τ_0^2 has no kernel. Thus the intertwiner $\tau_0\colon M^{\mathrm{gen}}_{(\frac12,-\frac12)} \to M^{\mathrm{gen}}_{(-\frac12,-\frac12)}$ is invertible and $M = L^+_{(\frac12,\frac12)}$ is irreducible. Similarly, the structure of $M = L^-_{(\frac12,\frac12)}$ is given by swapping $\frac12$ and $-\frac12$ in (4.6).

Case $(\gamma_1,\gamma_2) = (-t^{r_i}, -t^{r_i})$ *for* $i = 1$ *or* 2. Let $H_{\{0\}}$ be the subalgebra of H_2 generated by $T_0, W_1^{\pm1}, W_2^{\pm1}$. For each of $i=1$ and $i=2$, there are two irreducible modules of central character $\mathbf{c} = (r_i, r_i)$:

$$L^+_{(r_i,r_i)} = \mathrm{Ind}_{H_{\{0\}}}^{H_2}\left(\mathbb{C}_{(r_i,-r_i)}\right), \qquad \text{where } \mathbb{C}_{(r_i,-r_i)} = \mathbb{C}v \text{ with } \begin{array}{r} W_1v = -t^{r_i}v, \\ W_2v = -t^{-r_i}v, \\ T_0v = t_0^{\frac12}v, \end{array}$$

and

$$L^-_{(r_i,r_i)} = \mathrm{Ind}_{H_{\{0\}}}^{H_2}\left(\mathbb{C}_{(-r_i,r_i)}\right), \qquad \text{where } \mathbb{C}_{(-r_i,r_i)} = \mathbb{C}v \text{ with } \begin{array}{r} W_1v = -t^{-r_i}v, \\ W_2v = -t^{r_i}v, \\ T_0v = -t_0^{-\frac12}v. \end{array}$$

The irreducibility of $L^+_{(r_i,r_i)}$ and $L^-_{(r_i,r_i)}$ is not immediate. We will show that $M = L^+_{(r_i,r_i)}$ is irreducible; the irreducibility of $L^-_{(r_i,r_i)}$ is proved analogously.

The generalized weight space decomposition of $M = L^+_{(r_i,r_i)}$ is

$$M = M^{\mathrm{gen}}_{(r_i,-r_i)} \oplus M^{\mathrm{gen}}_{(-r_i,r_i)} \oplus M^{\mathrm{gen}}_{(r_i,r_i)} \qquad \text{with} \qquad \begin{gathered} \dim\left(M^{\mathrm{gen}}_{(r_i,-r_i)}\right) = \dim\left(M^{\mathrm{gen}}_{(-r_i,r_i)}\right) = 1, \\ \dim\left(M^{\mathrm{gen}}_{(r_i,r_i)}\right) = 2. \end{gathered}$$

The element $W_1W_2^{-1}$ acts on $M^{\mathrm{gen}}_{(r_i,-r_i)}$ with eigenvalue $t^{r_i-(-r_i)}$. Since the parameters are generic (see (4.1)), $t^{2r_i} \neq t^{\pm 1}$ and thus, by (2.42), τ_1^2 has no kernel. Thus the intertwiner $\tau_1 \colon M^{\mathrm{gen}}_{(r_i,-r_i)} \to M^{\mathrm{gen}}_{(-r_i,r_i)}$ is invertible. As a $H_{\{0\}}$-module, $M^{\mathrm{gen}}_{(r_i,r_i)}$ is irreducible (2-dimensional). So either $N = M^{\mathrm{gen}}_{(r_i,r_i)}$ is an H_2-submodule or M is irreducible.

For the purpose of deriving a contradiction, assume that $N = M^{\mathrm{gen}}_{(r_i,r_i)}$ is an H_2-submodule of M. The space N has a basis

$$\{n_\gamma, T_1n_\gamma\} \qquad \text{with} \qquad W_1n_\gamma = -t^{r_i}n_\gamma, \quad \text{and} \quad W_2n_\gamma = -t^{r_i}n_\gamma.$$

By (2.24), $W_1^{-1}T_1n_\gamma = T_1W_2^{-1}n_\gamma + (t^{\frac12} - t^{-\frac12})W_1^{-1}n_\gamma = T_1(-t^{-r_i})n_\gamma + (t^{\frac12} - t^{-\frac12})(-t^{-r_i})n_\gamma$ and the action of W_1^{-1} and W_1^{-2} on the basis $\{n_\gamma, T_1n_\gamma\}$ are given by the matrices

$$\begin{aligned} \rho\left(W_1^{-1}\right) &= (-t^{-r_i})\begin{pmatrix} 1 & \left(t^{\frac12} - t^{-\frac12}\right) \\ 0 & 1 \end{pmatrix}, \\ \rho\left(W_1^{-2}\right) &= \rho\left(W_1^{-1}\right)^2 = t^{-2r_i}\begin{pmatrix} 1 & 2\left(t^{\frac12} - t^{-\frac12}\right) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \rho\left(1 - W_1^{-2}\right) &= \left(1 - t^{-2r_i}\right)\begin{pmatrix} 1 & \frac{-2\left(t^{\frac12} - t^{-\frac12}\right)t^{-2r_i}}{1 - t^{-2r_i}} \\ 0 & 1 \end{pmatrix} \qquad \text{and} \\ \rho\left(1 - W_1^{-2}\right)^{-1} &= \frac{1}{(1 - t^{-2r_i})}\begin{pmatrix} 1 & \frac{2\left(t^{\frac12} - t^{-\frac12}\right)t^{-2r_i}}{1 - t^{-2r_i}} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Since N is a submodule of M then

$$0 = \tau_0 = T_0 - \frac{\left(t_0^{1/2} - t_0^{-1/2}\right) + \left(t_k^{1/2} - t_k^{-1/2}\right)W_1^{-1}}{1 - W_1^{-2}}$$

(see (2.34) for the formula for τ_0), and so

$$\begin{aligned}\rho(T_0) &= \left(\left(t_0^{1/2}-t_0^{-1/2}\right)+\left(t_k^{1/2}-t_k^{-1/2}\right)W_1^{-1}\right)\left(1-W_1^{-2}\right)^{-1}\\ &= \frac{\left(t_0^{1/2}-t_0^{-1/2}\right)+\left(t_k^{1/2}-t_k^{-1/2}\right)(-t^{-r_i})}{1-t^{-2r_i}}\\ &\quad\cdot\begin{pmatrix}1 & \frac{\left(t_k^{1/2}-t_k^{-1/2}\right)\left(t^{\frac12}-t^{-\frac12}\right)(-t^{-r_i})}{\left(t_0^{1/2}-t_0^{-1/2}\right)+\left(t_k^{1/2}-t_k^{-1/2}\right)(-t^{-r_i})}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{2\left(t^{\frac12}-t^{-\frac12}\right)t^{-2r_i}}{1-t^{-2r_i}}\\ 0 & 1\end{pmatrix}\\ &= \frac{\left(t_0^{1/2}-t_0^{-1/2}\right)+\left(t_k^{1/2}-t_k^{-1/2}\right)(-t^{-r_i})}{1-t^{-2r_i}}\\ &\quad\cdot\begin{pmatrix}1 & \frac{t^{\frac12}-t^{-\frac12}}{-t^{r_i}}\left(\frac{2(-t^{-r_i})}{1-t^{-2r_i}}+\frac{\left(t_k^{1/2}-t_k^{-1/2}\right)}{\left(t_0^{1/2}-t_0^{-1/2}\right)+\left(t_k^{1/2}-t_k^{-1/2}\right)(-t^{-r_i})}\right)\\ 0 & 1\end{pmatrix}.\end{aligned}$$

Recall, from (3.5), that $-t^{r_i}=\pm t_k^{\pm\frac12}t_0^{\frac12}$, so that

$$\begin{aligned}&\frac{\left(t_0^{1/2}-t_0^{-1/2}\right)+\left(t_k^{1/2}-t_k^{-1/2}\right)(-t^{-r_i})}{1-t^{-2r_i}}\\ &\qquad\qquad = \frac{\left(t_0^{1/2}-t_0^{-1/2}\right)+\left(t_k^{1/2}-t_k^{-1/2}\right)\left(\pm t_k^{\mp\frac12}t_0^{-\frac12}\right)}{1-t_k^{\mp1}t_0^{-1}} = t_0^{\frac12}.\end{aligned}$$

The eigenvalues of $\rho(T_0)$ are $t_0^{\frac12}$ and, since $(T_0-t_0^{\frac12})(T_0+t_0^{-\frac12})=0$, the Jordan blocks of $\rho(T_0)$ are of size 1, forcing

$$\begin{aligned}0 &= \frac{2(-t^{-r_i})}{1-t^{-2r_i}}+\frac{\left(t_k^{1/2}-t_k^{-1/2}\right)}{\left(t_0^{1/2}-t_0^{-1/2}\right)+\left(t_k^{1/2}-t_k^{-1/2}\right)(-t^{-r_i})} = \frac{2(-t^{-r_i})}{1-t^{-2r_i}}+\frac{\left(t_k^{1/2}-t_k^{-1/2}\right)}{(1-t^{-2r_i})\,t_0^{1/2}}\\ &= \frac{2(-t^{-r_i})\,t_0^{\frac12}+\left(t_k^{1/2}-t_k^{-1/2}\right)}{(1-t^{-2r_i})\,t_0^{1/2}} = \frac{2\left(\pm t_k^{\mp\frac12}t_0^{-\frac12}\right)t_0^{\frac12}+\left(t_k^{1/2}-t_k^{-1/2}\right)}{(1-t^{-2r_i})\,t_0^{1/2}} = \frac{\pm\left(t_k^{1/2}+t_k^{-1/2}\right)}{(1-t^{-2r_i})\,t_0^{1/2}}.\end{aligned}$$

This is a contradiction since, by the generic condition on parameters in (4.1), $1\neq(-t^{r_1})(-t^{r_2})=(-t_k^{\frac12}t_0^{-\frac12})(t_k^{\frac12}t_0^{\frac12})=-(t_k^{\frac12})^2$. Thus N is not a submodule of M, and so M is irreducible.

Case $(\gamma_1,\gamma_2)=(-1,-t)$. Let $H_{\{1\}}$ be the subalgebra of H_2 generated by $T_1,W_1^{\pm1},W_2^{\pm1}$. There are two irreducible modules of central character $\mathbf{c}=(0,1)$:

$$L^+_{(0,1)}=\mathrm{Ind}_{H_{\{1\}}}^{H_2}\left(\mathbb{C}_{(-1,0)}\right),\qquad \text{where } \mathbb{C}_{(-1,0)}=\mathbb{C}v \text{ with } \begin{array}{l}W_1v=-t^{-1}v,\\ W_2v=-v,\\ T_1v=t^{\frac12}v,\end{array}$$

and

$$L^-_{(0,1)} = \mathrm{Ind}_{H_{\{1\}}}^{H_2}\left(\mathbb{C}_{(1,0)}\right), \qquad \text{where } \mathbb{C}_{(1,0)} = \mathbb{C}v \text{ with } \begin{array}{l} W_1 v = -tv, \\ W_2 v = -v, \\ T_1 v = -t^{-\frac{1}{2}} v. \end{array}$$

The irreducibility of $L^+_{(0,1)}$ and $L^-_{(0,1)}$ is not immediate. We will show that $M = L^+_{(0,1)}$ is irreducible; the irreducibility of $L^-_{(0,1)}$ is proved analogously.

The generalized weight space decomposition of $M = L^+_{(0,1)}$ is

$$M = M^{\mathrm{gen}}_{(-1,0)} \oplus M^{\mathrm{gen}}_{(1,0)} \oplus M^{\mathrm{gen}}_{(0,1)} \qquad \text{with} \qquad \begin{array}{l} \dim\left(M^{\mathrm{gen}}_{(-1,0)}\right) = \dim\left(M^{\mathrm{gen}}_{(1,0)}\right) = 1, \\ \dim\left(M^{\mathrm{gen}}_{(0,1)}\right) = 2. \end{array}$$

The element W_1^{-1} acts on $M^{\mathrm{gen}}_{(-1,0)}$ with eigenvalue $-t$. Since the parameters are generic (see (4.1)), $-t \notin \{-t^{\pm r_1}, -t^{\pm r_2}\}$ and thus, by (2.41), τ_0^2 has no kernel. Thus the intertwiner $\tau_0 \colon M^{\mathrm{gen}}_{(-1,0)} \to M^{\mathrm{gen}}_{(1,0)}$ is invertible. Since $M^{\mathrm{gen}}_{(0,1)}$ is irreducible as a $H_{\{0\}}$-module, we have either $N = M^{\mathrm{gen}}_{(0,1)}$ is an H_2-submodule or M is irreducible.

For the purpose of deriving a contradiction, assume that $N = M^{\mathrm{gen}}_{(0,1)}$ is an H_2-submodule of M. The space N has a basis

$$\{n_\gamma, T_0 n_\gamma\} \qquad \text{with} \qquad W_1 n_\gamma = -n_\gamma, \quad \text{and} \quad W_2 n_\gamma = -t n_\gamma.$$

By (C2) and (B3),

$$\begin{aligned} &W_1 W_2^{-1} T_0 n_\gamma \\ &\qquad = T_0 W_1^{-1} W_2^{-1} n_\gamma + \left(\left(t_0^{1/2} - t_0^{-1/2}\right) + \left(t_k^{1/2} - t_k^{-1/2}\right) W_1^{-1}\right) \frac{W_1 - W_1^{-1}}{1 - W_1^{-2}} W_2^{-1} n_\gamma \\ &\qquad = T_0 t^{-1} n_\gamma + \left(\left(t_0^{1/2} - t_0^{-1/2}\right) + \left(t_k^{1/2} - t_k^{-1/2}\right)(-1)\right) t^{-1} n_\gamma, \end{aligned}$$

and the action of $W_1 W_2^{-1}$ on the basis $\{n_\gamma, T_0 n_\gamma\}$ is given by the matrix

$$\rho\left(W_1 W_2^{-1}\right) = \begin{pmatrix} t^{-1} & \left(\left(t_0^{1/2} - t_0^{-1/2}\right) + \left(t_k^{1/2} - t_k^{-1/2}\right)(-1)\right) t^{-1} \\ 0 & t^{-1} \end{pmatrix}.$$

Thus

$$\begin{aligned} \rho\left(1 - W_1 W_2^{-1}\right) &= \begin{pmatrix} 1 - t^{-1} & -\left(\left(t_0^{1/2} - t_0^{-1/2}\right) + \left(t_k^{1/2} - t_k^{-1/2}\right)(-1)\right) t^{-1} \\ 0 & 1 - t^{-1} \end{pmatrix} \\ &= \left(1 - t^{-1}\right) \begin{pmatrix} 1 & -\dfrac{\left(\left(t_0^{1/2} - t_0^{-1/2}\right) + \left(t_k^{1/2} - t_k^{-1/2}\right)(-1)\right) t^{-1}}{1 - t^{-1}} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and

$$\rho\left(1 - W_1 W_2^{-1}\right)^{-1} = \frac{1}{(1 - t^{-1})} \begin{pmatrix} 1 & \dfrac{\left(\left(t_0^{1/2} - t_0^{-1/2}\right) + \left(t_k^{1/2} - t_k^{-1/2}\right)(-1)\right) t^{-1}}{1 - t^{-1}} \\ 0 & 1 \end{pmatrix}.$$

If N is a submodule of M then $0 = \tau_1 = T_1 - \frac{t^{\frac{1}{2}}+t^{-\frac{1}{2}}}{1-W_1W_2^{-1}}$ (see (2.34) for the formula for τ_1). Thus

$$\rho(T_1) = t^{\frac{1}{2}} \begin{pmatrix} 1 & \dfrac{\left(\left(t_0^{1/2} - t_0^{-1/2}\right) + \left(t_k^{1/2} - t_k^{-1/2}\right)(-1)\right)t^{-1}}{1-t^{-1}} \\ 0 & 1 \end{pmatrix}.$$

Since $(T_1 - t^{\frac{1}{2}})(T_1 + t^{-\frac{1}{2}}) = 0$ the Jordan blocks of $\rho(T_1)$ are of size one, forcing

$$0 = \left(t_0^{1/2} - t_0^{-1/2}\right) - \left(t_k^{1/2} - t_k^{-1/2}\right) = t_0^{-\frac{1}{2}}\left(t_0^{1/2} + t_k^{-1/2}\right)\left(t_0^{1/2} - t_k^{1/2}\right).$$

This is a quadratic equation in $t_0^{\frac{1}{2}}$ with two solutions, $t_0^{\frac{1}{2}} = t_k^{\frac{1}{2}}$ and $t_0^{\frac{1}{2}} = -t_k^{-\frac{1}{2}}$. This is a contradiction since, by the generic condition on parameters in (4.1), $-t^{-r_1} = -t_0^{\frac{1}{2}}t_k^{-\frac{1}{2}} \neq -1$ and $-t^{r_2} = t_0^{\frac{1}{2}}t_k^{\frac{1}{2}} \neq -1$. Thus N is not a submodule of M, and so M is irreducible.

5. Representations of $\mathcal{B}_k^{\mathrm{ext}}$ in tensor space

In this section we give a Schur–Weyl duality approach to the representations of the two boundary Hecke algebras H_k^{ext}. More generally, in Theorem 5.1 we show that, for a quantum group or quasitriangular Hopf algebra $U_q\mathfrak{g}$ and three $U_q\mathfrak{g}$-modules M, N and V, there is an action of the two boundary braid group $\mathcal{B}_k^{\mathrm{ext}}$ on tensor space $M \otimes N \otimes V^{\otimes k}$ that commutes with the $U_q\mathfrak{g}$-action. This means that there is a weak Schur–Weyl duality pairing between $U_q\mathfrak{g}$-modules and $\mathcal{B}_k^{\mathrm{ext}}$-modules, so that if $M \otimes N \otimes V^{\otimes k}$ is completely reducible as a $U_q\mathfrak{g}$-module then

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_\lambda L(\lambda) \otimes B_k^\lambda \qquad \text{as } \left(U_q\mathfrak{g}, \mathcal{B}_k^{\mathrm{ext}}\right)\text{-modules},$$

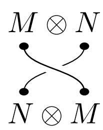
where $L(\lambda)$ are irreducible $U_q\mathfrak{g}$-modules and B_k^λ are $\mathcal{B}_k^{\mathrm{ext}}$-modules. In Section 5.4 we will explain that when $\mathfrak{g} = \mathfrak{gl}_n$ and M and N and V are appropriately chosen, the $\mathcal{B}^{\mathrm{ext}}$-action provides an action of the two boundary Hecke algebra H_k^{ext} (where the parameters depend on the choice of M and N). Our main theorem, Theorem 5.5, proves that the H_k^{ext}-modules B_k^λ that appear in tensor space $M \otimes N \otimes V^{\otimes k}$ are irreducible, and identifies them in terms of the classification of irreducible calibrated H_k^{ext}-modules which is given in Theorem 3.5.

5.1. **Quantum groups and R-matrices.** Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form. Assume that q is not a root of unity and let $U_q\mathfrak{g}$ be the Drinfeld-Jimbo quantum group corresponding to $\mathfrak{g}$. The quantum group $U_q\mathfrak{g}$ is a ribbon Hopf algebra with invertible R-matrix

$$R = \sum_R R_1 \otimes R_2 \quad \text{in} \quad U_q\mathfrak{g} \otimes U_q\mathfrak{g}, \quad \text{and ribbon element } v = q^{-2\rho}u,$$

where $u = \sum_R S(R_2)R_1$ and ρ is the staircase weight (see [15, Corollary (2.15)]). For $U_q\mathfrak{g}$-modules M and N, the map

$$\begin{array}{rccc} \check{R}_{MN}\colon & N \otimes M & \longrightarrow & M \otimes N \\ & n \otimes m & \longmapsto & \displaystyle\sum_R R_2 m \otimes R_1 n \end{array} \tag{5.1}$$



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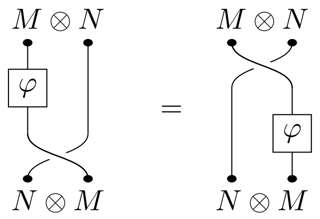
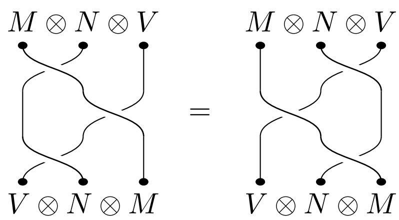
is a $U_q\mathfrak{g}$-module isomorphism. The quasitriangularity of a ribbon Hopf algebra provides the relations (see, for example, [21, (2.9), (2.10), and (2.12)]),

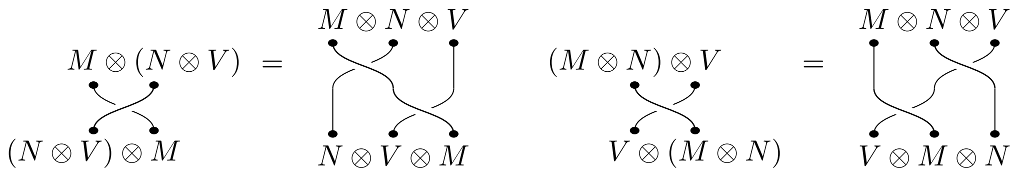


for any isomorphism $\varphi : M \to M$,

$$(\varphi \otimes \mathrm{id}_N)\check{R}_{MN} \quad = \check{R}_{MN}(\mathrm{id}_N \otimes \varphi),$$



$$(\check{R}_{MN} \otimes \mathrm{id}_V)(\mathrm{id}_N \otimes \check{R}_{MV})(\check{R}_{NV} \otimes \mathrm{id}_M) = (\mathrm{id}_M \otimes \check{R}_{NV})(\check{R}_{MV} \otimes \mathrm{id}_N)(\mathrm{id}_V \otimes \check{R}_{MN}),$$



(5.2)

$$(\check{R}_{M\otimes N,V}) = (\mathrm{id}_M \otimes \check{R}_{NV})(\check{R}_{MV} \otimes \mathrm{id}_N) \quad (\check{R}_{M\otimes N,V}) = (\mathrm{id}_M \otimes \check{R}_{NV})(\check{R}_{MV} \otimes \mathrm{id}_N).$$

For a $U_q\mathfrak{g}$-module M define

$$\begin{array}{rccc} C_M\colon & M & \longrightarrow & M \\ & m & \longmapsto & vm \end{array} \quad \text{so that} \quad C_{M\otimes N} = (\check{R}_{MN}\check{R}_{NM})^{-1}(C_M \otimes C_N) \tag{5.3}$$

(see [7, Proposition 3.2]). Let $L(\lambda)$ denote the simple $U_q\mathfrak{g}$-module generated by a highest weight vector v_λ^+ of weight λ. Then

$$C_{L(\lambda)} = q^{-\langle\lambda,\lambda+2\rho\rangle}\mathrm{id}_{L(\lambda)} \tag{5.4}$$

(see [15, Proposition 2.14] or [7, Proposition 5.1]). From (5.4) and (5.3), it follows that if $M = L(\mu)$ and $N = L(\nu)$ are finite-dimensional irreducible $U_q\mathfrak{g}$-modules of highest weights μ and ν respectively, then $\check{R}_{MN}\check{R}_{NM}$ acts on the $L(\lambda)$-isotypic component $L(\lambda)^{\oplus c^\lambda_{\mu\nu}}$ of the decomposition

$$L(\mu) \otimes L(\nu) = \bigoplus_\lambda L(\lambda)^{\oplus c^\lambda_{\mu\nu}} \qquad \text{by the scalar} \qquad q^{\langle\lambda,\lambda+2\rho\rangle - \langle\mu,\mu+2\rho\rangle - \langle\nu,\nu+2\rho\rangle}. \tag{5.5}$$

Proposition 5.1. *Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra with a symmetric nondegenerate* ad*-invariant bilinear form. Assume that q is not a root of unity and let $U_q\mathfrak{g}$ be the corresponding Drinfeld-Jimbo quantum group. Let $\mathcal{Z} = Z(U_q\mathfrak{g})$ be the center* *of $U_q\mathfrak{g}$ and let $\mathcal{Z}\mathcal{B}_k^{\mathrm{ext}}$ be the group algebra of $\mathcal{B}_k^{\mathrm{ext}}$ with coefficients in $\mathcal{Z}$. Let M, N, and V be $U_q\mathfrak{g}$-modules. Then $M \otimes N \otimes V^{\otimes k}$ is a $\mathcal{Z}\mathcal{B}_k^{\mathrm{ext}}$-module with action given by*

$$\begin{array}{rccll} \Phi\colon & \mathcal{Z}\mathcal{B}_k^{\mathrm{ext}} & \longrightarrow & \mathrm{End}_{U_q\mathfrak{g}}\Big(M \otimes N \otimes V^{\otimes k}\Big) & \\ & T_i & \longmapsto & \check{R}_i, & \text{for } i = 1, \dots k-1, \\ & X_1 & \longmapsto & \check{R}_M^2, & \\ & Y_1 & \longmapsto & \check{R}_N^2, & \\ & Z_1 & \longmapsto & \check{R}_0^2, & \\ & P & \longmapsto & \Big(\check{R}_{MN}\check{R}_{NM}\Big) \otimes \mathrm{id}_V^{\otimes(k)}, & \end{array} \tag{5.6}$$

where

$$\check{R}_0^2 = \Big(\check{R}_{(M\otimes N)V}\check{R}_{V(M\otimes N)}\Big) \otimes \mathrm{id}_V^{\otimes(k-1)}, \qquad \check{R}_i = \mathrm{id}_M \otimes \mathrm{id}_V^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \mathrm{id}_V^{\otimes(k-i-1)}$$

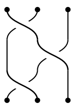
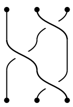
for $i = 1, \dots k-1$,

$$\check{R}_M^2 = \Big(\Big(\mathrm{id}_M \otimes \check{R}_{NV}\Big)\Big(\Big(\check{R}_{MV}\check{R}_{VM}\Big) \otimes \mathrm{id}_N\Big)\Big(\mathrm{id}_M \otimes \check{R}_{NV}^{-1}\Big)\Big) \otimes \mathrm{id}_V^{\otimes k-1}, \quad \text{and}$$

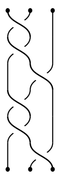
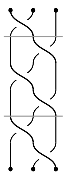
$$\check{R}_N^2 = \mathrm{id}_M \otimes \Big(\check{R}_{NV}\check{R}_{VN}\Big) \otimes \mathrm{id}_V^{\otimes(k-1)},$$

with $\check{R}_{MV}$ as in (5.1). *Moreover, this $\mathcal{Z}\mathcal{B}_k^{\mathrm{ext}}$ action commutes with the $U_q\mathfrak{g}$-action on $M \otimes N \otimes V^{\otimes k}$.*

Proof. This proof follows the proof of [21, Proposition 3.1], checking that the images of the generators T_i, X_1, Y_1, and Z_1 under the map Φ satisfy the relations of presentation (a) of the two boundary braid group in Theorem 2.1, as well as relations (2.15) and (2.16) for the extended two boundary braid group. For $i \in \{1, \dots, k-2\}$,

$$\Phi(T_i)\Phi(T_{i+1})\Phi(T_i) = \check{R}_i\check{R}_{i+1}\check{R}_i =$$

$$=$$

$$= \check{R}_{i+1}\check{R}_i\check{R}_{i+1} = \Phi(T_{i+1})\Phi(T_i)\Phi(T_{i+1}).$$

For $L = M$ or $L = N$ or $L = 0$,

$$\check{R}_L^2\check{R}_1\check{R}_L^2\check{R}_1 =$$

$$=$$

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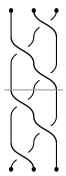
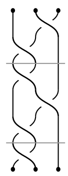

$$=$$

$$= \check{R}_1\check{R}_L^2\check{R}_1\check{R}_L^2,$$

which establishes

$$\Phi(A)\Phi(T_1)\Phi(A)\Phi(T_1) = \Phi(T_1)\Phi(A)\Phi(T_1)\Phi(A) \qquad \text{for } A = X_1, Y_1, \text{ and } Z_1, \text{ respectively.}$$

The formula

$$\Phi(Z_1) = \check{R}_0^2 = \check{R}_M^2\check{R}_N^2 = \Phi(X_1)\Phi(Y_1)$$

is a consequence of the third set of relations (cabling relations) in (5.2). Finally, the relations

$$\Phi(P)\Phi(Y_1)\Phi(P) = \Phi(Z_1^{-1})\Phi(Y_1)\Phi(Z_1) \qquad \text{and}$$
$$\Phi(P)\Phi(X_1)\Phi(P) = \Phi(Z_1^{-1})\Phi(X_1)\Phi(Z_1)$$

follow from the first and second sets of relations for $\check{R}$-matrices in (5.2) by the same braid computation by which the identities (2.13) were derived. The remainder of the relations (commuting generators) follow directly from the definitions of $\Phi(T_i)$, $\Phi(X_1)$, $\Phi(Y_1)$, $\Phi(Z_1)$, and $\Phi(P)$. ☐

5.2. **The $\mathcal{B}_k^{\mathrm{ext}}$-modules B_k^λ.** Assume that M, N, and V are finite-dimensional $U_q\mathfrak{g}$-modules and that ω is the highest weight of V so that

$$V = L(\omega) \qquad \text{is irreducible of highest weight } \omega.$$

Let $\mathcal{P}^{(j)}$ be an index set for the irreducible $U_q\mathfrak{g}$-modules that appear in $M \otimes N \otimes V^{\otimes j}$ and let $\mathcal{P}^{(-1)}$ be an index set for the irreducible $U_q\mathfrak{g}$-modules in M. The *Bratteli diagram* for the sequence of $U_q\mathfrak{g}$-modules

$$M, \quad M \otimes N, \quad M \otimes N \otimes V, \quad M \otimes N \otimes V \otimes V, \quad \cdots \tag{5.7}$$

is the graph with vertices on level j labeled by $\mu \in \mathcal{P}^{(j)}$, for $j \in \mathbb{Z}_{\geq -1}$, and labeled edges as follows.

(a) If $\mu \in \mathcal{P}^{(-1)}$ and $\lambda \in \mathcal{P}^{(0)}$ then there are $m_{\mu\lambda}$ edges $\mu \to \lambda$, where

$$L(\mu) \otimes N \cong \bigoplus_{\lambda \in \mathcal{P}^{(0)}} L(\lambda)^{\oplus m_{\mu\lambda}}.$$

(b) If $j \in \mathbb{Z}_{\geq 0}$ and $\mu \in \mathcal{P}^{(j)}$ and $\lambda \in \mathcal{P}^{(j+1)}$ then there are $m_{\mu\lambda}$ edges $\mu \to \lambda$, where

$$L(\mu) \otimes V \cong \bigoplus_{\lambda \in \mathcal{P}^{(j+1)}} L(\lambda)^{\oplus m_{\mu\lambda}}.$$

(c) Each edge $\mu \to \lambda$ is labeled with $\frac{1}{2}(\langle \lambda, \lambda + 2\rho\rangle - \langle \omega, \omega + 2\rho\rangle - \langle \mu, \mu + 2\rho\rangle)$.

A specific example in the case where $\mathfrak{g} = \mathfrak{gl}_n$ is given in Figure 5.1. The edges of the Bratteli diagram keep track of the decomposition of $M \otimes N \otimes V^{\otimes j}$ as j increases. The edge labels will then be used in Theorem 5.2 to keep track of the eigenvalues of the elements Z_i in the representation of $\mathcal{B}_k^{\mathrm{ext}}$. These eigenvalues play the role of contents of boxes; see (5.13).

If M and N are finite-dimensional then $M \otimes N \otimes V^{\otimes k}$ is completely decomposable as a $U_q\mathfrak{g}$-module. If B_k^λ is the space of highest weight vectors of weight λ in $M \otimes N \otimes V^{\otimes k}$, then

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B_k^\lambda, \qquad \text{as } (U_q\mathfrak{g}, \mathcal{B}_k^{\mathrm{ext}})\text{-bimodules.} \tag{5.8}$$

The $\mathcal{B}_k^{\mathrm{ext}}$-modules B_k^λ are not necessarily irreducible and not necessarily nonisomorphic, though they will be in the (mostly rare but very important) settings where $\Phi(\mathbb{C}\mathcal{B}_k^{\mathrm{ext}}) = \mathrm{End}_{U_q\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.

Recall from (2.9) that

$$Z_i = T_{i-1} \cdots T_1 Z_1 T_1 \cdots T_{i-1} \qquad \text{for } i = 1, \ldots, k.$$

The following proposition shows that, as operators on B_k^λ, the Z_i are simultaneously diagonalizable and have eigenvalues determined by the edges on the Bratteli diagram. The proof follows the same schematic that is used, for example, in the proof of [21, Proposition 3.2].

Proposition 5.2. *Assume M, N, and V are finite-dimensional $U_q\mathfrak{g}$ modules with V irreducible. For $\lambda \in \mathcal{P}^{(k)}$, let B_k^λ be the $\mathcal{B}_k^{\mathrm{ext}}$-module in* (5.8) *and let*

$$\mathcal{T}_k^\lambda = \left\{ \text{paths } S = \left(S^{(-1)} \xrightarrow{e_0} S^{(0)} \xrightarrow{e_1} \ldots \xrightarrow{e_k} S^{(k)} = \lambda \right) \text{ in the Bratteli diagram} \right\}.$$

Then

$$B_k^\lambda \quad \textit{has a basis} \qquad \left\{ v_S \;\middle|\; S \in \mathcal{T}_k^\lambda \right\}$$

of simultaneous eigenvectors for the action of $P, Z_1, \dots Z_k$*, with*

$$Pv_S = q^{2e_0} v_S \qquad \textit{and} \qquad Z_i v_S = q^{2e_i} v_S, \quad \textit{for } i = 1, \dots k,$$

so that the eigenvalues of P *and* $Z_1, \dots Z_k$ *on* v_S *are determined by the labels on the edges of the path* S*.*

Proof. The basis $\{v_S \mid S \in \mathcal{T}_k^\lambda\}$ is constructed inductively. For the initial case, choose any basis $\widehat{B}_{-1}$ of the highest weight vectors in M, and let $\widehat{B}_{-1}^\nu$ be the set of basis elements in $\widehat{B}_{-1}$ of weight ν. For the inductive step, assume that $\widehat{B}_{k-1}^\mu = \{v_T \mid T \in \mathcal{T}_{k-1}^\mu\}$ has been constructed so that

$$M \otimes N \otimes V^{\otimes(k-1)} = \bigoplus_{\mu \in P^{(k-1)}} L(\mu) \otimes B_{k-1}^\mu = \bigoplus_{\mu \in P^{(k-1)}} L(\mu) \otimes \left(\sum_{T \in \mathcal{T}_{k-1}^\mu} \mathbb{C} v_T \right),$$

The set $\widehat{B}_{k-1}^\mu = \{v_T \mid T \in \mathcal{T}_{k-1}^\mu\}$ is a basis of the vector space of highest weight vectors of weight μ in $M \otimes N \otimes V^{\otimes(k-1)}$ that is indexed by the paths $T = (T^{(-1)} \to \cdots \to T^{(k-1)} = \mu)$ of length k in the Bratteli diagram that end at μ. In this form $L(\mu) \otimes \mathbb{C}v_T$ denotes the irreducible $U_q\mathfrak{g}$-submodule of $M \otimes N \otimes V^{\otimes(k-1)}$ with highest weight vector v_T of weight μ.

Then, for each $T = (T^{(-1)} \to \cdots \to T^{(k-1)} = \mu)$ in $\mathcal{T}_{k-1}^\mu$, choose a basis

$$\widehat{B}_k^{T \to \lambda} = \left\{ v_S \;\middle|\; S = \left(T^{(-1)} \to \cdots \to T^{(k-1)} = \mu \to \lambda \right) \right\}$$

of highest weight vectors in the submodule of $M \otimes N \otimes V^{\otimes k}$ given by

$$(L(\mu) \otimes \mathbb{C}v_T) \otimes V = L(\mu) \otimes V \otimes \mathbb{C}v_T = \sum_{\mu \to \lambda} L(\lambda) \otimes \mathbb{C}v_S.$$

The basis $\widehat{B}_k^{T \to \lambda}$ is indexed by the edges in the Bratteli diagram from μ to a partition λ on level k. Then

$$\widehat{B}_k^\lambda = \bigsqcup_{\mu} \bigsqcup_{T \in \mathcal{T}_{k-1}^\mu} \mathcal{T}_k^{T \to \lambda} \qquad \text{is a basis of } B_k^\lambda.$$

The central element $q^{-2\rho}u$ in $U_q\mathfrak{g}$ acts on the submodule $L(\mu) \otimes \mathbb{C}v_T$ of $M \otimes N \otimes V^{\otimes(k-1)}$ by the constant $q^{-\langle \mu, \mu+2\rho \rangle}$. From (5.2), (5.3), and (5.4) it follows that Z_i acts on $M \otimes N \otimes V^{\otimes k}$ by

$$\begin{aligned}
\Phi(Z_i) &= \check{R}_{i-1} \cdots \check{R}_1 \check{R}_0^2 \check{R}_1 \cdots \check{R}_{i-1} \\
&= \check{R}_{M \otimes N \otimes V^{\otimes(i-1)}, V} \check{R}_{V, M \otimes N \otimes V^{\otimes(i-1)}} \otimes \mathrm{id}_V^{\otimes(k-i)} \\
&= (C_{M \otimes N \otimes V^{\otimes(i-1)}} \otimes C_V) C_{M \otimes N \otimes V^{\otimes i}}^{-1} \otimes \mathrm{id}_V^{\otimes(k-i)} \\
&= \sum_{\lambda, \mu, \nu} q^{\langle \lambda, \lambda+2\rho \rangle - \langle \mu, \mu+2\rho \rangle - \langle \omega, \omega+2\rho \rangle} \pi_{\mu\omega}^\lambda \otimes \mathrm{id}_V^{\otimes(k-i)},
\end{aligned}$$

where $\pi_{\mu\nu}^\lambda \colon M \otimes N \otimes \mathrm{id}_V^{\otimes i} \to M \otimes N \otimes \mathrm{id}_V^{\otimes i}$ is the projection onto the $L(\lambda)$ isotypic component of $(L(\mu) \otimes B_{i-1}^\mu) \otimes V$. Thus Z_i acts diagonally on the basis $\widehat{B}_k^\lambda$ and, by the definition of the labels of edges in the Bratteli diagram in (5.7), the eigenvalues of $Z_i v_S = q^{2e_i} v_S$ where e_i is the label on the edge $S^{(i)} \to S^{(i+1)}$ in the Bratteli diagram. □

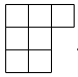
5.3. **Some tensor products for $\mathfrak{g} = \mathfrak{gl}_n$.** The finite-dimensional irreducible polynomial representations $L(\lambda)$ of $U_q\mathfrak{gl}_n$ are indexed by elements of

$$P^+_{poly} = \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, \mid \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$

Use

$$\rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} = \sum_{i=1}^{n}(n-i)\varepsilon_i, \tag{5.9}$$

as in [17, I (1.13)]. Identify each element $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ in P^+_{poly} with the corresponding partition having λ_i boxes in row i so that, for example,

$$\lambda = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 =$$  .

The *content* of the box in row i and column j of a partition λ is

$$c(\text{box}) = j - i = (\text{diagonal number of box}), \tag{5.10}$$

where the diagonals are numbered by the elements of $\mathbb{Z}$ from southwest to northeast, with the northwest corner box of a partition being in diagonal 0.

The representation $L(\varepsilon_1) = L(\square)$ is the standard n-dimensional representation of $U_q\mathfrak{gl}_n$. When $\nu = \varepsilon_1$, the decomposition in (5.5) is given by

$$L(\mu) \otimes L(\square) \cong \bigoplus_{\lambda \in \mu^+} L(\lambda), \tag{5.11}$$

where μ^+ is the set of partitions obtained by adding a box to μ. If $\lambda \in \mu^+$ and λ/μ is the box added to μ to obtain λ, then the action in (5.5) is given by

$$\begin{aligned}
&\langle\lambda, \lambda + 2\rho\rangle - \langle\mu, \mu + 2\rho\rangle - \langle\varepsilon_1, \varepsilon_1 + 2\rho\rangle \\
&\qquad = \langle\mu + \varepsilon_i, \mu + \varepsilon_i + 2\rho\rangle - \langle\mu, \mu + 2\rho\rangle - \langle\varepsilon_1, \varepsilon_1 + 2\rho\rangle \\
&\qquad = 2\mu_i + 1 + 2\rho_i - 1 - 2\rho_1 \\
&\qquad = 2\mu_i + 2(n-i) - 2(n-1) = 2\mu_i - 2i + 2 = 2c(\lambda/\mu)
\end{aligned} \tag{5.12}$$

(see [17, I (5.16) and (8.4)]). Since $\langle\varepsilon_1, \varepsilon_1 + 2\rho\rangle = 2(n-1) + 1 = 2n - 1$, it follows by induction on the number of boxes in a partition λ that

$$\langle\lambda, \lambda + 2\rho\rangle = (2n-1)|\lambda| + \sum_{\text{box} \in \lambda} 2c(\text{box}). \tag{5.13}$$

For $\mu, \nu \in P^+_{poly}$, the decomposition of the tensor product $L(\mu) \otimes L(\nu)$ can be calculated using the Littlewood–Richardson rule (see [17, Ch. I (9.2)]). When μ and ν are rectangles the decomposition is multiplicity free by the following theorem. In equation (5.14), $\mathcal{A}$ consists of the boxes that are in the union of the rectangles (a^c) and (b^d) (placed with northwest corner at $(1,1)$), and the dashed rectangular regions are the $\min(a,b) \times d$ rectangle $\mathcal{B}$ with northwest corner box at $(\max(a,b)+1, 1)$, and the $d \times \min(a,b)$ rectangle $\mathcal{B}'$ with northwest corner at $(1, c+1)$.

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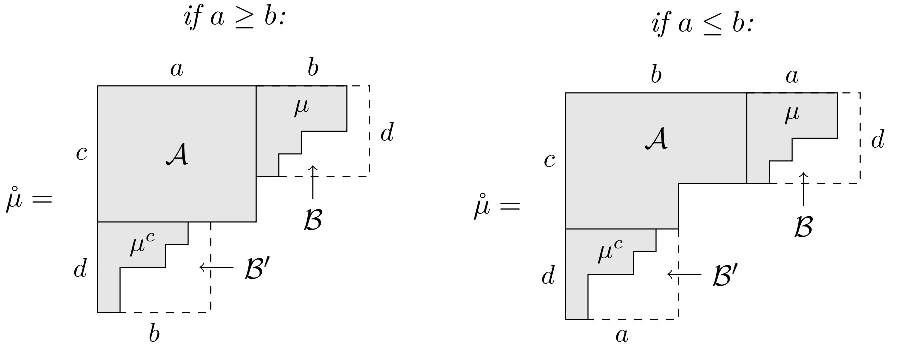
Proposition 5.3 (See [27, Lemma 3.3], [19, Theorem 2.4])**.** *Let* $a, b, c, d \in \mathbb{Z}_{\geq 0}$ *such that* $c \geq d$. *For* $\mu \subseteq (\min(a, b)^d)$ *let*



(5.14)

so that μ^c *is the* $180°$ *rotation of the complement of* μ *in a* $\min(a, b) \times d$ *rectangle. Denote the rectangular partition with* c *rows of length* a *by* (a^c). *Then*

$$L((a^c)) \otimes L((b^d)) \cong \bigoplus_{\mu \subseteq (\min(a,b)^d)} L(\mathring{\mu}) \cong \bigoplus_{\nu \in \mathcal{P}^{(0)}} L(\nu), \tag{5.15}$$

where $\mathcal{P}^{(0)} = \{\mathring{\mu} \mid \mu \subseteq ((\min(a, b)^d)\}$.

For an example of the decomposition in (5.15), see Figure 5.1, where the decomposition of $L(a^c) \otimes L(2^2)$ for $a, c \geq 2$ is indicated in level 0 of the Bratteli diagram (see the description following (5.22) for explanation of the Bratteli diagram).

The value in (5.5) for the product in (5.15) is given by using (5.13) to compute

$$\begin{aligned}\langle \mathring{\mu}, \mathring{\mu} + 2\rho\rangle - \langle (a^c), (a^c) + 2\rho\rangle - \left\langle (b^d), (b^d) + 2\rho \right\rangle \\ = (2n-1)(|\mathring{\mu}| - |(a^c)| - |(b^d)|) \\ + \left(\sum_{\mathrm{box} \in \mathring{\mu}} 2c(\mathrm{box})\right) - \sum_{\mathrm{box} \in (a^c)} 2c(\mathrm{box}) - \sum_{\mathrm{box} \in (b^d)} 2c(\mathrm{box}) \\ = 0 + \sum_{\mathrm{box} \in \mathring{\mu}} 2c(\mathrm{box}) - ac(a-c) - bd(b-d).\end{aligned} \tag{5.16}$$

5.4. **Irreducible** H_k^{ext}**-modules in** $M \otimes N \otimes V^{\otimes k}$**.** In this subsection we provide, for $\mathfrak{g} = \mathfrak{gl}_n$, specific highest weight modules M, N, and V such that the $\mathcal{B}_k^{\mathrm{ext}}$-action factors through the extended two boundary Hecke algebra H_k^{ext}. In these cases the $\mathcal{B}_k^{\mathrm{ext}}$-modules B_k^λ in (5.8) are calibrated H_k^{ext}-modules. Theorem 5.5 identifies the B_k^λ for these cases explicitly in terms of the indexings of calibrated H_k^{ext}-modules given in Theorem 3.5 and Proposition 3.2.

Recall that, as defined in Section 2.2, the *extended two boundary Hecke algebra* H_k^{ext} is the quotient of the group algebra of the extended two boundary braid group $\mathbb{C}\mathcal{B}_k^{\mathrm{ext}}$ by the relations

$$(X_1 - a_1)(X_1 - a_2) = 0, \qquad (Y_1 - b_1)(Y_1 - b_2) = 0,$$
$$\text{and} \quad \left(T_i - t^{\frac{1}{2}}\right)\left(T_i + t^{-\frac{1}{2}}\right) = 0, \tag{5.17}$$

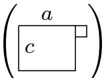
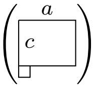
$i = 1, \ldots, k-1$, for fixed $a_1, a_2, b_1, b_2, t^{\frac{1}{2}} \in \mathbb{C}^\times$.

Theorem 5.4. *If $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^c))$, $N = L((b^d))$, and $V = L(\square)$,*

$$a_1 = q^{2a}, \quad a_2 = q^{-2c}, \quad b_1 = q^{2b}, \quad b_2 = q^{-2d}, \quad \text{and} \quad t^{\frac{1}{2}} = q, \tag{5.18}$$

then the map Φ from Proposition 5.1 gives an action of H_k^{ext} on $M \otimes N \otimes V^{\otimes k}$ commuting with that of $U_q\mathfrak{gl}_n$.

Proof. The module $M \otimes V$ decomposes as

$M \otimes V = L\Big($  $\Big) \oplus L\Big($  $\Big).$ (5.19)

By (5.5) and (5.12), $\check{R}_{MV}\check{R}_{VM}$ acts on the first summand by the constant q^{2a} and on the second summand by the constant q^{-2c}. So

$$\Big(\Phi(X_1) - q^{2a}\Big)\Big(\Phi(X_1) - q^{-2c}\Big) = 0; \quad \text{similarly} \quad \Big(\Phi(Y_1) - q^{2b}\Big)\Big(\Phi(Y_1) - q^{-2d}\Big) = 0$$

by replacing (a^c) with (b^d). The relation

$$(\Phi(T_i) - q)\Big(\Phi(T_i) + q^{-1}\Big) = 0$$

follows similarly by considering the tensor product $V \otimes V = L(\square) \otimes L(\square)$. $\square$

From (2.17), (5.18), and (3.5),

$$\begin{aligned}
&a_1 = q^{2a}, && a_2 = q^{-2c}, && b_1 = q^{2b}, \quad b_2 = q^{-2d}, \quad t^{\frac{1}{2}} = q,\\
&t_k^{\frac{1}{2}} = a_1^{\frac{1}{2}}(-a_2)^{-\frac{1}{2}} && = -iq^{a+c} && \text{and} \quad t_0^{\frac{1}{2}} = b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}} = -iq^{b+d},\\
&-t^{r_1} = -t_k^{\frac{1}{2}}t_0^{-\frac{1}{2}} && = -q^{(a+c)-(b+d)}, && \text{and} \quad -t^{r_2} = t_k^{\frac{1}{2}}t_0^{\frac{1}{2}} \qquad = -q^{a+c+b+d}.
\end{aligned} \tag{5.20}$$

Using these conversions, the generality conditions in (4.1) become requirements that q is not a root of unity and

$$-q^{(a+c)-(b+d)}, -q^{a+c+b+d} \notin \Big\{1, -1, q^{\pm 1}, -q^{\pm 1}, q^{\pm 2}, -q^{\pm 2}\Big\}$$
$$\text{and} \quad -q^{(a+c)-(b+d)} \neq -q^{\pm(a+c+b+d)}.$$

In the context of Theorem 5.4, these genericity conditions are

$$q \text{ is not a root of unity}, \quad a, b, c, d \in \mathbb{Z}_{>0} \quad \text{and} \quad (a+c) - (b+d) \notin \{0, \pm 1, \pm 2\}. \tag{5.21}$$

In the setting of Theorem 5.4, equation (5.8) provides H_k^{ext}-modules B_k^λ with

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B_k^\lambda, \qquad \text{as } \Big(U_q\mathfrak{g}, \mathcal{H}_k^{\mathrm{ext}}\Big)\text{-bimodules}. \tag{5.22}$$

Theorem 5.5 below will accomplish our primary goal for this paper by identifying the module B_k^λ explicitly as a calibrated H_k^{ext}-module $H_k^{(z,\mathbf{c},J)}$ as constructed in Theorem 3.5. The results of (5.11), (5.12), and Proposition 5.3 show that the Bratteli diagram of (5.7) has $\mathcal{P}^{(-1)} = \{(a^c)\}$, $\mathcal{P}^{(0)} = \{\mathring{\mu} \mid \mu \subseteq ((\min(a,b))^d)\}$ as in Proposition 5.3 and, for $j \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{P}^{(j)} = \Big\{ \text{partitions obtained by adding } j \text{ boxes to a partition in } \mathcal{P}^{(0)} \Big\}.$$

By (5.16), if $\mathring{\mu} \in \mathcal{P}^{(0)}$ then there is an edge

$$(a^c) \xrightarrow{e_0(\mathring{\mu})} \mathring{\mu} \text{ with label } \quad e_0(\mathring{\mu}) = -\frac{ac}{2}(a-c) - \frac{bd}{2}(b-d) + \sum_{\mathrm{box} \in \mathring{\mu}} c(\mathrm{box}). \tag{5.23}$$

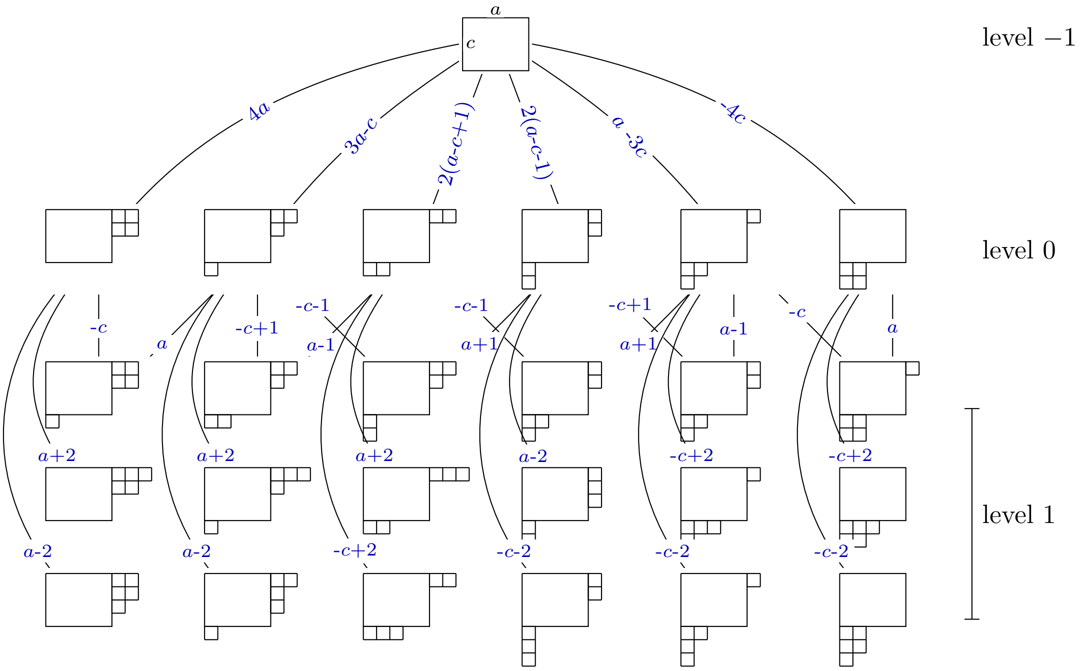


FIGURE 5.1. Levels −1, 0, and 1 of a Bratteli diagram encoding isotypic components of $M \otimes N \otimes V$ where $a, c > 2$ and $b = d = 2$. The edges from level −1 to level 0 are labeled by $e_0(T^{(0)})$ as in (5.16); the edges from level 0 to 1 are labeled by the content of the box added.

For $j \geq 0$, the edges $\mu \to \lambda$ from level j to level $j+1$ correspond to adding a single box to μ to get λ, and are labeled by $c(\lambda/\mu)$, the content of the box λ/μ:

$$\mu \xrightarrow{c(\lambda/\mu)} \lambda \quad \text{for edges from level } j \text{ to level } j+1. \tag{5.24}$$

The case when $M = L(a^c)$ and $N = L(2^2)$ with $a, c > 2$ is illustrated in Figure 5.1.

Let $\lambda \in \mathcal{P}^{(k)}$. Define

$$c_0 = -\frac{1}{2}(k(a-c+b-d)+ac(a-c)+bd(b-d))+\sum_{\text{box}\in\lambda} c(\text{box}) \quad \text{and} \quad z = (-1)^k q^{2c_0}. \tag{5.25}$$

Using notation as in (5.14), let

$$\mu^c = \lambda \cap \mathcal{B}' \quad \text{and let } S^{(0)}_{\max} \text{ be the corresponding } \mathring{\mu}. \tag{5.26}$$

Define the *shifted content* of a box by

$$\begin{gathered}\widetilde{c}(\text{box}) = c(\text{box}) - \frac{1}{2}(a-c+b-d), \\ \text{and let } \mathbf{c} = (c_1, \ldots, c_k) \quad \text{with} \quad 0 \leq c_1 \leq c_2 \leq \cdots \leq c_k\end{gathered} \tag{5.27}$$

be the sequence of absolute values of the shifted contents of the boxes in $\lambda/S^{(0)}_{\max}$ arranged in increasing order. Index the boxes of $\lambda/S^{(0)}_{\max}$ with $1, 2 \ldots, k$ so that

$$\text{if } i<j \text{ then } |\widetilde{c}(\mathrm{box}_i)| \le |\widetilde{c}(\mathrm{box}_j)|, \tag{5.28a}$$

$$\text{if } i<j \text{ and } \widetilde{c}(\mathrm{box}_i) = \widetilde{c}(\mathrm{box}_j) < 0 \text{ then } \mathrm{box}_i \text{ is SE of } \mathrm{box}_j, \tag{5.28b}$$

$$\text{if } i<j \text{ and } \widetilde{c}(\mathrm{box}_i) = \widetilde{c}(\mathrm{box}_j) \ge 0 \text{ then } \mathrm{box}_i \text{ is NW of } \mathrm{box}_j, \tag{5.28c}$$

$$\text{if } i<j \text{ and } \widetilde{c}(\mathrm{box}_i) = -\widetilde{c}(\mathrm{box}_j), \text{ then } \widetilde{c}(\mathrm{box}_i) \le 0 \le \widetilde{c}(\mathrm{box}_j), \tag{5.28d}$$

and define

$$\begin{aligned} J = \{\varepsilon_i \mid \widetilde{c}(\mathrm{box}_i) \in \{-r_1, -r_2\}\} \\ \sqcup \left\{ \varepsilon_j - \varepsilon_i \;\middle|\; \begin{array}{l} \widetilde{c}(\mathrm{box}_j) = \widetilde{c}(\mathrm{box}_i) + 1 > 0 \text{ and } \mathrm{box}_j \text{ is NW of } \mathrm{box}_i, \text{ or} \\ \widetilde{c}(\mathrm{box}_j) = \widetilde{c}(\mathrm{box}_i) - 1 < 0 \text{ and } \mathrm{box}_j \text{ is SE of } \mathrm{box}_i, \text{ or} \\ \widetilde{c}(\mathrm{box}_j) = -\widetilde{c}(\mathrm{box}_i) - 1 < 0 < \widetilde{c}(\mathrm{box}_i) \end{array} \right\} \\ \sqcup \left\{ \varepsilon_j + \varepsilon_i \;\middle|\; \begin{array}{l} \widetilde{c}(\mathrm{box}_j) = -1 \text{ and } \widetilde{c}(\mathrm{box}_i) = 0 \text{ and } \mathrm{box}_j \text{ is SE of } \mathrm{box}_i, \text{ or} \\ \widetilde{c}(\mathrm{box}_j) = \frac{1}{2} \text{ and } \widetilde{c}(\mathrm{box}_i) = -\frac{1}{2} \text{ and } \mathrm{box}_j \text{ is NW of } \mathrm{box}_i, \text{ or} \\ \widetilde{c}(\mathrm{box}_j) = -\frac{1}{2} \text{ and } \widetilde{c}(\mathrm{box}_i) = -\frac{1}{2} \end{array} \right\}, \end{aligned} \tag{5.29}$$

so that J is a subset of $P(\mathbf{c})$, where $P(\mathbf{c})$ is as defined in (3.7). See Examples 5.7 and 5.8 following the proof of Theorem 5.5.

Theorem 5.5. *Let $\mathfrak{g} = \mathfrak{gl}_n$ and let $M = L(a^c)$, $N = L(b^d)$ and $V = L(\square)$ so that H^{ext}_k acts on $M \otimes N \otimes V^{\otimes k}$ as in Theorem 5.4. Assume that the genericity conditions of* (5.21) *hold so that q is not a root of unity, $a, b, c, d \in \mathbb{Z}_{>0}$ and $(a+c)-(b+d) \notin \{0, \pm 1, \pm 2\}$. For $\lambda \in \mathcal{P}^{(k)}$, let B^λ_k be the H^{ext}_k-module of* (5.22) *and define z, $\mathbf{c}$ and J as in* (5.25), (5.27), *and* (5.29). *Then*

$$B^\lambda_k \cong H_k^{(z,\mathbf{c},J)} \text{ as } H^{\mathrm{ext}}_k\text{-modules.} \tag{5.30}$$

Proof. By Proposition 5.2, B^λ_k is a calibrated H^{ext}_k module. Therefore B^λ_k has a composition series with factors that are irreducible calibrated H^{ext}_k-modules. By Theorem 3.5, each factor is isomorphic to some $H_k^{(z,\mathbf{c},J)}$ where $(\mathbf{c}, J)$ is a skew local region, and $(z, \mathbf{c}, J)$ is determined by the eigenvalues of the action of $W_0, W_1, \ldots, W_k$. By Proposition 5.2, the simultaneous eigenbasis $\{v_S \mid S \in \mathcal{T}^\lambda_k\}$ B^λ_k is indexed by

$$\mathcal{T}^\lambda_k = \left\{ \begin{array}{c} \text{paths } S = \Big((a^c) \to S^{(0)} \to S^{(1)} \to \cdots \to S^{(k)} = \lambda\Big) \\ \text{in the Bratteli diagram} \end{array} \right\}. \tag{5.31}$$

To determine which $H_k^{(z,\mathbf{c},J)}$ appear as composition factors of B^λ_k it is necessary to compute the eigenvalues of the action of the W_i's on the basis vectors v_S as follows.

By (5.23), (5.24), and the formulas in Proposition 5.2,

$$\Phi(P)v_S = q^{2e_0(S^{(0)})} v_S \qquad \text{and} \qquad \Phi(Z_i) v_S = q^{2c(S^{(i)}/S^{(i-1)})} v_S \quad \text{for } i = 1, \ldots, k.$$

Using (2.18) and (5.18), $W_i = -(a_1a_2b_1b_2)^{-\frac{1}{2}} Z_i$ with $a_1 = q^{2a}$, $a_2 = q^{-2c}$, $b_1 = q^{2b}$, and $b_2 = q^{-2d}$, and thus

$$\begin{aligned} \Phi(W_i) v_S = -(a_1a_2b_1b_2)^{-\frac{1}{2}} \Phi(Z_i) v_S &= -q^{-(a-c+b-d)} q^{2c(S^{(i)}/S^{(i-1)})} v_S \\ &= -q^{2\tilde{c}(S^{(i)}/S^{(i-1)})} v_S. \end{aligned} \tag{5.32}$$

Then

$$\Phi(PW_1\cdots W_k)v_S=(-1)^k q^{2\left(e_0\left(S^{(0)}\right)+c\left(S^{(1)}/S^{(0)}\right)+\cdots+c\left(S^{(k)}/S^{(k-1)}\right)\right)-k(a-c+b-d)}v_S$$

so that, with c_0 and z as in (5.25),

$$\Phi(W_0)=\Phi(PW_1\cdots W_k)v_S=(-1)^k q^{2c_0}v_S=zv_S. \tag{5.33}$$

Let $S=((a^c)\to S^{(0)}\to S^{(1)}\to\cdots\to S^{(k)}=\lambda)$ be a path to λ in the Bratteli diagram. In the context of the diagrams in (5.14), the partitions $S^{(0)}$ and $S^{(0)}_{\max}$ differ by moving some boxes from μ to μ^c (from the NW border of $\lambda/S^{(0)}_{\max}$ in $\mathcal{B}$ to the NW border of $\lambda/S^{(0)}$ in $\mathcal{B}'$). Thus the sequence $\mathbf{c}=(c_1,\dots,c_k)$, where $c_1,\dots c_k$ are the values

$$\left|\widetilde{c}\left(S^{(1)}/S^{(0)}\right)\right|,\dots,\left|\widetilde{c}\left(S^{(k)}/S^{(k-1)}\right)\right|$$

arranged in increasing order, coincides with $\mathbf{c}$ as defined in (5.27). Let $w_S\in\mathcal{W}_0$ be the minimal length element such that

$$\begin{aligned} w_S\mathbf{c}=w_S(c_1,\dots c_k)&=\left(c_{w_S^{-1}(1)},\dots c_{w_S^{-1}(k)}\right)\\ &=\left(\widetilde{c}\left(S^{(1)}/S^{(0)}\right),\dots\widetilde{c}\left(S^{(k)}/S^{(k-1)}\right)\right),\end{aligned} \tag{5.34}$$

where $c_{-i}=-c_i$ for $i\in\{1,\dots k\}$. The signed permutation w_S is the unique signed permutation such that

$$w_S\mathbf{c}=\left(\widetilde{c}\left(S^{(1)}/S^{(0)}\right),\dots\widetilde{c}\left(S^{(k)}/S^{(k-1)}\right)\right)\qquad\text{and}\qquad R(w_S)\cap Z(\mathbf{c})=\emptyset,$$

where $Z(\mathbf{c})$ is as in (3.6). If the boxes of $\lambda/S^{(0)}$ are indexed according to the same conditions as just before (5.29), then w_S is the signed permutation given by

$$w_S(i)=\mathrm{sgn}(\widetilde{c}(\mathrm{box}_i))(\text{entry in box}_i\text{ of }S),$$

where the path S is identified with the standard tableau of shape $\lambda/S^{(0)}$ that has the box $S^{(j)}/S^{(j-1)}$ filled with j.

The basis vector v_S appears in a composition factor isomorphic to $H_k^{(z,\mathbf{c},J)}$ where

$$J=R(w_S)\cap P(\mathbf{c}),\quad\text{where}\quad R(w_S)=R_1\sqcup R_2\sqcup R_3\quad\text{and}\quad P(\mathbf{c})=P_1\sqcup P_2\sqcup P_3,$$

as defined in (3.2) and (3.7), are given by

$$\begin{aligned} R_1&=\{\varepsilon_i\mid i>0\text{ and }w_S(i)<0\}, & P_1&=\{\varepsilon_i\mid c_i\in\{r_1,r_2\}\},\\ R_2&=\{\varepsilon_j-\varepsilon_i\mid i<j\text{ and }w_S(i)>w_S(j)\}, & P_2&=\{\varepsilon_j-\varepsilon_i\mid 0<i<j, c_j=c_i+1\},\\ R_3&=\{\varepsilon_j+\varepsilon_i\mid i<j\text{ and }-w_S(i)>w_S(j)\}, & P_3&=\{\varepsilon_j+\varepsilon_i\mid 0<i<j, c_j=-c_i+1\}.\end{aligned}$$

To describe $J=(R_1\cap P_1)\sqcup(R_2\cap P_2)\sqcup(R_3\cap P_3)$ in terms of the boxes in λ, first record that

$$R_1\cap P_1=\{\varepsilon_i\mid i>0\text{ and }w_S(i)<0\}\cap\{\varepsilon_i\mid c_i\in\{r_1,r_2\}\}=\{\varepsilon_i\mid\widetilde{c}(\mathrm{box}_i)=\{-r_1,-r_2\}\}.$$

Next analyze

$$R_2\cap P_2=\{\varepsilon_j-\varepsilon_i\mid i<j\text{ and }w(i)>w(j)\}\cap\{\varepsilon_j-\varepsilon_i\mid 0<i<j,\ c_j=c_i+1\}.$$

Since $0\le c_i$ and $c_j=c_i+1$, we have $c_j\ge 1$.

(Case 1) $\widetilde{c}(\mathrm{box}_i)\ge 0$, so that $\widetilde{c}(\mathrm{box}_j)=\pm(\widetilde{c}(\mathrm{box}_i)+1)$.

(a) $\widetilde{c}(\mathrm{box}_j)=\widetilde{c}(\mathrm{box}_i)+1$.

- If box_j is NW of box_i then $w(j)<w(i)$ and $\varepsilon_j-\varepsilon_i\in J$.

- If box_j is SE of box_i then $w(j) > w(i)$ and $\varepsilon_j - \varepsilon_i \notin J$.

(b) $\widetilde{c}(\mathrm{box}_j) = -(\widetilde{c}(\mathrm{box}_i)+1)$.
- Then $w(j) < 0 < w(i)$ so that $w(j) < w(i)$ and $\varepsilon_j - \varepsilon_i \in J$.

(Case 2) $\widetilde{c}(\mathrm{box}_i) < 0$, so that $\widetilde{c}(\mathrm{box}_j) = \pm(-\widetilde{c}(\mathrm{box}_i)+1)$.

(a) $\widetilde{c}(\mathrm{box}_j) = \widetilde{c}(\mathrm{box}_i) - 1 < \widetilde{c}(\mathrm{box}_i) < 0$.
- If box_j is NW of box_i then $-w(j) < -w(i)$, so that $w(i) < w(j)$ and $\varepsilon_j - \varepsilon_i \notin J$.
- If box_j is SE of box_i then $-w(j) > -w(i)$ so that $w(i) > w(j)$ and $\varepsilon_j - \varepsilon_i \in J$.

(b) $\widetilde{c}(\mathrm{box}_j) = -\widetilde{c}(\mathrm{box}_i) + 1 > 0 > \widetilde{c}(\mathrm{box}_i)$.
- Then $w(i) < 0$ and $0 < w(j)$ so that $\varepsilon_j - \varepsilon_i \notin J$.

Finally, analyze

$$R_3 \cap P_3 = \{\varepsilon_j + \varepsilon_i \mid i < j \text{ and } -w(i) > w(j)\} \cap \{\varepsilon_j + \varepsilon_i \mid 0 < i < j,\ c_j = -c_i + 1\}.$$

Since $0 \le c_i$ and $c_j = -c_i + 1 \ge c_i$, we have $0 \le c_i \le 1/2$. Since the entries of $\mathbf{c}$ are in $\mathbb{Z}$ or in $\frac{1}{2}+\mathbb{Z}$, the possibilities for (c_i, c_j) are $(0,1)$ and $(\frac{1}{2},\frac{1}{2})$, and the possibilities for $(\widetilde{c}(\mathrm{box}_i), \widetilde{c}(\mathrm{box}_j))$ are $(0,1)$, $(0,-1)$, $(\frac{1}{2},\pm\frac{1}{2})$, or $(-\frac{1}{2},\pm\frac{1}{2})$.

(Case 1) $\widetilde{c}(\mathrm{box}_j) = 1$ and $\widetilde{c}(\mathrm{box}_i) = 0$.
- If box_j is NW of box_i then $0 < w(j) < w(i)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.
- If box_j is SE of box_i then $0 < w(i) < w(j)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.

(Case 2) $\widetilde{c}(\mathrm{box}_j) = -1$ and $\widetilde{c}(\mathrm{box}_i) = 0$.
- If box_j is NW of box_i then $-w(j) < w(i)$ so that $-w(i) < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.
- If box_j is SE of box_i then $-w(j) > w(i)$ so that $-w(i) > w(j)$ and $\varepsilon_j + \varepsilon_i \in J$.

(Case 3) $\widetilde{c}(\mathrm{box}_j) = \frac{1}{2}$ and $\widetilde{c}(\mathrm{box}_i) = \frac{1}{2}$.
- Then $0 < w(i) < w(j)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.

(Case 4) $\widetilde{c}(\mathrm{box}_j) = -\frac{1}{2}$ and $\widetilde{c}(\mathrm{box}_i) = \frac{1}{2}$.
- This case cannot occur since, when indexing the boxes of $\lambda/S^{(0)}$,
- the boxes of shifted content $-\frac{1}{2}$ are numbered before boxes of shifted content $\frac{1}{2}$.

(Case 5) $\widetilde{c}(\mathrm{box}_j) = \frac{1}{2}$ and $\widetilde{c}(\mathrm{box}_i) = -\frac{1}{2}$.
- If box_j is NW of box_i then $w(i) < 0$ and $w(j) < -w(i)$ so that $\varepsilon_j + \varepsilon_i \in J$.
- If box_j is SE of box_i then $w(i) < 0$ and $-w(i) < w(i)$ so that $\varepsilon_j + \varepsilon_i \notin J$.

(Case 6) $\widetilde{c}(\mathrm{box}_j) = -\frac{1}{2}$ and $\widetilde{c}(\mathrm{box}_i) = -\frac{1}{2}$.
- Then $0 < -w(j) < -w(i)$ and $w(j) < 0 < -w(i)$ so that $\varepsilon_j + \varepsilon_i \in J$.

This analysis shows that $J = R(w_S) \cap P(\mathbf{c}) = (R_1 \cap P_1) \sqcup (R_2 \cap P_2) \sqcup (R_3 \cap P_3)$ is as given in (5.29).

A consequence of the description of J in (5.29) is that $J = R(w_S) \cap P(\mathbf{c})$ is independent of the choice of $S \in \mathcal{T}_k^\lambda$. It follows that all composition factors of B_k^λ are isomorphic to $H_k^{(z,\mathbf{c},J)}$.

Let $S, T \in \mathcal{T}_k^\lambda$ such that v_S and v_T have the same eigenvalues for $W_0, \dots, W_k$. By definition of $\mathcal{T}_k^\lambda$, $S^{(k)} = T^{(k)} = \lambda$. Since

$$W_k v_S = -q^{\tilde{c}\left(S^{(k)}/S^{(k-1)}\right)} v_S = -q^{\tilde{c}\left(\lambda/S^{(k-1)}\right)} v_S$$

and

$$W_k v_T = -q^{\tilde{c}\left(T^{(k)}/T^{(k-1)}\right)} v_T = -q^{\tilde{c}\left(\lambda/T^{(k-1)}\right)} v_T,$$

we have $\widetilde{c}(\lambda/T^{(k-1)}) = \widetilde{c}(\lambda/S^{(k-1)})$ which implies that $T^{(k-1)} = S^{(k-1)}$. Using this and the fact that the eigenvalues of W_{k-1} on v_S and v_T are the same, implies similarly that $T^{(k-2)} = S^{(k-2)}$. Induction gives that

$$S^{(0)} = T^{(0)}, \quad \dots, \quad S^{(k)} = T^{(k)} \qquad \text{so that} \qquad S = T.$$

Thus $\dim((B_k^\lambda)_\gamma) \le 1$ (in the notation of (3.1)) and $B_k^\lambda \cong H_k^{(z,\mathbf{c},J)}$ as H_k^{ext}-modules. $\square$

In the course of the proof of Theorem 5.5 we have also established the following result, which deserves mention.

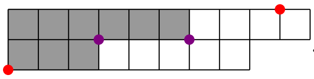
Corollary 5.6. *Keeping the notations of Theorem 5.5, let $\lambda \in \mathcal{P}^{(k)}$ and $S \in \mathcal{T}_k^\lambda$, and let w_S be the signed permutation defined in* (5.34)*. Then*

$$\begin{array}{ccc} \mathcal{T}_k^\lambda & \longrightarrow & \mathcal{F}^{(\mathbf{c},J)} \\ S & \longmapsto & w_S \end{array} \quad \textit{is a bijection.}$$

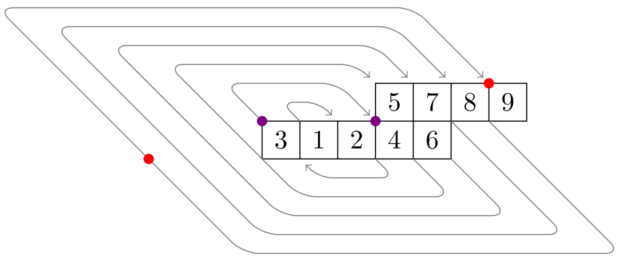
Example 5.7. Let $M = L(a^c) = L(6)$ and $N = L(b^d) = L(3)$ so that

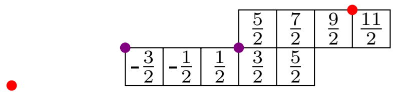
$$a = 6, \quad c = 1, \quad b = 3, \quad d = 1, \quad r_1 = \frac{3}{2}, \quad \text{and} \quad r_2 = \frac{11}{2}.$$

The partition $\lambda = (10, 8)$ is in $\mathcal{P}^{(k)}$ with $k = 9$. Then we draw λ as the (marked) partition

$\lambda = (10, 8) =$  . Here, $S_{\max}^{(0)} = (6, 3)$

is indicated by the shaded boxes. The boxes of $\lambda/S_{\max}^{(0)}$ have

indexing 

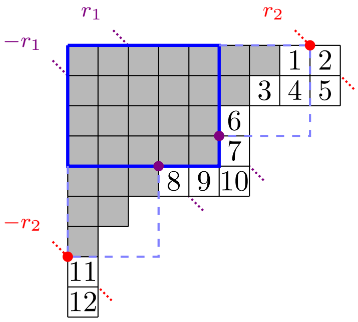
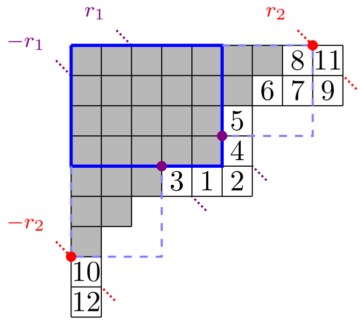
and shifted contents 

.

Example 5.8. Let $M = L(a^c) = L(5^4)$ and $N = L(b^d) = L(3^3)$ so that

$$a = 5, \quad c = 4, \quad b = 3 \quad d = 3, \quad r_1 = \frac{3}{2}, \quad \text{and} \quad r_2 = \frac{15}{2}.$$

The partition $\lambda = (9,9,6,6,6,2,1,1,1)$ is in $\mathcal{P}^{(k)}$ with $k = 12$. For this partition $S^{(0)}_{\max} = (7,6,5,5,3,2,1)$; and one tableau $S \in \mathcal{T}^\lambda_k$ with $S^{(0)} = S^{(0)}_{\max}$ (where the shaded portion of λ corresponds to $S^{(0)}$) is

$S =$  ; and 

indicates the indexing of the boxes in $\lambda/S^{(0)}_{\max}$. The contents of the boxes $S^{(i)}/S^{(i-1)}$ for $i = 1,\dots,k$ are $7,8,5,6,7,3,2,-1,0,1,-7,-8$; and since $-\frac{1}{2}(a-c+b-d) = -\frac{1}{2}$, the shifted contents $\widetilde{c}(S^{(i)}/S^{(i-1)})$ for $i = 1,\dots,k$ are

$$\frac{13}{2}, \frac{15}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{5}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{15}{2}, -\frac{17}{2},$$

respectively. The sum of the contents of the boxes in $S^{(0)}_{\max}$ is 1, the sum of the contents of the boxes in λ is 23, $c_0 = -\frac{1}{2}(12(5-4+3-3)+5\cdot 4(5-4)+3\cdot 3(3-3))+24 = 8$,

$$z = q^{16}, \qquad \text{and} \qquad \mathbf{c} = \left(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{9}{2},\frac{11}{2},\frac{13}{2},\frac{13}{2},\frac{15}{2},\frac{15}{2},\frac{17}{2}\right)$$

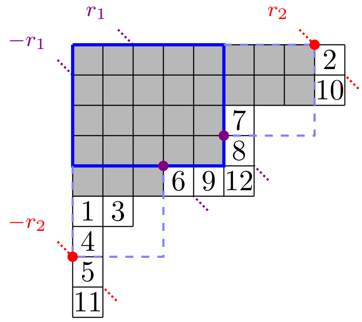
is the sequence of absolute values of the shifted contents, arranged in increasing order. Using (5.34),

$$w_S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ -9 & 10 & -8 & 7 & 6 & 3 & 4 & 1 & 5 & -11 & 2 & -12 \end{pmatrix},$$

$$P(\mathbf{c}) = \left\{ \begin{array}{l} \boxed{\varepsilon_3}, \varepsilon_4, \boxed{\varepsilon_{10}}, \varepsilon_{11}, \varepsilon_2 - \varepsilon_{-1}, \\ \varepsilon_3 - \varepsilon_1, \varepsilon_4 - \varepsilon_1, \boxed{\varepsilon_3 - \varepsilon_2}, \boxed{\varepsilon_4 - \varepsilon_2}, \varepsilon_5 - \varepsilon_3, \boxed{\varepsilon_5 - \varepsilon_4}, \\ \varepsilon_7 - \varepsilon_6, \boxed{\varepsilon_8 - \varepsilon_7}, \varepsilon_9 - \varepsilon_7, \\ \boxed{\varepsilon_{10} - \varepsilon_8}, \varepsilon_{11} - \varepsilon_8, \boxed{\varepsilon_{10} - \varepsilon_9}, \boxed{\varepsilon_{11} - \varepsilon_9}, \boxed{\varepsilon_{12} - \varepsilon_{10}}, \boxed{\varepsilon_{12} - \varepsilon_{11}}, \end{array} \right\},$$

$$R(w_S) = \left\{ \begin{array}{l} \varepsilon_1, \boxed{\varepsilon_3}, \boxed{\varepsilon_{10}}, \varepsilon_{12} \\ \varepsilon_{10} - \varepsilon_1, \varepsilon_{12} - \varepsilon_1, \boxed{\varepsilon_3 - \varepsilon_2}, \boxed{\varepsilon_4 - \varepsilon_2}, \varepsilon_5 - \varepsilon_2, \varepsilon_6 - \varepsilon_2, \varepsilon_7 - \varepsilon_2, \varepsilon_8 - \varepsilon_2, \\ \varepsilon_9 - \varepsilon_2, \varepsilon_{10} - \varepsilon_2, \varepsilon_{11} - \varepsilon_2, \varepsilon_{12} - \varepsilon_2, \\ \varepsilon_{10} - \varepsilon_3, \varepsilon_{12} - \varepsilon_3, \boxed{\varepsilon_5 - \varepsilon_4}, \varepsilon_6 - \varepsilon_4, \varepsilon_7 - \varepsilon_4, \varepsilon_8 - \varepsilon_4, \varepsilon_9 - \varepsilon_4, \varepsilon_{10} - \varepsilon_4, \\ \varepsilon_{11} - \varepsilon_4, \varepsilon_{12} - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_5, \varepsilon_8 - \varepsilon_5, \varepsilon_9 - \varepsilon_5, \varepsilon_{10} - \varepsilon_5, \varepsilon_{11} - \varepsilon_5, \\ \varepsilon_{12} - \varepsilon_5, \varepsilon_8 - \varepsilon_6, \varepsilon_{10} - \varepsilon_6, \varepsilon_{11} - \varepsilon_6, \varepsilon_{12} - \varepsilon_6, \\ \boxed{\varepsilon_8 - \varepsilon_7}, \varepsilon_{10} - \varepsilon_7, \varepsilon_{11} - \varepsilon_7, \varepsilon_{12} - \varepsilon_7, \boxed{\varepsilon_{10} - \varepsilon_8}, \varepsilon_{12} - \varepsilon_8, \\ \boxed{\varepsilon_{10} - \varepsilon_9}, \boxed{\varepsilon_{11} - \varepsilon_9}, \varepsilon_{12} - \varepsilon_9, \boxed{\varepsilon_{12} - \varepsilon_{10}}, \boxed{\varepsilon_{12} - \varepsilon_{11}}, \\ \varepsilon_3 + \varepsilon_1, \varepsilon_4 + \varepsilon_1, \varepsilon_5 + \varepsilon_1, \varepsilon_6 + \varepsilon_1, \varepsilon_7 + \varepsilon_1, \varepsilon_8 + \varepsilon_1, \varepsilon_9 + \varepsilon_1, \varepsilon_{10} + \varepsilon_1, \\ \varepsilon_{11} + \varepsilon_1, \varepsilon_{12} + \varepsilon_1, \ \varepsilon_{10} + \varepsilon_2, \varepsilon_{12} + \varepsilon_2, \\ \varepsilon_4 + \varepsilon_3, \varepsilon_5 + \varepsilon_3, \varepsilon_6 + \varepsilon_3, \varepsilon_7 + \varepsilon_3, \varepsilon_8 + \varepsilon_3, \varepsilon_9 + \varepsilon_3, \varepsilon_{10} + \varepsilon_3, \varepsilon_{11} + \varepsilon_3, \\ \varepsilon_{12} + \varepsilon_3, \varepsilon_{10} + \varepsilon_4, \varepsilon_{12} + \varepsilon_4, \varepsilon_{10} + \varepsilon_5, \varepsilon_{12} + \varepsilon_5, \varepsilon_{10} + \varepsilon_6, \varepsilon_{12} + \varepsilon_6, \varepsilon_{10} + \varepsilon_7, \\ \varepsilon_{12} + \varepsilon_7, \ \varepsilon_{10} + \varepsilon_8, \varepsilon_{12} + \varepsilon_8, \varepsilon_{10} + \varepsilon_9, \varepsilon_{12} + \varepsilon_9, \varepsilon_{11} + \varepsilon_{10}, \varepsilon_{12} + \varepsilon_{10}, \varepsilon_{12} + \varepsilon_{11} \end{array} \right\},$$

and $J = R(w_S) \cap P(\mathbf{c})$ consists of the outlined elements of $P(\mathbf{c})$ (which are the same as the outlined elements of $R(w_S)$). Another $T \in \mathcal{T}_k^\lambda$ is (again, with $T^{(0)}$ indicated by the shaded boxes)

$T =$ 

Keeping the setting of Theorem 5.5, Proposition 3.2 associates a configuration of $2k$ boxes to $(\mathbf{c}, J)$. This configuration can be described in terms of the data of $\lambda \in \mathcal{P}^{(k)}$ as follows. With $S^{(0)}_{\max}$ as defined just before (5.27), let $\mathrm{rot}(\lambda/S^{(0)}_{\max})$ be the $180°$ rotation of the skew shape $\lambda/S^{(0)}_{\max}$. Then the configuration of boxes κ corresponding to $(\mathbf{c}, J)$ is

$$\kappa = \mathrm{rot}\left(\lambda/S^{(0)}_{\max}\right) \cup \lambda/S^{(0)}_{\max}, \tag{5.35}$$

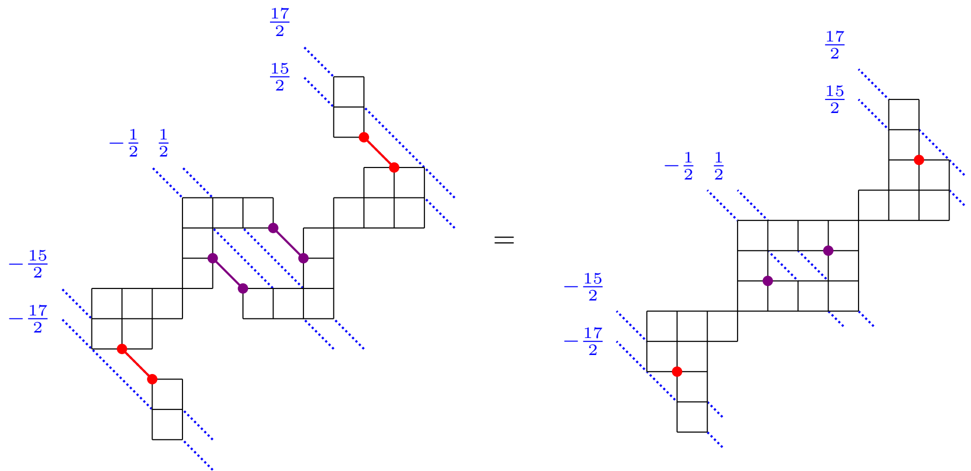
so that it is the (disjoint) union of two skew shapes $\lambda/S^{(0)}_{\max}$ and $\mathrm{rot}(\lambda/S^{(0)}_{\max})$, placed with

(a) $\mathrm{rot}(\lambda/S^{(0)})$ northwest of $\lambda/S^{(0)}$,

(b) $\lambda/S^{(0)}$ positioned so that the contents of its boxes are $(\widetilde{c}(S^{(1)}/S^{(0)}), \ldots, \widetilde{c}(S^{(k)}/S^{(k-1)}))$,

(c) $\mathrm{rot}(\lambda/S^{(0)})$ positioned so that the contents of its boxes are $(-\widetilde{c}(S^{(k)}/S^{(k-1)}), \ldots, -\widetilde{c}(S^{(1)}/S^{(0)}))$,

and with markings placed at the NE corners of the rectangles $\mathcal{B}$ and $\mathcal{B}'$ corresponding to $\lambda/S^{(0)}$ (in the notation of (5.14)). The resulting doubled skew shape is symmetric under the $180°$ rotation which sends a box on diagonal c_i to a box on diagonal $-c_i$. In the case of Example 5.8 the corresponding configuration of boxes is

$\kappa =$ 

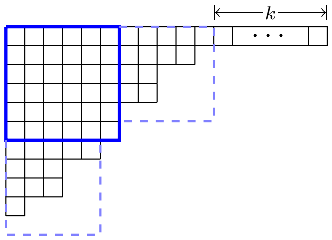
This configuration of boxes also appeared in Example 3.3.

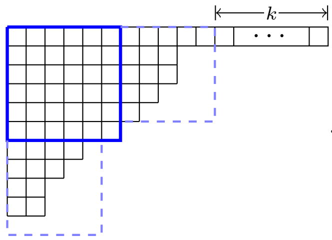
For generically large a, b, c, d, there will be examples of $\lambda, \mu \in \mathcal{P}^{(k)}$ with $\lambda \neq \mu$ and $B_k^\lambda \cong B_k^\mu$ as H_k^{ext}-modules; see Example 5.9. This is because the eigenvalues of P on

$M \otimes N$ are not sufficient to distinguish the components of $M \otimes N$ as a $\mathfrak{gl}_n$-module. It could be helpful to further extend H_k^{ext} and consider an algebra $Z(U_q\mathfrak{gl}_n) \otimes H_k$ acting on $M \otimes N \otimes V^{\otimes k}$.

Example 5.9. Let $a = c = 6$ and $b = d = 4$,

$$\lambda(k) = (11+k, 10, 8, 8, 6, 6, 5, 3, 3, 1) \qquad \text{and} \qquad \mu(k) = (11+k, 9, 9, 8, 7, 6, 4, 3, 2, 2), \text{ i.e.}$$

$\lambda(k) =$ 

and $\mu(k) =$ 

.

Then $\lambda(k) \neq \mu(k)$ but, as H_k^{ext}-modules,

$$B_k^{\lambda(k)} \cong B_k^{\mu(k)} \cong H_k^{(z,\mathbf{c},\emptyset)}, \quad \text{where } \mathbf{c} = (11, 12, \ldots, 11+k-1) \text{ and } z = q^{28+k(k+21)}.$$

Recall from (5.22) that

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B_k^{\lambda} \qquad \text{as } \left(U_q\mathfrak{g}, \mathcal{H}_k^{\mathrm{ext}}\right)\text{-bimodules.}$$

A consequence of Theorem 3.5 (b) is the following construction of the irreducible H_k^{ext}-modules B_k^{λ}. Keeping the setting and notation of (5.31), for $\lambda \in \mathcal{P}^{(k)}$ and $S \in \mathcal{T}_k^{\lambda}$, let

$$s_jS \text{ be the path from } (a^c) \text{ to } \lambda \text{ that differs from } S \text{ only at } S^{(j)}. \tag{5.36}$$

The path s_jS is unique if it exists: if $S = ((a^c) \to S^{(0)} \to S^{(1)} \to \cdots \to S^{(k)})$ then $S^{(j+1)}$ is obtained by adding a box to $S^{(j)}$, and $(s_jS)^{(j)}$ is obtained by moving a box of $S^{(j)}$ to the position of the added box in $S^{(j+1)}$. In the case that $j = 0$, the paths s_0S and S satisfy $(s_0S)^{(1)} = S^{(1)}$ and the partitions $(s_0S)^{(0)}$ and $S^{(0)}$ in $\mathcal{P}^{(0)}$ differ by the placement of one box, with

$$\widetilde{c}\left((s_0S)^{(1)}/(s_0S)^{(0)}\right) = -\widetilde{c}\left(S^{(1)}/S^{(0)}\right), \tag{5.37}$$

where the shifted content of a box $\widetilde{c}(\text{box})$ is as defined in (5.27).

Corollary 5.10. *Keep the conditions of Theorems 5.4 and 5.5. In particular, assume that the generality conditions of (5.21) hold so that q is not a root of unity, $a, b, c, d \in \mathbb{Z}_{>0}$ and $(a+c)-(b+d) \notin \{0, \pm 1, \pm 2\}$. Let $\lambda \in \mathcal{P}^{(k)}$. Then B_k^{λ} has a basis $\{v_S \mid S \in \mathcal{T}_k^{\lambda}\}$ such that the H_k^{ext}-action is given by*

$$Pv_S = q^{2e_0(T)}v_S, \qquad Z_iv_S = q^{2c\left(S^{(i)}/S^{(i-1)}\right)}v_S,$$

$$T_iv_S = [T_i]_{S,S}v_S + \sqrt{-([T_i]_{S,S} - q)\left([T_i]_{S,S} + q^{-1}\right)}\; v_{s_iS}, \qquad \textit{for } i = 1, \ldots, k-1,$$

$$Y_1v_S = [Y_1]_{S,S}v_S + \sqrt{-\left([Y_1]_{S,S} - q^{-2d}\right)\left([Y_1]_{S,S} - q^{2b}\right)}\; v_{s_0S},$$

$$X_1v_S = [X_1]_{S,S}v_S + q^{-2c\left(S^{(1)}/S^{(0)}\right)}q^{(a-c+b-d)}\sqrt{-\left([X_1]_{S,S} - q^{2a}\right)\left([X_1]_{S,S} - q^{-2c}\right)}\; v_{s_0S},$$

where $v_{s_jS} = 0$ *if* s_jS *does not exist, and*

$$[T_i]_{S,S} = \frac{q-q^{-1}}{1-q^{2\left(c\left(S^{(i)}/S^{(i-1)}\right)-c\left(S^{(i+1)}/S^{(i)}\right)\right)}},$$

$$[Y_1]_{S,S} = \frac{\left(q^{2b}+q^{-2d}\right)-\left(q^{2a}+q^{-2c}\right)q^{2(b-d)}q^{-2c\left(S^{(1)}/S^{(0)}\right)}}{1-q^{2(a-c+b-d)}q^{-4c\left(S^{(1)}/S^{(0)}\right)}},$$

$$[X_1]_{S,S} = \frac{\left(q^{2a}+q^{-2c}\right)-\left(q^{2b}+q^{-2d}\right)q^{2(a-c)}q^{-2c\left(S^{(1)}/S^{(0)}\right)}}{1-q^{2(a-c+b-d)}q^{-4c\left(S^{(1)}/S^{(0)}\right)}}.$$

Proof. The appropriate basis of B_k^λ is the one given in Proposition 5.2 and used also in the proof of Theorem 5.5. It is only necessary to convert from the notation v_w in Theorem 3.5 to the notation v_S using the bijection in Corollary 5.6. Recall from (5.20) that

$$a_1 = q^{2a}, \qquad a_2 = q^{-2c}, \qquad b_1 = q^{2b}, \qquad b_2 = q^{-2d},$$

$$t^{\frac{1}{2}} = q, \qquad t_k^{\frac{1}{2}} = a_1^{\frac{1}{2}}(-a_2)^{-\frac{1}{2}} = -iq^{a+c}, \quad \text{and} \quad t_0^{\frac{1}{2}} = b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}} = -iq^{b+d}.$$

From (3.12) and (5.32),

$$\gamma_{w^{-1}(i)}v_w = \Phi(W_i)v_S = -q^{-(a-c+b-d)}q^{2c\left(S^{(i)}/S^{(i-1)}\right)}v_S.$$

From (2.18), (2.9), and (h), $Y_1 = b_1^{\frac{1}{2}}(-b_2)^{\frac{1}{2}}T_0 = iq^{b-d}T_0$ and $X_1 = (a_1+a_2)-a_1a_2Y_1Z_1^{-1} = q^{2a}+q^{-2c}-q^{2(a-c)}Y_1Z_1^{-1}$. With these conversions, the formulas from (3.13) and (3.14) become

$$T_iv_S = T_iv_w = [T_i]_{S,S}v_S + [T_i]_{s_iS,S}v_{s_iS}, \qquad \text{for } i=1,\dots,k-1,$$

$$Y_1v_S = iq^{b-d}T_0v_w = [Y_1]_{S,S}v_S + [Y_1]_{s_0S,S}v_{s_0S}\ ,$$

and

$$\begin{aligned} X_1v_S &= \left(q^{2a}+q^{-2c}-q^{2(a-c)}Y_1Z_1^{-1}\right)v_S = \left(q^{2a}+q^{-2c}-q^{2(a-c)}q^{-2c\left(S^{(1)}/S^{(0)}\right)}Y_1\right)v_S \\ &= [X_1]_{S,S}v_S - [X_1]_{s_0S,S}v_{s_0S}, \end{aligned}$$

with

$$[T_i]_{S,S} = [T_i]_{ww} = \frac{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}{1-\gamma_{w^{-1}(i)}\gamma^{-1}_{w^{-1}(i+1)}} = \frac{q-q^{-1}}{1-q^{2\left(c\left(S^{(i)}/S^{(i-i)}\right)-c\left(S^{(i+1)}/S^{(i)}\right)\right)}},$$

$$\begin{aligned} [Y_1]_{S,S} &= iq^{b-d}[T_0]_{ww} = iq^{b-d}\frac{\left(t_0^{\frac{1}{2}}-t_0^{-\frac{1}{2}}\right)+\left(t_k^{\frac{1}{2}}-t_k^{-\frac{1}{2}}\right)\gamma^{-1}_{w^{-1}(1)}}{1-\gamma^{-2}_{w^{-1}(1)}} \\ &= iq^{b-d}(-i)\frac{\left(q^{(b+d)}+q^{-(b+d)}\right)-\left(q^{(a+c)}+q^{-(a+c)}\right)q^{a-c+b-d}q^{-2c\left(S^{(1)}/S^{(0)}\right)}}{1-q^{2(a-c+b-d)}q^{-4c\left(S^{(1)}/S^{(0)}\right)}} \\ &= \frac{\left(q^{2b}+q^{-2d)}\right)-\left(q^{2a}+q^{-2c}\right)q^{2(b-d)}q^{-2c\left(S^{(1)}/S^{(0)}\right)}}{1-q^{2(a-c+b-d)}q^{-4c\left(S^{(1)}/S^{(0)}\right)}}, \end{aligned}$$

and

$$\begin{aligned}[X_1]_{S,S} &= q^{2a}+q^{-2c}-q^{2(a-c)}q^{-2c\left(S^{(1)}/S^{(0)}\right)}[Y_1]_{S,S} \\ &= q^{2a}+q^{-2c}-q^{2(a-c)}q^{-2c\left(S^{(1)}/S^{(0)}\right)} \\ &\qquad \frac{\left(q^{2b}+q^{-2d}\right)-\left(q^{2a}+q^{-2c}\right)q^{2(b-d)}q^{-2c\left(S^{(1)}/S^{(0)}\right)}}{1-q^{2(a-c+b-d)}q^{-4c\left(S^{(1)}/S^{(0)}\right)}} \\ &= \frac{\left(q^{2a}+q^{-2c}\right)-\left(q^{2b}+q^{-2d}\right)q^{2(a-c)}q^{-2c\left(S^{(1)}/S^{(0)}\right)}}{1-q^{2(a-c+b-d)}q^{-4c\left(S^{(1)}/S^{(0)}\right)}}.\end{aligned}$$

On the two-dimensional subspace $\mathrm{span}_{\mathbb{C}}\{v_S, v_{s_0S}\}$ the action of T_0 in the basis $\{v_S, v_{s_0S}\}$ is a symmetric matrix $[T_0]$, and so the matrix of Y_1 in this basis is $[Y_1]=iq^{b-d}[T_0]$ is also symmetric. The action of Z_1 is by a diagonal matrix $[Z_1]$, so $[Z_1]^t=[Z_1]$. Therefore, using $X_1=Z_1Y_1^{-1}$ from (2.9) and $X_1=(a_1+a_2)-a_1a_2X_1^{-1}$ from (h), we have $([X_1]^{-1})^t = ([Y_1][Z_1]^{-1})^t=([Z_1]^{-1})^t[Y_1]^t=[Z_1]^{-1}[Y_1]$ and so

$$[Z_1][X_1]^t[Z_1]^{-1}=[Z_1]\left((a_1+a_2)-a_1a_2[Z_1]^{-1}[Y_1]\right)[Z_1]^{-1}=[X_1].$$

Thus

$$[Z_1]_{S,S}[X_1]_{s_0S,S}[Z_1^{-1}]_{s_0L,s_0S}=[X_1]_{S,s_0S}$$

and

$$-[X_1]_{S,s_0S}[X_1]_{s_0S,S}=([X_1]_{S,S}-a_1)([X_1]_{S,S}-a_2),$$

since $[X_1]$ is a 2×2 matrix with eigenvalues a_1 and a_2 (as in the proof of Theorem 3.5). Thus

$$\begin{aligned}[X_1]_{s_0S,S} &= \sqrt{([X_1]_{s_0S,S})^2}=\sqrt{[X_1]_{S,s_0S}[Z_1]^{-1}_{S,S}[X_1]_{s_0S,S}[Z_1]_{s_0S,s_0S}} \\ &= \sqrt{[Z_1]^{-1}_{S,S}[Z_1]_{s_0Ss_0S}}\sqrt{-([X_1]_{S,S}-q^{2a})([X_1]_{S,S}-q^{-2c})}.\end{aligned}$$

By (5.37), $c((s_0S)^{(1)}/(s_0S)^{(0)})=-c(S^{(1)}/S^{(0)})+(a-c+b-d)$, so that

$$\sqrt{[Z_1]^{-1}_{S,S}[Z_1]_{s_0S,s_0S}}=q^{-c\left(S^{(1)}/S^{(0)}\right)}q^{c\left((s_0S)^{(1)}/(s_0S)^{(0)}\right)}=q^{-2c\left(S^{(1)}/S^{(0)}\right)+(a-c+b-d)}.$$

Thus

$$[X_1]_{s_0S,S}=q^{-2c\left(S^{(1)}/S^{(0)}\right)}q^{(a-c+b-d)}\sqrt{-\left([X_1]_{S,S}-q^{2a}\right)\left([X_1]_{S,S}-q^{-2c}\right)}.$$

□

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