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Zajj Daugherty and Arun Ram

In memory of a friend and an inspiration, Vladimir Rittenberg 1934–2018.

Abstract. The two boundary Temperley–Lieb algebra TL_k is a quotient of the type C_k affine Hecke algebra H_k. The algebra H_k has a diagrammatic presentation by braids with k strands and two poles and TL_k has a presentation via non-crossing diagrams with boundaries. The algebra TL_k plays a role in the analysis of Heisenberg spin chains with boundaries. A calibrated representation of TL_k is a TL_k-module for which all the Murphy elements (integrals) are simultaneously diagonalizable. In this paper we give a combinatorial classification and construction of all irreducible calibrated TL_k-modules and explain how these modules also arise from a Schur–Weyl duality with the quantum group $U_q\mathfrak{gl}_2$.

1. Introduction

The paper [3] studied the calibrated representations of affine Hecke algebras of type C with unequal parameters and developed their combinatorics and their role in Schur–Weyl duality. This paper applies that information to the study of two boundary Temperley–Lieb algebras. The two boundary Temperley–Lieb algebras appear in statistical mechanics for analysis of spin chains with generalized boundary conditions [5, 6]. Knowledge of the representation theory of the two boundary Temperley–Lieb algebras is useful for the determination of the spectrum of the Hamiltonian for these spin chains with boundaries. In fact, the need to understand the representation theory of the two boundary Temperley–Lieb algebra better was a primary motivation for our preceding papers [2, 3] on two boundary Hecke algebras.

In Section 2 we review the definition and structure of the two boundary Hecke algebra H_k (the affine Hecke algebra of type C with unequal parameters). Then we carefully

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analyze certain idempotents which, as we prove in Theorem 3.1, generate the ideal that one must quotient by to obtain the two boundary Temperley–Lieb algebra TL_k from the two boundary Hecke algebra H_k. It is the expression of these idempotents in terms of the intertwiner presentation of H_k that eventually provides understanding of the weights that can appear in TL_k-modules.

In Section 3 we define the two boundary Temperley–Lieb algebra TL_k and the symplectic blob algebra $TL_k(b)$ following [4, 8, 9, 10, 12, 15, 16] and review the diagram algebra calculus for these algebras. Part of our contribution is to extend this calculus to make its connection to the diagrammatic calculus of the Hecke algebra H_k via braids.

Comparing (3.8) and Corollary 3.3 gives that the *symplectic blob algebras* $TL_k(b)$ for $b \in \mathbb{C}$ and the two boundary Temperley–Lieb algebra TL_k produce the same irreducible representations. The symplectic blob algebras function as quotients of the two boundary Temperley–Lieb algebra TL_k by a central character. The key computation appears in Theorem 3.2 where we use the diagrammatics to give a proof of a result of [4] that provides an expansion of a certain central element of H_k inside TL_k. Using the Hecke algebra point of view, this result enables us to understand that the center of TL_k is a polynomial ring in one variable $Z(TL_k) = \mathbb{C}[Z]$, and that TL_k is of finite rank over this center. In retrospect, the algebra H_k has a similar structure and so perhaps this should not be surprising but, nonetheless, it is pleasant to see it come out in such a vivid and explicit form. Let us clarify that, although the realisation that $Z(TL_k) = \mathbb{C}[Z]$ is visible from Theorem 3.2 and the indexing of irreducible representations by central character that one gets from the Hecke algebra, we have not written a proper proof of this statement in this paper.

We have used a different normalization of the parameters of the two boundary Hecke and Temperley–Lieb algebra from those used in [4, 9]. Our normalization will be helpful, for example, for future applications of these algebras to the theory of Macdonald polynomials and to the study of the exotic nilpotent cone. In both of these cases the affine Hecke algebra of type C_n plays an important role: the Koornwinder polynomials are the Macdonald polynomials for type $(C_n^\vee, C_n)$ [14], and the K-theory of the Steinberg variety of the exotic nilpotent cone provides a geometric construction of the representations of the two boundary Hecke and Temperley–Lieb algebras at unequal parameters (see [11]).

The calibrated representations are the irreducible representations of the two boundary Hecke algebra for which a large family of commuting operators (integrals, or Murphy elements) have a simple (joint) spectrum. This property makes these representations particularly attractive, and the detailed combinatorics of these representations has been worked out in [3]. In Section 4 we use the detailed analysis of the idempotents done in Section 2 to determine exactly which calibrated irreducible representations of the two boundary Hecke algebra are representations of the two boundary Temperley–Lieb algebra (Theorem 4.1). In consequence, we obtain a full classification of the calibrated irreducible representations of the two boundary Temperley–Lieb algebras. Theorem 4.1 proves that the calibrated irreducible representations are classified by triples $(z, \mathbf{c}, J)$ which convert (via the conversion in [3, § 3.1]) to a (possibly marked) skew shape with ≤ 2 rows. By [3, Theorem 3.5], the dimension and structure of the representation indexed by $(z, \mathbf{c}, J)$ is the same for many different choices of z and $\mathbf{c}$; it depends only on three sets $Z(\mathbf{c})$, $P(\mathbf{c})$ and J.

As explained in [3], there is a Schur–Weyl type duality between the two boundary Hecke algebra and the quantum group $U_q\mathfrak{gl}_n$. The classical Schur–Weyl duality between $U_q\mathfrak{gl}_n$ and the finite Hecke algebra of type A becomes a Schur–Weyl duality for the finite

Temperley–Lieb algebra when $n = 2$. In Theorem 5.1 we show that at $n = 2$ the Schur–Weyl duality of [3] gives a Schur–Weyl duality for the two boundary Temperley–Lieb algebra. This method (coming from R-matrices for the quantum group $U_q\mathfrak{gl}_2$) provides calibrated representations of the two boundary Temperley–Lieb algebra TL_k. Using our results from Section 4, we determine exactly which irreducible calibrated representations of TL_k occur in the Schur–Weyl duality context. A consequence of the result of Theorem 5.1 is that (for $t_k \neq t_0$ and $t_k \neq t_0 t^{\pm 1}$ and $t_k \neq t^{\pm 2}$ and t^2 is not a root of unity) a representative of each of the possible structures of calibrated irreducible TL_k representations (determined by the skew local region $(\mathbf{c}, J)$) does appear in the Schur–Weyl duality context.

2. THE TWO BOUNDARY HECKE ALGEBRA H_k

The two boundary Hecke algebra is sometimes called the affine Hecke algebra of type $(C^\vee, C)$. In this section we shall follow our previous paper [3] for the extended affine Hecke algebra H_k^{ext} of type C_k and define idempotents

$$p_i^{(1^3)}, p_0^{(\emptyset,1^2)}, p_0^{(1^2,\emptyset)}, p_{0^\vee}^{(\emptyset,1^2)}, p_{0^\vee}^{(1^2,\emptyset)},$$

which we will need to quotient by in order to obtain the two boundary Temperley–Lieb algebra. We derive expressions of these elements in terms of the different choices of generators: the braid generators T_i, the cap/cup generators e_i, and the intertwiner generators τ_i and W_j.

The *affine Hecke algebra of type C with unequal parameters* H_k is the subalgebra of H_k^{ext} generated by $T_0, \ldots, T_k$ which satisfy braid relations for the affine Dynkin diagram of type C and the quadratic relations

$$\begin{gathered}\left(T_0 - t_0^{\frac{1}{2}}\right)\left(T_0 + t_0^{-\frac{1}{2}}\right) = 0, \quad \left(T_i - t^{\frac{1}{2}}\right)\left(T_i + t^{-\frac{1}{2}}\right) = 0, \\ \left(T_k - t_k^{\frac{1}{2}}\right)\left(T_k + t_k^{-\frac{1}{2}}\right) = 0,\end{gathered} \tag{2.1}$$

for $i \in \{1, \ldots, k-1\}$. The larger algebra H_k^{ext} also contains an element W_0 and *Murphy elements for* H_k^{ext} given by

$$W_1 = T_1^{-1} T_2^{-1} \cdots T_{k-1}^{-1} T_k T_{k-1} \cdots T_2 T_1 T_0 \quad \text{and} \quad W_j = T_{j-1} W_{j-1} T_{j-1},$$

for $j \in \{2, \ldots, k\}$. These elements $W_0, W_1, \ldots, W_k$ commute with each other and satisfy relations with the $T_0, \ldots, T_k$ as in [3, Theorem 2.2]. The *intertwining operators* for H_k^{ext} are

$$\tau_0 = T_0 - \frac{\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right) + \left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right) W_1^{-1}}{1 - W_1^{-2}}, \quad \text{and} \quad \tau_i = T_i - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - W_i W_{i+1}^{-1}}, \tag{2.2}$$

for $i \in \{1, \ldots, k-1\}$, and satisfy

$$\tau_0 W_1 = W_1^{-1} \tau_0 \qquad \text{and} \qquad \tau_0 W_j = W_j \tau_0 \text{ for } j \neq 1; \tag{2.3}$$

for $i = 1, \ldots, k-1$,

$$\begin{aligned} \tau_i W_i = W_{i+1} \tau_i \quad &\text{and} \quad \tau_i W_{i+1} = W_i \tau_i \quad \text{for} \quad i > 0, \\ &\text{and} \quad \tau_i W_j = W_j \tau_i \quad \text{for} \quad j \neq i, i+1; \end{aligned} \tag{2.4}$$

(see [3, Proposition 2.5]).

Let $a, a_0, a_k \in \mathbb{C}^\times$ and define

$$a_0 e_0 = T_0 - t_0^{\frac{1}{2}}, \qquad a e_i = T_i - t^{\frac{1}{2}}, \qquad a_k e_k = T_k - t_k^{\frac{1}{2}}, \tag{2.5}$$

for $i \in \{1, \ldots, k-1\}$. The relations in (2.1) are equivalent to

$$T_0 e_0 = -t_0^{-\frac{1}{2}} e_0, \qquad T_i e_i = -t^{-\frac{1}{2}} e_i, \qquad T_k e_k = -t_k^{-\frac{1}{2}} e_k, \tag{2.6}$$

and to

$$e_0^2 = \frac{-\left(t_0^{\frac{1}{2}} + t_0^{-\frac{1}{2}}\right)}{a_0} e_0, \quad e_i^2 = \frac{-\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right)}{a} e_i, \quad e_k^2 = \frac{-\left(t_k^{\frac{1}{2}} + t_k^{-\frac{1}{2}}\right)}{a_k} e_k, \tag{2.7}$$

for $i \in \{1, \ldots, k-1\}$.

Remark 2.1. The coefficients a, a_0, and a_k are chosen somewhat differently across the literature on Temperley–Lieb algebras, and we keep them general here to match to any given convention. Some favorite choices for a, a_0, and a_k can traced to the following computations.

For $i \in \{1, \ldots, k-2\}$, using $T_i = a e_i + t^{\frac{1}{2}}$ to expand $T_i T_{i+1} T_i$ and $T_{i+1} T_i T_{i+1}$ in terms of the e_i shows that in the presence of the relations (2.1),

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{is equivalent to} \quad a^3 e_i e_{i+1} e_i - a e_i = a^3 e_{i+1} e_i e_{i+1} - a e_{i+1}.$$

Similarly, $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$ is equivalent to

$$a_0^2 a^2 e_0 e_1 e_0 e_1 - a_0 a \left(t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}\right) e_0 e_1 = a_0^2 a^2 e_1 e_0 e_1 e_0 - a_0 a \left(t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}\right) e_1 e_0.$$

In the case that $a_0^2 a^2 = a_0 a (t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}})$ then

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0 \quad \text{is equivalent to} \quad e_0 e_1 e_0 e_1 - e_0 e_1 = e_1 e_0 e_1 e_0 - e_1 e_0.$$

In the case that $a^3 = a$ then

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{is equivalent to} \quad e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1}.$$

This is the explanation for why conventions often fix a, a_0 and a_k to satisfy

$$a = \pm 1, \quad a_0 a = t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}} = \left[\!\left[t_0 t^{-1} \right]\!\right] \quad \text{and} \quad a_k a = t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}} = \left[\!\left[t_k t^{-1} \right]\!\right],$$

where we use the notation

$$[x] = \frac{t^{\frac{x}{2}} - t^{-\frac{x}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \quad \text{and} \quad [\![t^s]\!] = \left(t^{\frac{s}{2}} + t^{-\frac{s}{2}}\right) = \left(\frac{t^s - t^{-s}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}\right)\left(\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{\frac{s}{2}} - t^{-\frac{s}{2}}}\right) = \frac{[2s]}{[s]}. \tag{2.8}$$

In order to make the statement of Proposition 2.2 (below) more manageable, we invoke the notation associated to the type C root system appearing in [3, § 3] and let

$$\begin{aligned} f_{2\varepsilon_i} &= 1 - W_i^{-2}, & & \\ f_{\varepsilon_i - r_2} &= \left(1 - t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} W_i^{-1}\right), & f_{\varepsilon_i - r_1} &= \left(1 + t_0^{\frac{1}{2}} t_k^{-\frac{1}{2}} W_i^{-1}\right), \\ f_{-\varepsilon_i - r_2} &= \left(1 - t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} W_i\right), & f_{-\varepsilon_i - r_1} &= \left(1 + t_0^{\frac{1}{2}} t_k^{-\frac{1}{2}} W_i\right), \\ f_{\varepsilon_i - \varepsilon_j} &= 1 - W_i W_j^{-1}, & f_{\varepsilon_i - \varepsilon_j + 1} &= 1 - t W_i W_j^{-1}, \\ f_{-\varepsilon_i - \varepsilon_j} &= 1 - W_i^{-1} W_j^{-1}, & f_{-\varepsilon_i - \varepsilon_j + 1} &= 1 - t W_i^{-1} W_j^{-1}, \end{aligned} \tag{2.9}$$

for $i, j \in \{1, \dots, k\}$. Then

$$a_0 e_0 = \tau_0 - t_0^{-\frac{1}{2}} \frac{f_{\varepsilon_1 - r_1} f_{\varepsilon_1 - r_2}}{f_{2\varepsilon_1}} \quad \text{and} \quad a_i e_i = \tau_i - t^{-\frac{1}{2}} \frac{f_{\varepsilon_i - \varepsilon_{i+1} + 1}}{f_{\varepsilon_i - \varepsilon_{i+1}}}, \tag{2.10}$$

and equations (2.42) and (2.43) from [3, Proposition 2.5] become

$$\tau_0^2 = W_1^{-2} t_0^{-1} \frac{f_{\varepsilon_1 - r_1} f_{-\varepsilon_1 - r_1} f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_2}}{f_{2\varepsilon_1}^2}, \quad \text{and} \quad \tau_i^2 = t^{-1} \frac{f_{\varepsilon_i - \varepsilon_{i+1} + 1} f_{\varepsilon_{i+1} - \varepsilon_i + 1}}{f_{\varepsilon_i - \varepsilon_{i+1}} f_{\varepsilon_{i+1} - \varepsilon_i}},$$

for $i = 1, \dots, k-1$.

2.1. The idempotents $p_i^{(1^3)}$, $p_0^{(\emptyset,1^2)}$ and $p_0^{(1^2,\emptyset)}$.

Fix $i \in \{1, \dots, k-2\}$. Let

$HS_3^{(i)}$ be the subalgebra of H_k^{ext} generated by T_i and T_{i+1}, and let
HB_2 be the subalgebra of H_k^{ext} generated by T_0 and T_1.

Define elements $p_i^{(1^3)} \in HS_3^{(i)}$ and $p_0^{(\emptyset,1^2)}, p_0^{(1^2,\emptyset)} \in HB_2$ by

$$\left(p_i^{(1^3)}\right)^2 = p_i^{(1^3)}, \qquad \left(p_0^{(\emptyset,1^2)}\right)^2 = p_0^{(\emptyset,1^2)} \qquad \left(p_0^{(1^2,\emptyset)}\right)^2 = p_0^{(1^2,\emptyset)}, \tag{2.11}$$

and

$$\begin{aligned} T_i p_i^{(1^3)} &= -t^{-\frac{1}{2}} p_i^{(1^3)}, & T_{i+1} p_i^{(1^3)} &= -t^{-\frac{1}{2}} p_i^{(1^3)}, \\ T_0 p_0^{(\emptyset,1^2)} &= -t_0^{-\frac{1}{2}} p_0^{(\emptyset,1^2)} & T_1 p_0^{(\emptyset,1^2)} &= -t^{-\frac{1}{2}} p_0^{(\emptyset,1^2)}, \\ T_0 p_0^{(1^2,\emptyset)} &= t_0^{\frac{1}{2}} p_0^{(1^2,\emptyset)}, & T_1 p_0^{(1^2,\emptyset)} &= -t^{-\frac{1}{2}} p_0^{(1^2,\emptyset)}. \end{aligned} \tag{2.12}$$

The idempotent $p_i^{(1^3)}$ projects onto the irreducible $HS_3^{(i)}$-module indexed by the partition $(1^3) = (1,1,1)$ and the idempotents $p_0^{(\emptyset,1^2)}$ and $p_0^{(1^2,\emptyset)}$ in HB_2 project onto the irreducible $HS_3^{(i)}$-modules indexed by $(\emptyset, 1^2)$ and $(1^2, \emptyset)$, respectively. The conditions in (2.12) are equivalent to

$$\begin{aligned} ae_i p_i^{(1^3)} &= -\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) p_i^{(1^3)}, & ae_{i+1} p_i^{(1^3)} &= -\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) p_i^{(1^3)}, \\ a_0 e_0 p_0^{(\emptyset,1^2)} &= -\left(t_0^{\frac{1}{2}} + t_0^{-\frac{1}{2}}\right) p_0^{(\emptyset,1^2)}, & ae_1 p_0^{(\emptyset,1^2)} &= -\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) p_0^{(\emptyset,1^2)}, \\ a_0 e_0 p_0^{(1^2,\emptyset)} &= 0, & ae_1 p_0^{(1^2,\emptyset)} &= -\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) p_0^{(1^2,\emptyset)}. \end{aligned} \tag{2.13}$$

The following proposition gives an expansion of these projection operators in terms of three important families of generators of H_k^{ext}: the braid T_i generators, the cup/cap e_i generators and the intertwining operators τ_i. As we will see in Section 3.1, the two boundary Temperley–Lieb algebra is precisely a quotient of H_k^{ext} by the ideal generated by these idempotents.

Proposition 2.2. *Let* $p_i^{(1^3)}$, $p_0^{(\emptyset,1^2)}$ *and* $p_0^{(1^2,\emptyset)}$ *be as defined in* (2.11) *and* (2.12) *and let*

$$N = t^{-\frac{1}{2}}(1+t)\left(1+t+t^2\right) \quad \textit{and} \quad N_0 = t_0^{-1} t^{-1} (1+t_0)(1+t)(1+t_0 t).$$

Then the expansions of these idempotents in terms of the three favored generating sets is given by

$$\begin{aligned}
Np_i^{(1^3)} &= T_iT_{i+1}T_i - t^{\frac{1}{2}}T_iT_{i+1} - t^{\frac{1}{2}}T_{i+1}T_i + tT_i + tT_{i+1} - t^{\frac{3}{2}} \\
&= a^3e_ie_{i+1}e_i - ae_i = a^3e_{i+1}e_ie_{i+1} - ae_{i+1} \\
&= \tau_i\tau_{i+1}\tau_i - t^{-\frac{1}{2}}\tau_{i+1}\tau_i\frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}} - t^{-\frac{1}{2}}\tau_i\tau_{i+1}\frac{f_{\varepsilon_{i+1}-\varepsilon_i+1}}{f_{\varepsilon_{i+1}-\varepsilon_i}} \\
&\qquad + t^{-1}\tau_i\frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}f_{\varepsilon_{i+2}-\varepsilon_i+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}f_{\varepsilon_{i+2}-\varepsilon_i}} + t^{-1}\tau_{i+1}\frac{f_{\varepsilon_{i+2}-\varepsilon_i+1}f_{\varepsilon_{i+1}-\varepsilon_i+1}}{f_{\varepsilon_{i+2}-\varepsilon_i}f_{\varepsilon_{i+1}-\varepsilon_i}} \\
&\qquad - t^{-\frac{3}{2}}\frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}f_{\varepsilon_{i+2}-\varepsilon_i+1}f_{\varepsilon_{i+1}-\varepsilon_i+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}f_{\varepsilon_{i+2}-\varepsilon_i}f_{\varepsilon_{i+1}-\varepsilon_i+1}},
\end{aligned}$$

$$\begin{aligned}
N_0p_0^{(\emptyset,1^2)} &= T_0T_1T_0T_1 - t_0^{\frac{1}{2}}T_1T_0T_1 - t^{\frac{1}{2}}T_0T_1T_0 \\
&\qquad + t_0^{\frac{1}{2}}t^{\frac{1}{2}}T_0T_1 + t_0^{\frac{1}{2}}t^{\frac{1}{2}}T_1T_0 - t_0t^{\frac{1}{2}}T_1 - t_0^{\frac{1}{2}}tT_0 + t_0t \\
&= a_0^2a^2e_0e_1e_0e_1 - a_0a\left(t_0^{-\frac{1}{2}}t^{\frac{1}{2}} + t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\right)e_0e_1 \\
&= a_0^2a^2e_1e_0e_1e_0 - a_0a\left(t_0^{-\frac{1}{2}}t^{\frac{1}{2}} + t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\right)e_1e_0 \\
&= \tau_0\tau_1\tau_0\tau_1 - t_0^{\frac{1}{2}}\tau_1\tau_0\tau_1\frac{f_{\varepsilon_1-r_2}f_{\varepsilon_1-r_1}}{f_{2\varepsilon_1}} - t^{-\frac{1}{2}}\tau_0\tau_1\tau_0\frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}} \\
&\qquad + t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\tau_0\tau_1\frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}}\frac{f_{\varepsilon_1-r_2}f_{\varepsilon_1-r_1}}{f_{2\varepsilon_1}} - t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\tau_1\tau_0\frac{f_{\varepsilon_2-r_2}f_{\varepsilon_2-r_1}}{f_{2\varepsilon_2}}\frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}} \\
&\qquad - t_0t^{-\frac{1}{2}}\tau_1\frac{f_{\varepsilon_2-r_2}f_{\varepsilon_2-r_1}}{f_{2\varepsilon_2}}\frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}}\frac{f_{\varepsilon_1-r_2}f_{\varepsilon_1-r_1}}{f_{2\varepsilon_1}} \\
&\qquad - t_0^{\frac{1}{2}}t^{-1}\tau_0\frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}}\frac{f_{\varepsilon_2-r_2}f_{\varepsilon_2-r_1}}{f_{2\varepsilon_2}}\frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}} \\
&\qquad + t_0t^{-1}\frac{f_{\varepsilon_1-r_2}f_{\varepsilon_1-r_1}}{f_{2\varepsilon_1}}\frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}}\frac{f_{\varepsilon_2-r_2}f_{\varepsilon_2-r_1}}{f_{2\varepsilon_2}}\frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}},
\end{aligned}$$

and

$$\begin{aligned}
N_0p_0^{(1^2,\emptyset)} &= T_0T_1T_0T_1 + t_0^{-\frac{1}{2}}T_1T_0T_1 - t^{\frac{1}{2}}T_0T_1T_0 \\
&\qquad - t_0^{-\frac{1}{2}}t^{\frac{1}{2}}T_0T_1 - t_0^{-\frac{1}{2}}t^{\frac{1}{2}}T_1T_0 - t_0t^{\frac{1}{2}}T_1 + t_0^{-\frac{1}{2}}tT_0 + t_0t \\
&= \left(a_0^2a^2e_0e_1e_0e_1 - a_0a\left(t_0^{-\frac{1}{2}}t^{\frac{1}{2}} + t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\right)e_0e_1\right) \\
&\qquad - \left(a_0a^2e_1e_0e_1 - a\left(t_0^{-\frac{1}{2}}t^{\frac{1}{2}} + t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\right)e_1\right) \\
&= \tau_0\tau_1\tau_0\tau_1 - t_0^{\frac{1}{2}}\tau_1\tau_0\tau_1W_1^{-2}\frac{f_{-\varepsilon_1-r_2}f_{-\varepsilon_1-r_1}}{f_{2\varepsilon_1}} - t^{-\frac{1}{2}}\tau_0\tau_1\tau_0\frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}} \\
&\qquad + t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\tau_0\tau_1W_1^{-2}\frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}}\frac{f_{-\varepsilon_1-r_2}f_{-\varepsilon_1-r_1}}{f_{2\varepsilon_1}} \\
&\qquad + t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\tau_1\tau_0W_2^{-2}\frac{f_{-\varepsilon_2-r_2}f_{-\varepsilon_2-r_1}}{f_{2\varepsilon_2}}\frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}}
\end{aligned}$$

$$
\begin{aligned}
&- t_0 t^{-\frac{1}{2}}\tau_1 W_1^{-2} W_2^{-2} \frac{f_{-\varepsilon_2-r_2} f_{-\varepsilon_2-r_1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}} \frac{f_{-\varepsilon_1-r_2}f_{-\varepsilon_1-r_1}}{f_{2\varepsilon_1}} \\
&- t_0^{\frac{1}{2}} t^{-1}\tau_0 W_2^{-2} \frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}} \frac{f_{-\varepsilon_2-r_2}f_{-\varepsilon_2-r_1}}{f_{2\varepsilon_2}}\frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}} \\
&+ t_0 t^{-1} W_1^{-2}W_2^{-2} \frac{f_{-\varepsilon_1-r_2}f_{-\varepsilon_1-r_1}}{f_{2\varepsilon_1}}\frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}}\frac{f_{-\varepsilon_2-r_2}f_{-\varepsilon_2-r_1}}{f_{2\varepsilon_2}}\frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}}.
\end{aligned}
$$

Proof. The expressions in terms of T_i are proved by using the relations $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1$ and $T_0^2 = (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})T_0 + 1$ to show that the equations in (2.12) are satisfied. In view of the conditions (2.11), using the equations (2.12) to compute the product of the expansion in terms of the T_i with each element $p_i^{(1^3)}$, $p_0^{(\emptyset,1^2)}$ and $p_0^{(1^2,\emptyset)}$ respectively, determines the normalizing constants

$$N = -t^{-\frac{3}{2}} - t^{-\frac{1}{2}} - t^{-\frac{1}{2}} - t^{\frac{1}{2}} - t^{\frac{1}{2}} - t^{\frac{3}{2}} = t^{-\frac{1}{2}}(1+t)\left(1+t+t^2\right),$$

and

$$N_0 = t_0^{-1}t^{-1} + t^{-1} + t_0^{-1} + 1 + 1 + t_0 + t + t_0 t = t_0^{-1}t^{-1}(1+t_0)(1+t)(1+t_0 t).$$

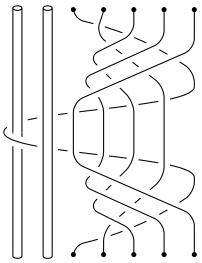
Checking the conditions (2.13) verifies that the expressions in terms of the e_i for the elements $Np_i^{(1^3)}$, $N_0 p_0^{(\emptyset,1^2)}$ and $N_0 p_0^{(1^2,\emptyset)}$ are correct. Similarly, using the expressions for $a_0 e_0$ and ae_i in terms of τ_i given in (2.10) to check these same conditions verifies that the expressions for the elements $Np_i^{(1^3)}$, $N_0 p_0^{(\emptyset,1^2)}$ and $N_0 p_0^{(1^2,\emptyset)}$ in terms of the τ_i are correct. ☐

2.2. The idempotents $p_{0^\vee}^{(\emptyset,1^2)}$ and $p_{0^\vee}^{(1^2,\emptyset)}$.

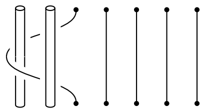
In Section 2.1 we produced idempotents $p_0^{(\emptyset,1^2)}$ and $p_0^{(1^2,\emptyset)}$ in the subalgebra of H_k^{ext} generated by T_0 and T_1. To take advantage of the symmetry of the type C situation, we need to have available the corresponding idempotents $p_{0^\vee}^{(\emptyset,1^2)}$ and $p_{0^\vee}^{(1^2,\emptyset)}$ coming from the subalgebra of H_k^{ext} generated by T_k and T_{k-1}. These are produced as follows.

Let w_A be the longest element of $WA_k = \langle s_1, \dots, s_{k-1}\rangle$. Let Let

$$T_{0^\vee} = T_{w_A}^{-1} T_k T_{w_A} = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}$$



$$= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}$$



$$= W_1 T_0^{-1},$$

and note that $T_{w_A}^{-1} T_{k-1} T_{w_A} = T_1$. Then

$$\left(T_{0^\vee} - t_k^{\frac{1}{2}}\right)\left(T_{0^\vee} + t_k^{-\frac{1}{2}}\right) = 0 \qquad \text{and} \qquad T_{0^\vee}T_1T_{0^\vee}T_1 = T_1T_{0^\vee}T_1T_{0^\vee}.$$

Let $HB_2^\vee$ be the subalgebra of H_k^{ext} generated by $T_{0^\vee}$ and T_1 and define idempotents $p_{0^\vee}^{(\emptyset,1^2)}$ and $p_{0^\vee}^{(1^2,\emptyset)}$ in $HB_2^\vee$ by the equations

$$\left(p_{0^\vee}^{(\emptyset,1^2)}\right)^2 = p_{0^\vee}^{(\emptyset,1^2)}, \qquad \left(p_{0^\vee}^{(1^2,\emptyset)}\right)^2 = p_{0^\vee}^{(1^2,\emptyset)}; \tag{2.14}$$

and

$$\begin{aligned} T_{0^\vee} p_{0^\vee}^{(\emptyset,1^2)} &= -t_k^{-\frac{1}{2}} p_{0^\vee}^{(\emptyset,1^2)}, & T_1 p_{0^\vee}^{(\emptyset,1^2)} &= -t^{-\frac{1}{2}} p_{0^\vee}^{(\emptyset,1^2)}, \\ T_{0^\vee} p_{0^\vee}^{(1^2,\emptyset)} &= t_k^{\frac{1}{2}} p_{0^\vee}^{(1^2,\emptyset)}, \qquad \text{and} & T_1 p_{0^\vee}^{(1^2,\emptyset)} &= -t^{-\frac{1}{2}} p_{0^\vee}^{(1^2,\emptyset)}. \end{aligned} \tag{2.15}$$

In effect, the conjugation by T_{w_A} replaces T_k and T_{k-1} by $T_{0^\vee}$ and T_1, respectively.

Let $a_k \in \mathbb{C}^\times$ and define

$$a_k e_{0^\vee} = T_{0^\vee} - t_k^{\frac{1}{2}}, \qquad \text{so that} \quad e_{0^\vee} = T_{w_A}^{-1} e_k T_{w_A} \quad \text{and} \quad e_1 = T_{w_A}^{-1} e_{k-1} T_{w_A}. \tag{2.16}$$

The conditions in (2.15) are equivalent to

$$\begin{aligned} a_k e_{0^\vee} p_{0^\vee}^{(\emptyset,1^2)} &= -\left(t_k^{\frac{1}{2}} + t_k^{-\frac{1}{2}}\right) p_{0^\vee}^{(\emptyset,1^2)}, & a e_1 p_{0^\vee}^{(\emptyset,1^2)} &= -\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) p_{0^\vee}^{(\emptyset,1^2)}, \\ a_k e_{0^\vee} p_{0^\vee}^{(1^2,\emptyset)} &= 0, \qquad \text{and} & a e_1 p_{0^\vee}^{(1^2,\emptyset)} &= -\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) p_{0^\vee}^{(1^2,\emptyset)}. \end{aligned} \tag{2.17}$$

Using $a_k e_{0^\vee} = W_1 T_0^{-1} - t_k^{\frac{1}{2}} = W_1(T_0 - (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})) - t_k^{\frac{1}{2}} = W_1(\tau_0 + t_0^{\frac{1}{2}} - c_{\alpha_0} - (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})) - t_k^{\frac{1}{2}}$, a short computation gives

$$a_k e_{0^\vee} = \tau_0 W_1^{-1} - t_0^{-\frac{1}{2}} W_1^{-1} \frac{f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_1}}{f_{2\varepsilon_1}}.$$

Letting $N_k = t_k^{-1} t^{-1}(1+t_k)(1+t)(1+t_k t)$, then

$$\begin{aligned} N_k p_{0^\vee}^{(\emptyset,1^2)} &= a_k^2 a^2 e_{0^\vee} e_1 e_{0^\vee} e_1 - a_k a \left(t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}\right) e_{0^\vee} e_1 \\ &= a_k^2 a^2 e_1 e_{0^\vee} e_1 e_{0^\vee} - a_k a \left(t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}\right) e_1 e_{0^\vee} \\ &= \tau_0 \tau_1 \tau_0 \tau_1 (W_1 W_2)^{-1} - t_0^{-\frac{1}{2}} \tau_1 \tau_0 \tau_1 (W_1 W_2)^{-1} \frac{f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} \\ &\quad + t^{-\frac{1}{2}} \tau_0 \tau_1 \tau_0 (W_1 W_2)^{-1} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &\quad - t_0^{-\frac{1}{2}} t^{-\frac{1}{2}} \tau_0 \tau_1 (W_1 W_2)^{-1} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} \\ &\quad - t_0^{-\frac{1}{2}} t^{-\frac{1}{2}} \tau_1 \tau_0 (W_1 W_2)^{-1} \frac{f_{\varepsilon_2 - r_2} f_{-\varepsilon_2 - r_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &\quad + t_0^{-1} t^{-\frac{1}{2}} \tau_1 (W_1 W_2)^{-1} \frac{f_{\varepsilon_2 - r_2} f_{-\varepsilon_2 - r_1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} \\ &\quad - t_0^{-\frac{1}{2}} t^{-1} \tau_0 (W_1 W_2)^{-1} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{\varepsilon_2 - r_2} f_{-\varepsilon_2 - r_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &\quad + t_0^{-1} t^{-1} (W_1 W_2)^{-1} \frac{f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{\varepsilon_2 - r_2} f_{-\varepsilon_2 - r_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}}, \end{aligned}$$

and

$$\begin{aligned}
N_k p_{0^\vee}^{(1^2,\emptyset)} &= \left(a_k^2 a^2 e_{0^\vee} e_1 e_{0^\vee} e_1 - a_k a \left(t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}\right) e_{0^\vee} e_1\right) \\
&\quad - \left(a_k a^2 e_1 e_{0^\vee} e_1 - a \left(t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}\right) e_1\right) \\
&= \tau_0\tau_1\tau_0\tau_1 (W_1W_2)^{-1} - t_0^{-\frac{1}{2}} \tau_1\tau_0\tau_1 (W_1W_2)^{-1} \frac{f_{-\varepsilon_1 - r_2} f_{\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} \\
&\quad + t^{-\frac{1}{2}} \tau_0\tau_1\tau_0 (W_1W_2)^{-1} \frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}} \\
&\quad - t_0^{-\frac{1}{2}} t^{-\frac{1}{2}} \tau_0\tau_1 (W_1W_2)^{-1} \frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}} \frac{f_{-\varepsilon_1-r_2} f_{\varepsilon_1-r_1}}{f_{2\varepsilon_1}} \\
&\quad - t_0^{-\frac{1}{2}} t^{-\frac{1}{2}} \tau_1\tau_0 (W_1W_2)^{-1} \frac{f_{-\varepsilon_2-r_2} f_{\varepsilon_2-r_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}} \\
&\quad + t_0^{-1} t^{-\frac{1}{2}} \tau_1 (W_1W_2)^{-1} \frac{f_{-\varepsilon_2-r_2} f_{\varepsilon_2-r_1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}} \frac{f_{-\varepsilon_1-r_2} f_{\varepsilon_1-r_1}}{f_{2\varepsilon_1}} \\
&\quad - t_0^{-\frac{1}{2}} t^{-1} \tau_0 \left(W_1^{-1} W_2\right)^{-1} \frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}} \frac{f_{-\varepsilon_2-r_2} f_{\varepsilon_2-r_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}} \\
&\quad + t_0^{-1} t^{-1} (W_1W_2)^{-1} \frac{f_{-\varepsilon_1-r_2} f_{\varepsilon_1-r_1}}{f_{2\varepsilon_1}} \frac{f_{-\varepsilon_2-\varepsilon_1+1}}{f_{-\varepsilon_2-\varepsilon_1}} \frac{f_{-\varepsilon_2-r_2} f_{\varepsilon_2-r_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2-\varepsilon_1+1}}{f_{\varepsilon_2-\varepsilon_1}},
\end{aligned}$$

in analogy with (and with the same proof as) Proposition 2.2.

3. THE TWO BOUNDARY TEMPERLEY–LIEB ALGEBRA TL_k

In this section we define the two boundary Temperley–Lieb algebra TL_k and review its diagrammatic calculus. The algebra TL_k is closely related to the symplectic blob algebras $TL_k(b)$, see (3.8)) and [8, 9, 10, 12, 15, 16]. We extend the diagrammatic calculus to make clear the relationship to the two boundary Hecke algebra and to set the stage for the proof of Theorem 3.2. Although Theorem 3.2 takes the form of a computation, it is a computation that has amazing consequences as it determines the relationship between the center of H_k^{ext} and the center of TL_k. The center of H_k^{ext} is a ring of symmetric functions (see [3, Theorem 2.3]) and the center of TL_k turns out to be a polynomial ring $\mathbb{C}[Z]$ in a single variable Z. In the same way that H_k^{ext} is finite rank over its center, the algebra TL_k is finite rank over $\mathbb{C}[Z]$. The algebra H_k^{ext} has rank $(2^k k!)^2$ over its center. A closed form formula for these ranks, in terms of sums of products of blobbed Catalan numbers, is given in [1, Theorem 4.2, Theorem 6.1 and Theorem 7.12]. These ranks coincide with the dimensions of the symplectic blob algebras $TL_k(b)$ defined in Section 3.6.

3.1. The extended two boundary Temperley–Lieb algebra TL_k^{ext}.

Let H_k^{ext} be the extended two boundary Hecke algebra as defined in (2.1). The *extended two boundary Temperley–Lieb algebra* TL_k^{ext} is the quotient of H_k^{ext} by the relations

$$p_{0^\vee}^{(\emptyset,1^2)} = p_{0^\vee}^{(1^2,\emptyset)} = 0, \qquad p_0^{(\emptyset,1^2)} = p_0^{(1^2,\emptyset)} = 0 \quad \text{and} \quad p_i^{(1^3)} = 0 \quad \text{for } i \in \{1, \ldots, k-2\}.$$

Theorem 3.1. *Assume that N, N_0 and N_k are invertible. The algebra TL_k^{ext} is the quotient of H_k^{ext} by the relations*

$$a_k a^2 e_{k-1} e_k e_{k-1} - a\left(t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}\right) e_{k-1} = 0, \quad a_0 a^2 e_1 e_0 e_1 - a\left(t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}\right) e_1 = 0,$$

and

$$a^3e_ie_{i+1}e_i - ae_i = a^3e_{i+1}e_ie_{i+1} - ae_{i+1} = 0 \qquad \text{for } i \in \{1,\dots,k-2\}.$$

Proof. Let $F_i = a^3e_ie_{i+1}e_i - ae_i = a^3e_{i+1}e_ie_{i+1} - ae_{i+1}$ for $i \in \{1,\dots,k-2\}$,

$$F_k = a_ka^2e_{k-1}e_ke_{k-1} - a\left(t_k^{-\frac{1}{2}}t^{\frac{1}{2}} + t_k^{\frac{1}{2}}t^{-\frac{1}{2}}\right)e_{k-1},$$

and

$$F_0 = a_0a^2e_1e_0e_1 - a\left(t_0^{-\frac{1}{2}}t^{\frac{1}{2}} + t_0^{\frac{1}{2}}t^{-\frac{1}{2}}\right)e_1.$$

By Proposition 2.2,

$$N_0p_0^{(\emptyset,1^2)} = a_0e_0F_0, \qquad N_0p_0^{(1^2,\emptyset)} = (a_0e_0-1)F_0,$$

$$F_0 = N_0\left(p_0^{(\emptyset,1^2)} - p_0^{(1^2,\emptyset)}\right), \quad \text{and} \qquad Np_i^{(1^3)} = F_i;$$

and, by (2.16),

$$T_{w_A}F_kT_{w_A}^{-1} = N_0\left(p_{0^\vee}^{(\emptyset,1^2)} - p_{0^\vee}^{(1^2,\emptyset)}\right), \qquad T_{w_A}^{-1}p_{0^\vee}^{(\emptyset,1^2)}T_{w_A} = a_ke_kF_k,$$

and

$$T_{w_A}^{-1}p_{0^\vee}^{(1^2,\emptyset)}T_{w_A} = (a_ke_k-1)F_k.$$

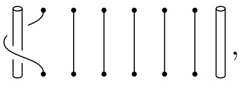
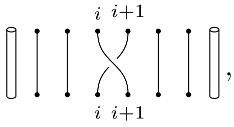
Thus, provided N, N_0 and N_k are invertible, the ideal $H_k^{\mathrm{ext}}F_kH_k^{\mathrm{ext}}$ is the same as the ideal generated by $p_{0^\vee}^{(1^2,\emptyset)}$ and $p_{0^\vee}^{(\emptyset,1^2)}$; the ideal $H_k^{\mathrm{ext}}F_0H_k^{\mathrm{ext}}$ is the same as the ideal generated by $p_0^{(1^2,\emptyset)}$ and $p_0^{(\emptyset,1^2)}$; and $H_k^{\mathrm{ext}}p_i^{(1^3)}H_k^{\mathrm{ext}} = H_k^{\mathrm{ext}}F_iH_k^{\mathrm{ext}}$. □

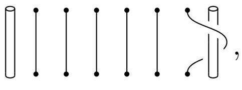
3.2. **The two boundary Temperley–Lieb algebra TL_k.** The *two boundary Temperley–Lieb algebra TL_k* is the subalgebra of TL_k^{ext} generated by $a_0e_0, ae_1, \dots, ae_{k-1}, a_ke_k$ (as defined in (2.5)). As in [3, Theorem 2.4], where $H_k^{\mathrm{ext}} = H_k \otimes \mathbb{C}[W_0^{\pm1}]$, the extended two boundary Temperley–Lieb algebra is

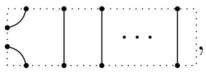
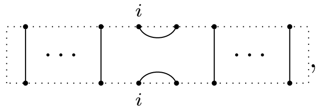
$$TL_k^{\mathrm{ext}} = TL_k \otimes \mathbb{C}\left[W_0^{\pm1}\right], \quad \text{as algebras, where} \quad W_0 = PW_1\cdots W_k.$$

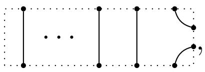
This structure guarantees that $Z(TL_k^{\mathrm{ext}}) = \mathbb{C}[W_0^{\pm1}] \otimes Z(TL_k)$ and that every finite dimensional irreducible representation of TL_k^{ext} is a tensor product $L \otimes V$ of an irreducible representation L of TL_k and an irreducible representation V of $\mathbb{C}[W_0^{\pm1}]$. Since $\mathbb{C}[W_0^{\pm1}]$ is the group algebra of the group $\mathcal{D}$ then V must be one dimensional, determined by the action of W_0 on the one-dimensional vector space V. The constant by which W_0 acts on V is the constant z appearing in Section 4.1.

3.3. **Diagrammatic calculus for TL_k.** Pictorially, identify

$T_0 =$

, $T_i =$


, $T_k =$

,

$e_0 =$

, $ae_i =$


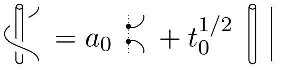
, and $e_k =$

,

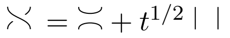
for $i \in \{1, \ldots, k-1\}$. Recall the notation

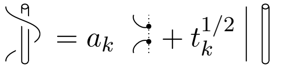
$$[\![x]\!] = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

from (2.8). With $i \in \{1, \ldots, k-1\}$, the relations (2.5), (2.6) and (2.7) are

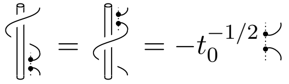
$$T_0 = a_0 e_0 + t_0^{\frac{1}{2}}, \qquad T_i = a e_i + t^{\frac{1}{2}}, \qquad T_k = a_k e_k + t_k^{\frac{1}{2}},$$

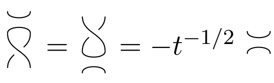


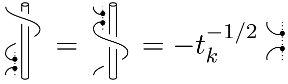




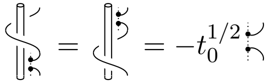
$$T_0 e_0 = -t_0^{-\frac{1}{2}} e_0, \qquad T_i(ae_i) = -t^{-\frac{1}{2}}(ae_i), \qquad T_k e_k = -t_k^{-\frac{1}{2}} e_k,$$

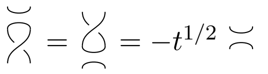


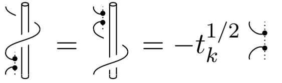




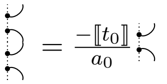
$$T_0^{-1} e_0 = -t_0^{\frac{1}{2}} e_0, \qquad T_i^{-1}(ae_i) = -t^{\frac{1}{2}}(ae_i), \qquad T_k^{-1} e_k = -t_k^{\frac{1}{2}} e_k,$$

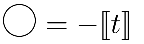


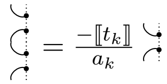




$$e_0^2 = \frac{-[\![t_0]\!]}{a_0} e_0, \qquad (ae_i)^2 = -[\![t]\!](ae_i), \qquad \text{and} \qquad e_k^2 = \frac{-[\![t_k]\!]}{a_k} e_k.$$

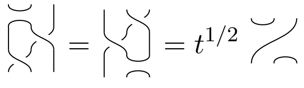


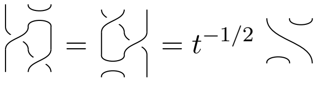


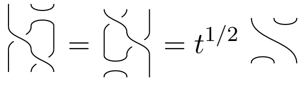


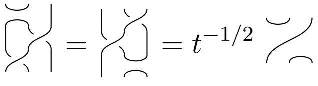
In the quotient by $(ae_i)(ae_{i+1})(ae_i) = ae_i$, we have

$$\begin{aligned} ae_i T_{i+1} T_i = aT_{i+1} T_i e_{i+1} &= t^{\frac{1}{2}} a^2 e_i e_{i+1}, \\ ae_i T_{i+1}^{-1} T_i^{-1} = aT_{i+1}^{-1} T_i^{-1} e_{i+1} &= t^{-\frac{1}{2}} a^2 e_i e_{i+1}, \\ ae_{i+1} T_i T_{i+1} = aT_i T_{i+1} e_i &= t^{\frac{1}{2}} a^2 e_{i+1} e_i, \\ ae_{i+1} T_i^{-1} T_{i+1}^{-1} = aT_i^{-1} T_{i+1}^{-1} e_i &= t^{-\frac{1}{2}} a^2 e_{i+1} e_i, \end{aligned} \tag{3.1}$$





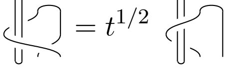


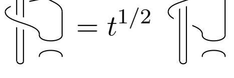


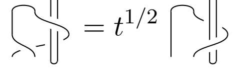
which are proved by using $T_i^{\pm 1} = ae_i + t^{\pm\frac{1}{2}}$ to expand both sides in terms of e_i.

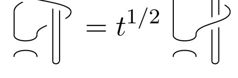
When $a_0(ae_1)e_0(ae_1) - [\![t_0 t^{-1}]\!](ae_1) = 0$ and $a_k(ae_{k-1})e_k(ae_{k-1}) - [\![t_k t^{-1}]\!]ae_{k-1} = 0$, then

$$\begin{aligned} (ae_1)T_0T_1 &= t^{\frac{1}{2}}(ae_1)T_0^{-1}, & T_1T_0(ae_1) &= t^{\frac{1}{2}}T_0^{-1}(ae_1), \\ (ae_{k-1})T_kT_{k-1} &= t^{\frac{1}{2}}(ae_{k-1})T_k^{-1}, & T_{k-1}T_k(ae_{k-1}) &= t^{\frac{1}{2}}T_k^{-1}(ae_{k-1}), \end{aligned} \tag{3.2}$$

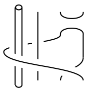




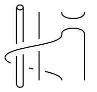




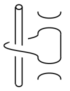
$$(ae_2)T_1T_0T_1T_2 = t^{\frac{3}{2}}(ae_2)T_1^{-1}T_0^{-1}T_1^{-1}, \tag{3.3}$$



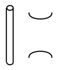
$$= t^{3/2}$$

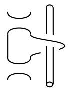


$$
\begin{aligned}
(ae_1)T_0(ae_1) &= -t^{\frac{1}{2}}\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right)(ae_1),\\
(ae_{k-1})T_k(ae_{k-1}) &= -t^{\frac{1}{2}}\left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right)(ae_{k-1}),
\end{aligned}
\tag{3.4}
$$

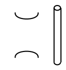


$$= -t^{\frac{1}{2}}\left(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}\right)$$



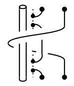


$$= -t^{\frac{1}{2}}\left(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}\right)$$

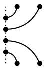


and

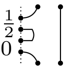
$$e_0T_1^{-1}T_0^{-1}T_1^{-1}e_0 = -t^{-\frac{1}{2}}[\![t_0]\!]e_0(ae_1)e_0 - t^{-1}t_0^{\frac{1}{2}}e_0^2.$$



$$= -t^{-\frac{1}{2}}[\![t_0]\!]$$



$$- t^{-1}t_0^{\frac{1}{2}}$$

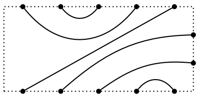


3.4. TL_k **as a diagram algebra.** The algebra TL_k is generated by $e_0, e_1, \ldots, e_k$, which are represented pictorially by diagrams as in Seciton 3.3. Using the pictorial notation, the algebra TL_k has a basis (see [9, Theorem 3.4] and [4, Definition 3.4]) of non-crossing diagrams with k dots in the top row, k dots in the bottom row, edges connecting a dot to a dot or connecting a dot to either the left or the right boundary, an even number of left boundary to right boundary edges, and

$$(-1)^{\#\{\text{left boundary edges}\}} = 1 \qquad \text{and} \qquad (-1)^{\#\{\text{right boundary edges}\}} = 1.$$

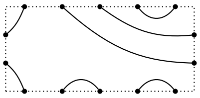
For example,

$$d_1 =$$



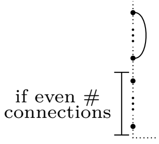
and

$$d_2 =$$

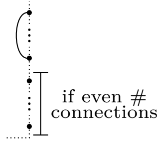


are both basis elements of TL_k. Multiplication of basis elements can be computed pictorially by vertical concatenation, with self-connected loops and strands with both ends on the left or on the right replaced by constant coefficients according to the following local rules (applied simultaneously, not iteratively):

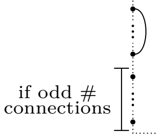
$$= -[\![t]\!],$$



$$= \frac{[\![t_0t^{-1}]\!]}{a_0},$$

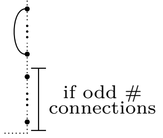


$$= \frac{[\![t_kt^{-1}]\!]}{a_k},$$



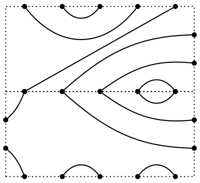
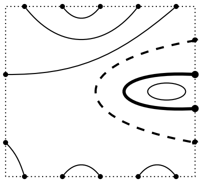
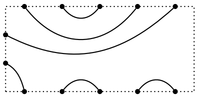
$$= \frac{-[\![t_0]\!]}{a_0},$$

and



$$= \frac{-[\![t_k]\!]}{a_k}.$$

For example with d_1 and d_2 as above,

$d_1 d_2 =$  $=$  $= (-[\![t]\!]) \left(\frac{-[\![t_k]\!]}{a_k}\right) \left(\frac{[\![t_k t^{-1}]\!]}{a_k}\right)$ 

(where the dashed strand is removed with a coefficient of $\frac{[\![t_k t^{-1}]\!]}{a_k}$, and the thick strand is removed with a coefficient of $\frac{-[\![t_k]\!]}{a_k}$).

3.5. The through-strand filtration of TL_k.

A *through-strand* is an edge that connects a top vertex to a bottom vertex. Define the ideals

$$TL_k^{(\leq j)} = \mathbb{C}\text{-span}\{\text{diagrams with} \leq j \text{ through-strands}\}.$$

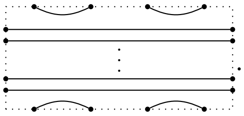
Then the algebra TL_k is filtered by ideals as

$$TL_k = TL_k^{(\leq k)} \supseteq TL_k^{(\leq k-1)} \supseteq \cdots \supseteq TL_k^{(\leq 1)} \supseteq TL_k^{(\leq 0)} \supseteq 0. \tag{3.5}$$

If

$$TL_k^{(j)} = \frac{TL_k^{(\leq j)}}{TL_k^{(\leq j-1)}}, \quad \text{then} \quad \dim\left(TL_k^{(j)}\right) < \infty, \quad \text{for } j \geq 1, \quad \text{and} \quad \dim\left(TL_k^{(\leq 0)}\right) = \infty,$$

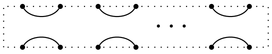
as there can be an arbitrarily large number of edges which connect the left and right sides in diagrams with no through strands:

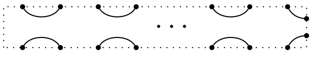


3.6. The elements I_1 and I_2.

As in [4, § 3.2], define

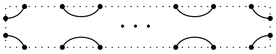
$$I_1 = \begin{cases} (ae_1)(ae_3)\cdots(ae_{k-1}), & \text{if } k \text{ is even,} \\ (ae_1)(ae_3)\cdots(ae_{k-2})e_k, & \text{if } k \text{ is odd,} \end{cases}$$

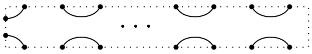
$=$  if k is even,

 if k is odd, (3.6)

and

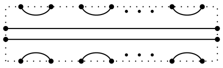
$$I_2 = \begin{cases} e_0(ae_2)\cdots(ae_{k-2})e_k, & \text{if } k \text{ is even,} \\ e_0(ae_2)\cdots(ae_{k-1}), & \text{if } k \text{ is odd.} \end{cases}$$

$=$  if k is even,

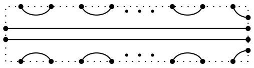
 if k is odd. (3.7)

Up to a constant multiple the elements I_1 and I_2 are idempotents (provided $[2] \neq 0$) and

$I_1 I_2 I_1 =$



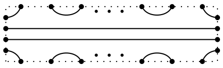
if k is even,



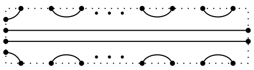
if k is odd,

and

$I_2 I_1 I_2 =$



if k is even,



if k is odd.

Corollary 3.3 below gives another striking formula for the elements $I_1I_2I_1$ and $I_2I_1I_2$.

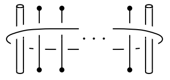
For a constant $b \in \mathbb{C}$ the *symplectic blob algebra* $TL_k(b)$, or *double quotient* of TL_k, (see [1, Definition 2.7] and [4, Definition 3.7]) is the quotient of TL_k by the relations

$$I_1 I_2 I_1 = bI_1 \quad \text{and} \quad I_2 I_1 I_2 = bI_2. \tag{3.8}$$

These quotients, as b varies, capture all of the irreducible representations of TL_k. There is an easy conceptual explanation for this. The center of TL_k is a polynomial ring in one variable $\mathbb{C}[Z]$, the value b corresponds to the choice of a central character, and the algebra $TL_k(b)$ is the quotient of TL_k by the central character so that the irreducible representations of $TL_k(b)$ are exactly those irreducible representations of TL_k on which the central element Z acts by a specific value (determied from b by the formulas in Corollary 3.3).

3.7. **The element ZI_1 in TL_k.** Conceptually, the diagram

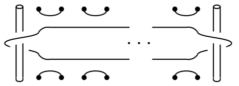
$F =$



would be a central element of H_k

if it represented a true element of the algebra H_k. Though the diagram F does not naturally represent an element of H_k, the diagrams

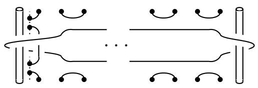
$$D^{\text{even}} = I_1 \left(T_0^{-1}(ae_2)(ae_4)\cdots(ae_{k-2})T_k\right) I_1 =$$



,

and

$$D^{\text{odd}} = I_2 \left(T_1^{-1}T_0^{-1}T_1^{-1}(ae_3)(ae_5)\cdots(ae_{k-2})T_k\right) I_2 =$$



do appear in the algebra TL_k and play an important role in the proof of the following theorem. See also [4, Theorem 4.1], using Remark 3.4 below as a guide to the differences in notations.

It is proved in [3, Theorem 2.3] that the element

$$Z = W_1 + W_1^{-1} + W_2 + W_2^{-1} + \cdots + W_k + W_k^{-1} \qquad \text{is central in } H_k^{\text{ext}}. \tag{3.9}$$

Since Z does not include W_0 (and hence the element P is not needed in the definition of Z) then Z is actually an element of H_k, and its image in the Temperley–Lieb quotient is an element of TL_k.

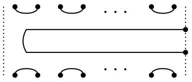
Theorem 3.2. *As elements of* TL_k,

if k *is even, then* $D^{even} = a_0a_kI_1I_2I_1 + \llbracket t_0t_kt^{-1}\rrbracket I_1$ *and* $ZI_1 = [k]D^{even}$, *and* *if* k *is odd, then*

$$D^{odd} = t^{-\frac{1}{2}}\left(\frac{-\llbracket t_0\rrbracket}{a_0}\right)\left(a_0a_kI_2I_1I_2 - \llbracket t_0t_k^{-1}\rrbracket I_2\right) \quad \text{and} \quad t^{-\frac{1}{2}}\left(\frac{-\llbracket t_0\rrbracket}{a_0}\right)ZI_2 = [k]D^{odd}.$$

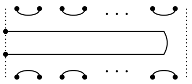
Proof. ***Case 1.*** k **even.**Let

$$L^{\text{even}} = I_1\big((ae_2)(ae_4)\cdots(ae_{k-2})e_k\big)I_1 =$$



$$= \left(\frac{\llbracket t_kt^{-1}\rrbracket}{a_k}\right)I_1,$$

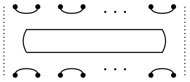
$$M^{\text{even}} = I_1\big(e_0(ae_2)(ae_4)\cdots(ae_{k-2})\big)I_1 =$$



$$= \left(\frac{\llbracket t_0t^{-1}\rrbracket}{a_0}\right)I_1,$$

and

$$P^{\text{even}} = I_1\big((ae_2)(ae_4)\cdots(ae_{k-2})\big)I_1 =$$



$$= -\llbracket t\rrbracket I_1.$$

Using $T_0^{-1} = a_0e_0 + t_0^{-\frac{1}{2}}$ for the left pole and $T_k = a_ke_k + t_k^{\frac{1}{2}}$ for the right pole,

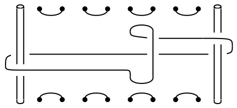
$$\begin{aligned} D^{\text{even}} &= a_0a_kI_1I_2I_1 + a_0t_k^{\frac{1}{2}}M^{\text{even}} + a_kt_0^{-\frac{1}{2}}L^{\text{even}} + t_0^{\frac{1}{2}}t_k^{\frac{1}{2}}P^{\text{even}} \\ &= a_0a_kI_1I_2I_1 + \left(t_k^{\frac{1}{2}}\llbracket t_0t^{-1}\rrbracket + t_0^{-\frac{1}{2}}\llbracket t_kt^{-1}\rrbracket - t_0^{-\frac{1}{2}}t_k^{\frac{1}{2}}\llbracket t\rrbracket\right)I_1 \\ &= a_0a_kI_1I_2I_1 + \llbracket t_0t_kt^{-1}\rrbracket I_1, \end{aligned}$$

which completes the proof of the first statement.

Using $(ae_1)T_1^{-1} = (-t^{\frac{1}{2}})(ae_1)$ and $(ae_1)T_1T_0(ae_1) = t^{\frac{1}{2}}(ae_1)T_0^{-1}(ae_1)$ gives

$$R^{\text{even}} = I_1\left(T_1^{-1}(ae_2)(ae_4)\cdots(ae_{k-2})T_kT_1T_0\right)I_1$$

$$=$$

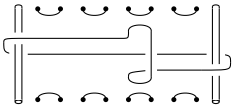


$$= \left(-t^{\frac{1}{2}}\right)t^{\frac{1}{2}}D^{\text{even}},$$

and using $T_{k-1}(ae_{k-1}) = (-t^{-\frac{1}{2}})(ae_{k-1})$ and $T_{k-1}^{-1}T_k^{-1}(ae_{k-1}) = t^{-\frac{1}{2}}T_k(ae_{k-1})$ gives

$$S^{\text{even}} = I_1\left(T_0(ae_2)(ae_4)\cdots(ae_{k-2})T_{k-1}^{-1}T_k^{-1}T_{k-1}\right)I_1$$

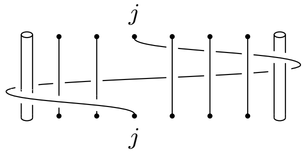
$$=$$



$$= \left(-t^{-\frac{1}{2}}\right)t^{-\frac{1}{2}}D^{\text{even}}.$$

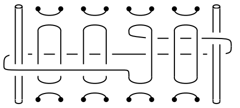
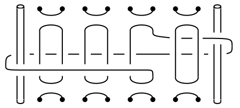
Pictorially,

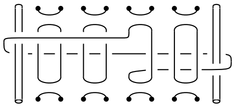
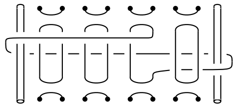
$$W_j =$$



for $j \in \{1, \ldots, k\}$.

Thus, pictorially,

$I_1W_{1+2i}I_1 =$  , $I_1W_{2+2i}I_1 =$  ,

$I_1W^{-1}_{1+2i}I_1 =$  , and $I_1W^{-1}_{2+2i}I_1 =$  .

Working left to right removing loops,

$$I_1W_{1+2i}I_1 = \left(t^{\frac12}t^{\frac12}(-[\![t]\!])\right)^i \left(t^{-\frac12}t^{\frac12}(-[\![t]\!])\right)^{\frac{k}{2}-1-i} R^{\text{even}} = (-[\![t]\!])^{\frac{k}{2}-1}t^{i+\frac12}\left(-t^{\frac12}\right)D^{\text{even}},$$

$$I_1W^{-1}_{1+2i}I_1 = \left(t^{-\frac12}t^{-\frac12}(-[\![t]\!])\right)^i \left(t^{-\frac12}t^{\frac12}(-[\![t]\!])\right)^{\frac{k}{2}-1-i} S = (-[\![t]\!])^{\frac{k}{2}-1}t^{-\left(i+\frac12\right)}\left(-t^{-\frac12}\right)D^{\text{even}},$$

for $i \in \{0,\dots,\frac{k}{2}-1\}$. Since $I_1W_{1+2i}I_1$ and $I_1W_{2+2i}I_1$ only differ by two twists (similarly $I_1W^{-1}_{1+2i}I_1$ and $I_1W^{-1}_{2+2i}I_1$ only differ by two twists) the relations $T_i^{\pm1}(ae_i) = (ae_i)T_i^{\pm1} = (-t^{\mp\frac12})(ae_i)$ give

$$I_1W_{2+2i}I_1 = \left(-t^{-\frac12}\right)\left(-t^{-\frac12}\right)t^{-1}I_1W_{1+2i}I_1 = (-[\![t]\!])^{\frac{k}{2}-1}t^{i+\frac12}\left(-t^{-\frac12}\right)D^{\text{even}}$$

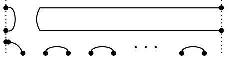
and

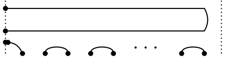
$$I_1W^{-1}_{2+2i}I_1 = \left(-t^{\frac12}\right)\left(-t^{-\frac12}\right)I_1W^{-1}_{1+2i}I_1 = (-[\![t]\!])^{\frac{k}{2}-1}t^{-\left(i+\frac12\right)}\left(-t^{\frac12}\right)D^{\text{even}},$$

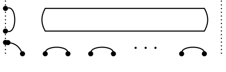
for $i \in \{0,\dots,\frac{k}{2}-1\}$. Thus

$$\begin{aligned}(-[\![t]\!])^{\frac{k}{2}}ZI_1 = ZI_1^2 = I_1ZI_1 &= \sum_{i=0}^{\frac{k}{2}-1} I_1\left(W_{1+2i}+W_{2+2i}+W^{-1}_{1+2i}+W^{-1}_{2+2i}\right)I_1\\ &= -(-[\![t]\!])^{\frac{k}{2}-1}D^{\text{even}}\sum_{i=0}^{\frac{k}{2}-1}\left(t^{i+\frac12}\left(t^{\frac12}+t^{-\frac12}\right)+t^{-\left(i+\frac12\right)}\left(t^{\frac12}+t^{-\frac12}\right)\right)\\ &= (-[\![t]\!])^{\frac{k}{2}}D^{\text{even}}\left(\frac{t^{\frac{k}{2}}-t^{-\frac{k}{2}}}{t^{\frac12}-t^{-\frac12}}\right) = (-[\![t]\!])^{\frac{k}{2}}[k]D^{\text{even}}.\end{aligned}$$

Case 2. k **odd.** Let

$L^{\text{odd}} = I_2\big((ae_3)(ae_5)\cdots(ae_{k-2})e_k\big)I_2 =$  $= \left(\dfrac{-[\![t_0]\!]}{a_0}\right)\left(\dfrac{[\![t_kt^{-1}]\!]}{a_k}\right)I_2,$

$M^{\text{odd}} = I_2\big((ae_1)(ae_3)\cdots(ae_{k-2})\big)I_2 =$  $= \left(\dfrac{-[\![t_0]\!]}{a_0}\right)I_2,$ and

$P^{\text{odd}} = I_2\big((ae_3)(ae_5)\cdots(ae_{k-2})\big)I_2 =$  $= \left(\dfrac{-[\![t_0]\!]}{a_0}\right)(-[\![t]\!])I_2.$

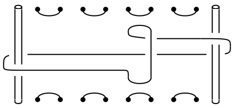
Using $e_0T_1^{-1}T_0^{-1}T_1^{-1}e_0 = -t^{-\frac{1}{2}}[\![t_0]\!]e_0(ae_1)e_0 - t^{-1}t_0^{\frac{1}{2}}e_0^2$ and $T_k = a_ke_k + t_k^{\frac{1}{2}}$ gives

$$\begin{aligned}D^{\mathrm{odd}} &= -t^{-\frac{1}{2}}[\![t_0]\!]a_kI_2I_1I_2 - t^{-1}t_0^{\frac{1}{2}}a_kL^{\mathrm{odd}} - t^{-\frac{1}{2}}[\![t_0]\!]t_k^{\frac{1}{2}}M^{\mathrm{odd}} - t^{-1}t_0^{\frac{1}{2}}t_k^{\frac{1}{2}}P^{\mathrm{odd}}\\ &= t^{-\frac{1}{2}}\left(\frac{-[\![t_0]\!]}{a_0}\right)\left(a_0a_kI_2I_1I_2 + \left(-t^{-\frac{1}{2}}t_0^{\frac{1}{2}}[\![t_kt^{-1}]\!] - t_k^{\frac{1}{2}}[\![t_0]\!] + t^{-\frac{1}{2}}t_0^{\frac{1}{2}}t_k^{\frac{1}{2}}[\![t]\!]\right)I_2\right)\\ &= t^{-\frac{1}{2}}\left(\frac{-[\![t_0]\!]}{a_0}\right)\left(a_0a_kI_2I_1I_2 - [\![t_0t_k^{-1}]\!]I_2\right),\end{aligned}$$

which completes the proof of the first statement.

Using $(ae_2)T_2^{-1} = -t^{\frac{1}{2}}(ae_2)$ and $T_2T_1T_0T_1(ae_2) = t^{\frac{3}{2}}T_1^{-1}T_0^{-1}T_1^{-1}(ae_2)$,

$$R^{\mathrm{even}} = I_1\left(T_1^{-1}(ae_2)(ae_4)\cdots(ae_{k-2})T_kT_1T_0\right)I_1 =$$

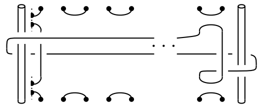


$$= \left(-t^{\frac{1}{2}}\right)t^{\frac{1}{2}}D^{\mathrm{even}},$$

Using $T_{k-1}(ae_{k-1}) = -t^{-\frac{1}{2}}(ae_{k-1})$ and $T_{k-1}^{-1}T_k^{-1}(ae_{k-1}) = t^{-\frac{1}{2}}T_k(ae_{k-1})$ gives

$$S^{\mathrm{odd}} = I_2\left(T_1^{-1}T_0^{-1}T_1^{-1}(ae_3)(ae_5)\cdots(ae_{k-2})T_{k-1}^{-1}T_k^{-1}T_{k-1}\right)I_2$$

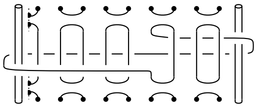
$$=$$



$$= \left(-t^{-\frac{1}{2}}\right)t^{-\frac{1}{2}}D^{\mathrm{odd}}.$$

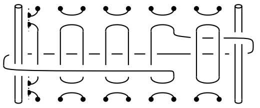
Pictorially,

$$I_2W_{1+2i}I_2 =$$



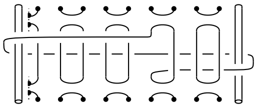
,

$$I_2W_{2+2i}I_2 =$$



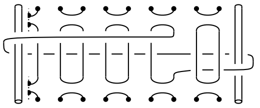
,

$$I_2W_{1+2i}^{-1}I_2 =$$



, and

$$I_2W_{2+2i}^{-1}I_2 =$$



.

Working left to right removing loops,

$$\begin{aligned}I_2W_{2+2i}I_2 &= \left(t^{\frac{1}{2}}t^{\frac{1}{2}}(-[\![t]\!])\right)^i\left(t^{-\frac{1}{2}}t^{\frac{1}{2}}(-[\![t]\!])\right)^{\frac{k-1}{2}-1-i}R^{\mathrm{odd}} = (-[\![t]\!])^{\frac{k-3}{2}}t^{i+1}(-t)D^{\mathrm{odd}},\\ I_2W_{2+2i}^{-1}I_2 &= \left(t^{-\frac{1}{2}}t^{-\frac{1}{2}}(-[\![t]\!])\right)^i\left(t^{-\frac{1}{2}}t^{\frac{1}{2}}(-[\![t]\!])\right)^{\frac{k-1}{2}-1-i}S^{\mathrm{odd}} = (-[\![t]\!])^{\frac{k-3}{2}}t^{-(i+1)}(-1)D^{\mathrm{odd}},\end{aligned}$$

for $i \in \{0,\ldots,\frac{k-1}{2}-1\}$. Since $I_2W_{2+2i}I_2$ and $I_2W_{3+2i}I_2$ only differ by two twists (similarly $I_2W_{2+2i}^{-1}I_2$ and $I_2W_{3+2i}^{-1}I_2$ only differ by two twists) the relations $T_i^{\pm1}e_i = e_iT_i^{\pm1} = (-t^{\mp\frac{1}{2}})e_i$ give

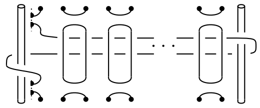
$$I_2W_{3+2i}I_2 = \left(-t^{-\frac{1}{2}}\right)\left(-t^{-\frac{1}{2}}\right)I_2W_{2+2i}I_2 = (-[\![t]\!])^{\frac{k-3}{2}}t^{i+1}(-1)D^{\mathrm{odd}},$$

and

$$I_2W_{3+2i}^{-1}I_2 = \left(-t^{\frac{1}{2}}\right)\left(-t^{\frac{1}{2}}\right)I_2W_{2+2i}^{-1}I_2 = (-[\![t]\!])^{\frac{k-3}{2}}t^{-(i+1)}(-t)D^{\mathrm{odd}},$$

for $i \in \{0, \dots, \frac{k-1}{2} - 1\}$. Next,

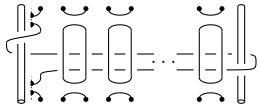
$$I_2W_1I_2 =$$



$$= \left(-t_0^{-\frac{1}{2}}\right)(-[\![t]\!])^{\frac{k-1}{2}} I_2((ae_1)(ae_3)\cdots(ae_{k-2})T_k)I_2,$$

and

$$I_2W_1^{-1}I_2 =$$



$$= \left(-t_0^{\frac{1}{2}}\right)(-[\![t]\!])^{\frac{k-1}{2}} I_2\left((ae_1)(ae_3)\cdots(ae_{k-2})T_k^{-1}\right)I_2.$$

Using $-t_0^{-\frac{1}{2}}T_k - t_0^{\frac{1}{2}}T_k^{-1} = -t_0^{-\frac{1}{2}}(a_ke_k + t_k^{\frac{1}{2}}) - t_0^{\frac{1}{2}}(a_ke_k + t_k^{-\frac{1}{2}}) = -[\![t_0]\!]a_ke_k - [\![t_ot_k^{-1}]\!]$,

$$\begin{aligned}
&I_2\left(W_1 + W_1^{-1}\right)I_2\\
&\qquad = (-[\![t]\!])^{\frac{k-1}{2}}\left(-[\![t_0]\!]a_kI_2I_1I_2 - [\![t_0t_k^{-1}]\!]M^{\mathrm{odd}}\right)\\
&\qquad = (-[\![t]\!])^{\frac{k-1}{2}}\left(-[\![t_0]\!]a_kI_2I_1I_2 - [\![t_0t_k^{-1}]\!]\left(\frac{-[\![t_0]\!]}{a_0}\right)I_2\right)\\
&\qquad = (-[\![t]\!])^{\frac{k-1}{2}}\left(\frac{-[\![t_0]\!]}{a_0}\right)\left(a_0a_kI_2I_1I_2 - [\![t_0t_k^{-1}]\!]I_2\right) = -(t+1)(-[\![t]\!])^{\frac{k-3}{2}}D^{\mathrm{odd}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left(\frac{-[\![t_0]\!]}{a_0}\right)(-[\![t]\!])^{\frac{k-1}{2}}ZI_2 = ZI_2^2 = I_2ZI_2\\
&\quad = I_2\left(W_1 + W_1^{-1}\right)I_2 + \sum_{i=0}^{\frac{k-1}{2}-1} I_2\left(W_{2+2i} + W_{3+2i} + W_{2+2i}^{-1} + W_{3+2i}^{-1}\right)I_2\\
&\quad = -(t+1)(-[\![t]\!])^{\frac{k-3}{2}}t^{\frac{1}{2}}D^{\mathrm{odd}} + (-[\![t]\!])^{\frac{k-3}{2}}\left(\sum_{i=0}^{\frac{k-3}{2}}\left(t^{i+1} - t^{-(i+1)}\right)(-t-1)D^{\mathrm{odd}}\right)\\
&\quad = -(-[\![t]\!])^{\frac{k-3}{2}}(t+1)D^{\mathrm{odd}}\left(1 + \sum_{i=0}^{\frac{k-3}{2}}\left(t^{i+1} - t^{-(i+1)}\right)\right) = (-[\![t]\!])^{\frac{k-1}{2}}t^{\frac{1}{2}}D^{\mathrm{odd}}[k]. \quad \square
\end{aligned}$$

The following corollary explains how the defining relations for the symplectic blob algebra $TL_k(b)$ come from relations that hold in the algebra TL_k.

Corollary 3.3. *Let* $Z = W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1}$ *and let* I_1 *and* I_2 *be as defined in* (3.6) *and* (3.7). *If k is even, then*

$$a_0a_kI_1I_2I_1 = \left(\frac{1}{[k]}Z - [\![t_0t_kt^{-1}]\!]\right)I_1 \qquad \textit{and} \quad a_0a_kI_2I_1I_2 = \left(\frac{1}{[k]}Z - [\![t_0t_kt^{-1}]\!]\right)I_2.$$

If k is odd, then

$$a_0a_kI_1I_2I_1 = \left(\frac{1}{[k]}Z + [\![t_0t_k^{-1}]\!]\right)I_1 \qquad \textit{and} \quad a_0a_kI_2I_1I_2 = \left(\frac{1}{[k]}Z + [\![t_0t_k^{-1}]\!]\right)I_2.$$

Proof. As observed in the proof of Theorem 3.2, the products I_1ZI_1 and I_2Z_2 reduce to computation of the diagram with a single string going around all the poles (D^{even} or D^{odd}). These diagrammatics give that there are constants C, C_1, C_2 and D, D_1, D_2 such that

$$I_1^2 = CI_1, \quad I_1I_2I_1 = (C_1Z + C_2)I_1, \quad I_2^2 = DI_2, \quad I_2I_1I_2 = (D_1Z + D_2)I_2.$$

Then, computing $(I_1I_2I_1)^2$ in two different ways, we have

$$I_1I_2I_1I_1I_2I_1 = CI_1I_2I_1I_2I_1 = C(D_1Z + D_2)I_1I_2I_1,$$

and

$$I_1I_2I_1I_1I_2I_1 = (C_1Z + C_2)I_1I_1I_2I_1 = C(C_1Z + C_2)I_1I_2I_1,$$

which indicates that $C_1Z + C_2 = D_1Z + D_2$.

Theorem 3.2 gives that, if k is even, then

$$a_0a_kI_1I_2I_1 = D^{\text{even}} - [\![t_0t_kt^{-1}]\!]I_1 = \frac{1}{[k]}ZI_1 - [\![t_0t_kt^{-1}]\!]I_1,$$

and if k is odd, then

$$a_0a_kI_2I_1I_2 = t^{\frac{1}{2}}\left(\frac{a_0}{-[\![t_0]\!]}\right)D^{\text{odd}} + [\![t_0t_k^{-1}]\!]I_2 = \frac{1}{[k]}ZI_2 + [\![t_0t_k^{-1}]\!]I_2. \qquad \square$$

Remark 3.4 (Comparison to de Gier–Nichols.)**.** Let us explain how to relate the constants in Corollary 3.3 and Proposition 4.2 to the values which appear in [4]. Let

$$\begin{aligned} t_0^{\frac{1}{2}} &= -iq^{\omega_1}, & t^{\frac{1}{2}} &= q^{-1}, & t_k^{\frac{1}{2}} &= -iq^{\omega_2}, \\ T_0 &= -i\mathscr{g}_0, & T_i &= -\mathscr{g}_i, & T_k &= -i\mathscr{g}_k, \\ e_0 &= \mathscr{e}_0, & e_i &= \mathscr{e}_i, & e_k &= \mathscr{e}_k. \end{aligned}$$

Then

$$(\mathscr{g}_0 - q^{\omega_1})(\mathscr{g}_0 - q^{-\omega_1}) = 0, \quad (\mathscr{g}_i + q^{-1})(\mathscr{g}_i - q) = 0, \quad (\mathscr{g}_k - q^{\omega_1})(\mathscr{g}_{-}q^{-\omega_1}) = 0,$$

as in [4, Definitions 2.4, 2.6, and 2.8], and

$$\mathscr{g}_0 = q^{\omega_1} - \left(q^{1+\omega_1} - q^{-(1+\omega_1)}\right)\mathscr{e}_0, \quad \mathscr{g}_i = \mathscr{e}_i - q^{-1}, \quad \mathscr{g}_k = q^{\omega_2} - \left(q^{1+\omega_2} - q^{-(1+\omega_2)}\right)e_k,$$

as in [4, (5)]. Following [4, Definitions 2.8 and (9)],

$$\begin{aligned} \mathscr{J}_0^{(C)} &= \mathscr{g}_1^{-1}\cdots\mathscr{g}_{k-1}^{-1}\mathscr{g}_k\mathscr{g}_{k-1}\cdots\mathscr{g}_2\mathscr{g}_1\mathscr{g}_0 \\ &= (-1)^{k-1}(-i)(-i)(-1)^{k-1}T_1^{-1}\cdots T_{k-1}^{-1}T_kT_{k-1}\cdots T_1T_0 = -W_1, \\ \mathscr{J}_i^{(C)} &= \mathscr{g}_i\mathscr{J}_{i-1}^{(C)}\mathscr{g}_i = (-1)^2T_i(-W_i)T_i = -W_{i+1} \quad \text{for } i \in \{1, \dots, k-1\}, \text{ and} \\ \mathscr{Z}_k &= \sum_{i=0}^{k-1}\left(J_i^{(C)} + \left(J_i^{(C)}\right)^{-1}\right) = -\left(W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1}\right) = -Z. \end{aligned}$$

Use the notation $[x] = \frac{t^{\frac{x}{2}} - t^{-\frac{x}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \frac{q^x - q^{-x}}{q - q^{-1}}$ as in (2.8), and let a_0, a and a_k take the favorite values from Remark 2.1 so that

$$a = -1, \quad a_0 = -[\![t_0t^{-1}]\!], \quad \text{and} \quad a_k = -[\![t_kt^{-1}]\!],$$

and set

$$\theta = c + \frac{k-1}{2} \quad \text{and} \quad z = [\![t^{\theta}]\!][k],$$

as in Proposition 4.2. Following [4, Theorem 4.1] and remembering that $\mathcal{Z}_k = -Z$, let

$$\Theta = \theta + \frac{1}{\log q} i\pi \quad \text{so that} \quad \begin{aligned} -[k][\![t^\theta]\!] &= -[k]\left(t^{\frac{\theta}{2}} + t^{-\frac{\theta}{2}}\right) = [k]\left(-q^{-\theta} - q^{\theta}\right) \\ &= [k]\left(q^{-\left(\theta + \frac{1}{\log q} i\pi\right)} + q^{\theta + \frac{1}{\log q} i\pi}\right) \\ &= [k]\left(q^{-\Theta} + q^{\Theta}\right) = [k]\frac{[2\Theta]}{[\Theta]}. \end{aligned}$$

Note that

$$\begin{aligned} a_0 a_k = [\![t_0 t^{-1}]\!]\,[\![t_k t^{-1}]\!] &= \left(t_0^{\frac{1}{2}} t^{-\frac{1}{2}} + t_0^{-\frac{1}{2}} t^{\frac{1}{2}}\right)\left(t_k^{\frac{1}{2}} t^{-\frac{1}{2}} + t_k^{-\frac{1}{2}} t^{\frac{1}{2}}\right) \\ &= \left(-iq^{-\omega_1 - 1} + iq^{\omega_1 + 1}\right)\left(-iq^{-\omega_2 - 1} + iq^{\omega_2 + 1}\right) = -[\omega_1 + 1][\omega_2 + 1]\left(q - q^{-1}\right)^2. \end{aligned}$$

Then the constant b that appears in [4, Definition 3.6 and Theorem 4.1] to make $I_1 I_2 I_1 = bI_1$ and $I_2 I_1 I_2 = bI_2$ as operators on a simple TL_k-module is computed from Corollary 3.3 as follows:

$$\begin{aligned} b = \frac{\frac{1}{[k]} z - [\![t_0 t_k t^{-1}]\!]}{a_0 a_k} &= \frac{\frac{1}{[k]}[k][\![t^\theta]\!] - [\![t_0 t_k t^{-1}]\!]}{[\![t_0 t^{-1}]\!]\,[\![t_k t^{-1}]\!]} = \frac{[\![t^\theta]\!] - [\![t_0 t_k t^{-1}]\!]}{[\![t_0 t^{-1}]\!]\,[\![t_k t^{-1}]\!]} \\ &= -\frac{\left(q^{\Theta} + q^{-\Theta}\right) + \left(-iq^{\omega_1}\right)\left(-iq^{\omega_2}\right) q + \left(iq^{-\omega_1}\right)\left(iq^{-\omega_2}\right) q^{-1}}{-[\omega_1 + 1][\omega_2 + 1]\left(q - q^{-1}\right)^2} \\ &= \frac{q^{\Theta} + q^{-\Theta} - q^{\omega_1 + \omega_2 + 1} - q^{-(\omega_1 + \omega_2 + 1)}}{[\omega_1 + 1][\omega_2 + 1]\left(q - q^{-1}\right)^2} \\ &= \frac{\left((q^{\omega_1 + \omega_2 + 1 + \Theta})^{\frac{1}{2}} - (q^{\omega_1 + \omega_2 + 1 + \Theta})^{-\frac{1}{2}}\right)\left((q^{\omega_1 + \omega_2 + 1 - \Theta})^{\frac{1}{2}} - (q^{\omega_1 + \omega_2 + 1 - \Theta})^{-\frac{1}{2}}\right)}{[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2} \\ &= \frac{\left[\frac{1}{2}(\omega_1 + \omega_2 + 1 + \Theta)\right]\left[\frac{1}{2}(\omega_1 + \omega_2 + 1 - \Theta)\right]}{[\omega_1 + 1][\omega_2 + 1]} \quad \text{when } k \text{ is even, and} \end{aligned}$$

$$\begin{aligned} b = \frac{\frac{1}{[k]} z + [\![t_0 t_k^{-1}]\!]}{a_0 a_k} &= \frac{\frac{1}{[k]}[k][\![t^\theta]\!] + [\![t_0 t_k^{-1}]\!]}{[\![t_0 t^{-1}]\!]\,[\![t_k t^{-1}]\!]} = \frac{[\![t^\theta]\!] + [\![t_0 t_k^{-1}]\!]}{[\![t_0 t^{-1}]\!]\,[\![t_k t^{-1}]\!]} \\ &= \frac{-\left(q^{\Theta} + q^{-\Theta}\right) + (-iq^{\omega_1})(iq^{-\omega_2}) + (iq^{-\omega_1})(-iq^{\omega_2})}{-[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2} \\ &= \frac{-q^{\Theta} - q^{-\Theta} + q^{\omega_1 - \omega_2} + q^{-(\omega_1 - \omega_2)}}{-[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2} \\ &= -\frac{\left((q^{\omega_1 - \omega_2 - \Theta})^{\frac{1}{2}} - (q^{\omega_1 - \omega_2 - \Theta})^{-\frac{1}{2}}\right)\left((q^{\omega_1 - \omega_2 + \Theta})^{\frac{1}{2}} - (q^{\omega_1 - \omega_2 + \Theta})^{-\frac{1}{2}}\right)}{[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2} \\ &= -\frac{[\frac{1}{2}(\omega_1 - \omega_2 - \Theta)][\frac{1}{2}(\omega_1 - \omega_2 + \Theta)]}{[\omega_1 + 1][\omega_2 + 1]} \qquad \text{when } k \text{ is odd.} \end{aligned}$$

4. CALIBRATED REPRESENTATIONS OF H_k^{ext} AND TL_k^{ext}

In this section we classify and construct all irreducible calibrated representations of the extended two boundary Temperley–Lieb algebras TL_k^{ext}. This is done by using the classification of irreducible calibrated H_k^{ext}-modules from [3]. Using the formulas for the elements $p_i^{(1^3)}$, $p_0^{(\emptyset,1^2)}$, $p_0^{(1^2,\emptyset)}$, $p_{0^\vee}^{(\emptyset,1^2)}$, and $p_{0^\vee}^{(1^2,\emptyset)}$ that one quotients H_k^{ext} by to obtain TL_k^{ext}, we determine exactly which irreducible calibrated representations of H_k^{ext} factor through the quotient, thus providing a full classification of irreducible calibrated representations of TL_k^{ext}.

4.1. **Calibrated representations of H_k^{ext}.** A *calibrated* H_k^{ext}*-module* is an H_k^{ext}-module M such that $W_0, W_1, \ldots, W_k$ are simultaneously diagonalizable as operators on M.

In [3, Theorem 3.5] we showed that the irreducible calibrated H_k^{ext}-modules can be classified by combinatorial data called skew local regions so that the map

$$\begin{array}{ccc} \mathbb{C}^\times \times \{\text{skew local regions } (\mathbf{c}, J)\} & \longleftrightarrow & \{\text{irreducible calibrated } H_k^{\mathrm{ext}}\text{-modules}\} \\ (z, \mathbf{c}, J) & \longmapsto & H_k^{(z,\mathbf{c},J)} \end{array}$$

is a bijection. Furthermore, [3, Theorem 3.5] gives an explicit construction of the irreducible representation $H_k^{(z,\mathbf{c},J)}$ from the combinatorial data in the skew local region $(\mathbf{c}, J)$ and the parameter $z \in \mathbb{C}^\times$.

The data of a skew local region $(\mathbf{c}, J)$ is equivalent to a configuration of boxes κ, where the boxes in κ satisfy the conditions $(\kappa1)$–$(\kappa4)$ of [3, § 3.1].

The dimension of the irreducible representation $H_k^{(z,\mathbf{c},J)}$ is the cardinality of the set $\mathcal{F}^{(\mathbf{c},J)}$ defined in [3, (3.8)] and the map

$$\begin{array}{ccc} \mathcal{F}^{(\mathbf{c},J)} & \longrightarrow & \{\text{standard fillings } S \text{ of the boxes of } \kappa\} \\ w & \longmapsto & S_w \end{array} \quad \text{is a bijection}$$

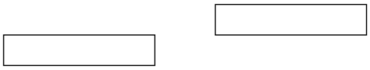
(see [3, Proposition 3.2]). The effect is that the standard fillings S of the boxes of κ provide a powerful tool for viewing the combinatorial structure of the representation $H_k^{(z,\mathbf{c},J)}$.

4.2. **Calibrated representations of TL_k^{ext}.** The following theorem determines which calibrated irreducible representations of H_k^{ext} are TL_k^{ext}-modules. In Theorem 4.1 the answer is stated in terms of the configuration of boxes κ. By $(\kappa1)$–$(\kappa4)$ of [3, § 3.1], the local region $(\mathbf{c}, J)$ is determined by κ. See Theorem 5.1 for the explicit conversion from κ to $(\mathbf{c}, J)$ for the irreducible calibrated TL_k-modules. The form of the skew shapes that appear in (4.2) forces that

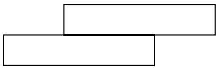
$$\mathbf{c} = (c, c+1, \ldots, c+k-1) \quad \text{with } c \in \mathbb{C}, \tag{4.1}$$

for any H_k^{ext}-module $H_k^{(z,\mathbf{c},J)}$ which is a TL_k^{ext}-module (i.e. which factors through the quotient that defines TL_k^{ext}).

Theorem 4.1. *Assume that if $r_1, r_2 \in \mathbb{Z}$ or $r_1, r_2 \in \mathbb{Z} + \frac{1}{2}$, then $r_2 > r_1 + 1$. Let κ be the configuration of boxes corresponding to a skew local region $(\mathbf{c}, J)$ with $\mathbf{c} \in \mathbb{Z}^k$ or $\mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k$. The irreducible calibrated H_k^{ext}-module $H_k^{(z,\mathbf{c},J)}$ is a TL_k^{ext}-module if and only if κ is a 180° rotationally symmetric skew shape with two rows of k boxes each (with or without markings),*



or



. (4.2)

Proof. Let $P = \{p_0^{(\emptyset,1^2)}, p_0^{(1^2,\emptyset)}, p_{0^\vee}^{(\emptyset,1^2)}, p_{0^\vee}^{(1^2,\emptyset)}, p_1^{(1^3)}, p_2^{(1^3)}, \ldots, p_{k-2}^{(1^3)}\}$ so that TL_k is the quotient of H_k by the ideal generated by the set P. For $w \in \mathcal{F}^{(\mathbf{c},J)}$ let S_w be the standard tableau of shape κ corresponding to w as given in [3, Proposition 3.2]. For $j \in \{-k, \dots, -1, 1, \dots, k\}$,

$$(w\mathbf{c})_j \text{ is the diagonal number of } S_w(j),$$

where $S_w(j)$ is the box containing j in S_w.

Step 1. Rewriting of the conditions for $pv_w = 0$ for $p \in P$. By the construction of $H_k^{(z,\mathbf{c},J)}$ in [3, Theorem 3.5], the module $H_k^{(z,\mathbf{c},J)}$ has basis $\{v_w \mid w \in \mathcal{F}^{(\mathbf{c},J)}\}$ and, if $w \in \mathcal{F}^{(\mathbf{c},J)}$ then

$$\begin{aligned}
\tau_i v_w = 0 &\quad \text{if and only if} \quad (w\mathbf{c})_{i+1} = (w\mathbf{c})_i \pm 1, \\
f_{\varepsilon_i - r_2} v_w = 0 &\quad \text{if and only if} \quad (w\mathbf{c})_i = r_2, \quad \text{and} \\
f_{\varepsilon_i - \varepsilon_j + 1} v_w = 0 &\quad \text{if and only if} \quad (w\mathbf{c})_i = (w\mathbf{c})_j - 1.
\end{aligned}$$

Let $i \in \{1, \dots, k-2\}$. Using the expansion of $p_i^{(1^3)}$ in terms of the τ_i from Proposition 2.2,

$$\begin{aligned}
p_i^{(1^3)} v_w = \tau_i \tau_{i+1} \tau_i v_w &- t^{-\frac{1}{2}} \tau_{i+1} \tau_i \frac{f_{\varepsilon_{i+1} - \varepsilon_{i+2} + 1}}{f_{\varepsilon_{i+1} - \varepsilon_{i+2}}} v_w - t^{-\frac{1}{2}} \tau_i \tau_{i+1} \frac{f_{\varepsilon_{i+1} - \varepsilon_i + 1}}{f_{\varepsilon_{i+1} - \varepsilon_i}} v_w \\
&+ t^{-1} \tau_i \frac{f_{\varepsilon_{i+1} - \varepsilon_{i+2} + 1} f_{\varepsilon_{i+2} - \varepsilon_i + 1}}{f_{\varepsilon_{i+1} - \varepsilon_{i+2}} f_{\varepsilon_{i+2} - \varepsilon_i}} v_w + t^{-1} \tau_{i+1} \frac{f_{\varepsilon_{i+2} - \varepsilon_i + 1} f_{\varepsilon_{i+1} - \varepsilon_i + 1}}{f_{\varepsilon_{i+2} - \varepsilon_i} f_{\varepsilon_{i+1} - \varepsilon_i}} v_w \\
&- t^{-\frac{3}{2}} \frac{f_{\varepsilon_{i+1} - \varepsilon_{i+2} + 1} f_{\varepsilon_{i+2} - \varepsilon_i + 1} f_{\varepsilon_{i+1} - \varepsilon_i + 1}}{f_{\varepsilon_{i+1} - \varepsilon_{i+2}} f_{\varepsilon_{i+2} - \varepsilon_i} f_{\varepsilon_{i+1} - \varepsilon_i + 1}} v_w,
\end{aligned}$$

we consider the condition $p_i^{(1^3)} v_w = 0$ term-by-term. First, $\tau_i \tau_{i+1} \tau_i v_w = 0$ exactly when $(wc)_{i+1} = (wc)_i \pm 1$ or $(s_i wc)_{i+2} = (s_i wc)_{i+1} \pm 1$ or $(s_{i+1} s_i w)_{i+1} = (s_{i+1} s_i w)_i = \pm 1$, i.e. when

$$(w\mathbf{c})_{i+1} = (w\mathbf{c})_i \pm 1 \quad \text{or} \quad (w\mathbf{c})_{i+2} = (w\mathbf{c})_i \pm 1 \quad \text{or} \quad (w\mathbf{c})_{i+2} = (w\mathbf{c})_{i+1} \pm 1.$$

Next, $-t^{-\frac{3}{2}} \frac{f_{\varepsilon_{i+1} - \varepsilon_{i+2} + 1} f_{\varepsilon_{i+2} - \varepsilon_i + 1} f_{\varepsilon_{i+1} - \varepsilon_i + 1}}{f_{\varepsilon_{i+1} - \varepsilon_{i+2}} f_{\varepsilon_{i+2} - \varepsilon_i} f_{\varepsilon_{i+1} - \varepsilon_i + 1}} v_w = 0$ exactly when

$$(w\mathbf{c})_{i+1} = (w\mathbf{c})_i + 1 \quad \text{or} \quad (w\mathbf{c})_{i+2} = (w\mathbf{c})_i + 1 \quad \text{or} \quad (w\mathbf{c})_{i+1} = (w\mathbf{c})_{i+2} + 1.$$

Thus $-t^{-\frac{3}{2}} \frac{f_{\varepsilon_{i+1} - \varepsilon_{i+2} + 1} f_{\varepsilon_{i+2} - \varepsilon_i + 1} f_{\varepsilon_{i+1} - \varepsilon_i + 1}}{f_{\varepsilon_{i+1} - \varepsilon_{i+2}} f_{\varepsilon_{i+2} - \varepsilon_i} f_{\varepsilon_{i+1} - \varepsilon_i + 1}} v_w = 0$ already implies $\tau_i \tau_{i+1} \tau_i v_w = 0$, and similarly for the other terms in the expansion of $p_i^{(1^3)} v_w = 0$. Thus $p_i^{(1^3)} v_w = 0$ if and only if

$$(w\mathbf{c})_i = (w\mathbf{c})_{i+1} - 1 \ \text{ or } \ (w\mathbf{c})_i = (w\mathbf{c})_{i+2} - 1 \ \text{ or } \ (w\mathbf{c})_{i+1} = (w\mathbf{c})_{i+2} - 1. \tag{4.3}$$

Similarly, $p_0^{(\emptyset, 1^2)} v_w = 0$ if and only if

$$\begin{aligned}
(w\mathbf{c})_1 \in \{r_1, r_2\} \quad &\text{or} \quad (w\mathbf{c})_2 \in \{r_1, r_2\} \\
\text{or} \quad (w\mathbf{c})_2 = (w\mathbf{c})_1 + 1 \quad &\text{or} \quad (w\mathbf{c})_2 = (w\mathbf{c})_{-1} + 1;
\end{aligned} \tag{4.4}$$

$p_0^{(1^2, \emptyset)} v_w = 0$ if and only if

$$\begin{aligned}
(w\mathbf{c})_1 \in \{-r_1, -r_2\} \quad &\text{or} \quad (w\mathbf{c})_2 \in \{-r_1, -r_2\} \\
\text{or} \quad (w\mathbf{c})_2 = (w\mathbf{c})_1 + 1 \quad &\text{or} \quad (w\mathbf{c})_2 = (w\mathbf{c})_{-1} + 1;
\end{aligned} \tag{4.5}$$

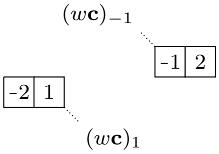
$p_{0^\vee}^{(\emptyset, 1^2)} v_w = 0$ if and only if

$$\begin{aligned}
(w\mathbf{c})_1 \in \{-r_1, r_2\} \quad &\text{or} \quad (w\mathbf{c})_2 \in \{-r_1, r_2\} \\
\text{or} (w\mathbf{c})_2 = (w\mathbf{c})_1 + 1 \quad &\text{or} \quad (w\mathbf{c})_2 = (w\mathbf{c})_{-1} + 1;
\end{aligned} \tag{4.6}$$

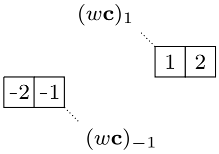
and $p_{0^\vee}^{(1^2,\emptyset)} v_w = 0$ if and only if

$$\begin{aligned} & (w\mathbf{c})_1 \in \{r_1, -r_2\} \quad \text{or} \quad (w\mathbf{c})_2 \in \{r_1, -r_2\} \\ \text{or} \quad & (w\mathbf{c})_2 = (w\mathbf{c})_1 + 1 \quad \text{or} \quad (w\mathbf{c})_2 = (w\mathbf{c})_{-1} + 1. \end{aligned} \tag{4.7}$$

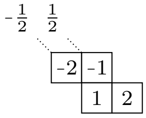
Step 2. If κ is as in (4.2) and $w \in \mathcal{F}^{(\mathbf{c},J)}$ and $p \in P$ then $pv_w = 0$. Assume κ has the form given in (4.2) and let $w \in \mathcal{F}^{(\mathbf{c},J)}$. Since κ has only two rows the positions of $(-2,-1,1,2)$ in S_w take one of the following forms:



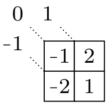
$(wc)_1 < -\frac{1}{2}$,



$(wc)_1 > \frac{1}{2}$,

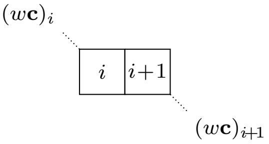


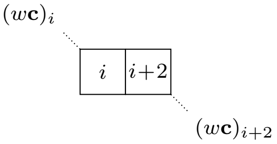
$(wc)_1 = -\frac{1}{2}$,



$(wc)_1 = 0$.

In each of these cases, the conditions in (4.4)–(4.7) give that $p_0^{(\emptyset,1^2)} v_w = 0$, $p_0^{(1^2,\emptyset)} v_w = 0$, $p_{0^\vee}^{(\emptyset,1^2)} v_w = 0$ and $p_{0^\vee}^{(1^2,\emptyset)} v_w = 0$. Next, let $i \in \{1, \dots, k-2\}$. Since κ has only two rows, then either i or $i+1$ are in the same row





or i and $i+2$ are in the same row. Thus, by (4.3), $p_i v_w = 0$. This completes the proof that if κ is of the form (4.2) then $H_k^{(z,\mathbf{c},J)}$ is a TL_k^{ext}-module.

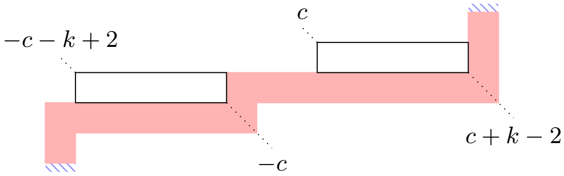
Step 3. If κ is not as in (4.2) then there exists $w \in \mathcal{F}^{(\mathbf{c},J)}$ and $p \in P$ such that $pv_w \neq 0$. Let $2k$ be the number of boxes in κ. The proof is by induction on k.

First, if $k = 2$, then the condition (4.3) does not apply. If $\mathbf{c} = (r_1, r_2)$ then there are 8 possibilities for $w\mathbf{c}$: (r_1, r_2), $(-r_1, r_2)$, $(r_1, -r_2)$, $(-r_1, -r_2)$, (r_2, r_1), $(-r_2, r_1)$, $(r_2, -r_1)$ and $(-r_2, -r_1)$. None of these satisfy all of the conditions (4.4)–(4.7). If $\mathbf{c} = (c_1, c_1 + 1)$, then $s_1\mathbf{c} = (c_1 + 1, c_1)$ does not satisfy (4.4) and $s_0 s_1 s_0 s_1 \mathbf{c} = (-c, -c-1)$ does not satisfy (4.7). Thus only the darker blue shaded local regions in Figure 4.1 can have $pv_w = 0$ for all $p \in P$ and all $w \in \mathcal{F}^{(\mathbf{c},J)}$. For all of these, κ is as in (4.2).

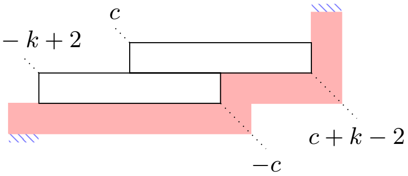
Next, assume $k > 2$ and proceed inductively. If $H_k^{(z,\mathbf{c},J)}$ is a calibrated TL_k^{ext}-module then

$$\mathrm{Res}^{TL_k^{\mathrm{ext}}}_{TL_{k-1}^{\mathrm{ext}}} \left(H_k^{(z,\mathbf{c},J)} \right)$$

is calibrated TL_{k-1}^{ext}-module. This means that if S_w is a standard tableau of shape κ and S'_w is S_w except with the boxes $S_w(k)$ and $S_w(-k)$ removed and κ' is the shape of S'_w, then κ' must be as in (4.2) and have only two rows. The box $S_w(k)$ is in a SE corner of κ and the box $S_w(-k)$ is in a NW corner of κ.



or



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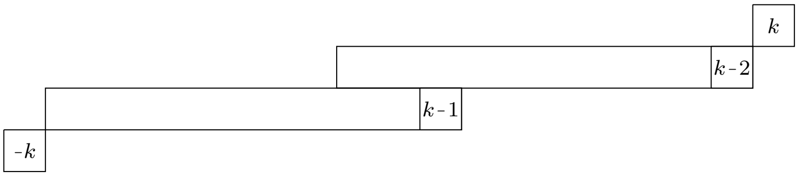
Given that κ' has only two rows and κ is obtained from κ' by adding boxes that could contain k and $-k$ in a standard tableau, the following are possibilities that we discard for κ:



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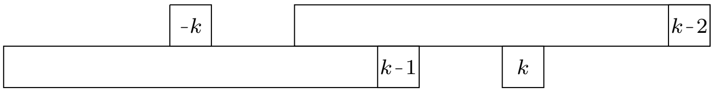
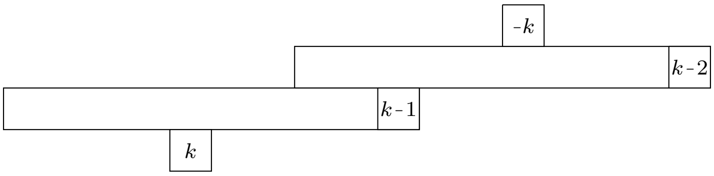


, and



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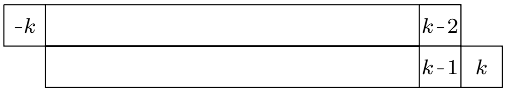
Namely, in each case there is a standard tableaux that has $k-2$, $k-1$ and k in positions that do not satisfy the conditions in (4.3). Thus, in these cases, there exists an S_w of shape κ for which $p_{k-2}^{(1^3)}v_w \neq 0$. In the remaining case



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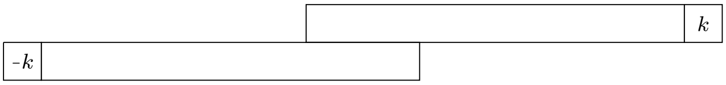
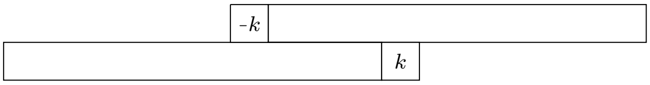
the shape κ does not satisfy the $(w\mathbf{c})_{k-2} \neq (w\mathbf{c})_k$ from [3, (3.10)] and the module $H_k^{(z,\mathbf{c},J)}$ is not calibrated. In summary, unless κ is of the form given in (4.2)



or



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then either $H_k^{(z,\mathbf{c},J)}$ is not calibrated or there exists an S_w of shape κ for which $p_{k-2}^{(1^3)}v_w \neq 0$.

□

The following proposition determines the action of the central element Z on each of the irreducible calibrated TL_k^{ext}-modules. As noted in (4.1), if an irreducible H_k^{ext}-module $H_k^{(z,\mathbf{c},J)}$ is a TL_k^{ext}-module $\mathbf{c} = (c, c+1, \ldots, c+k-1)$ for some $c \in \mathbb{C}$.

Proposition 4.2. *Let* $Z = W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1}$ *be the central element of* TL_k^{ext} *studied in Theorem 3.2. Assume that* $\mathbf{c} = (c, c+1, \ldots, c+k-1)$ *and* $H_k^{(z,\mathbf{c},J)}$ *is an irreducible calibrated* TL_k^{ext} *as in Theorem 4.1. If* $v \in H_k^{(z,\mathbf{c},J)}$ *then*

$$Zv = [\![t^\theta]\!][k]v, \qquad \text{where} \quad \theta = c + \frac{k-1}{2}, \quad [\![t^\theta]\!] = t^{\frac{\theta}{2}} + t^{-\frac{\theta}{2}} \quad \text{and} \quad [k] = \frac{t^{\frac{k}{2}} - t^{-\frac{k}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}.$$

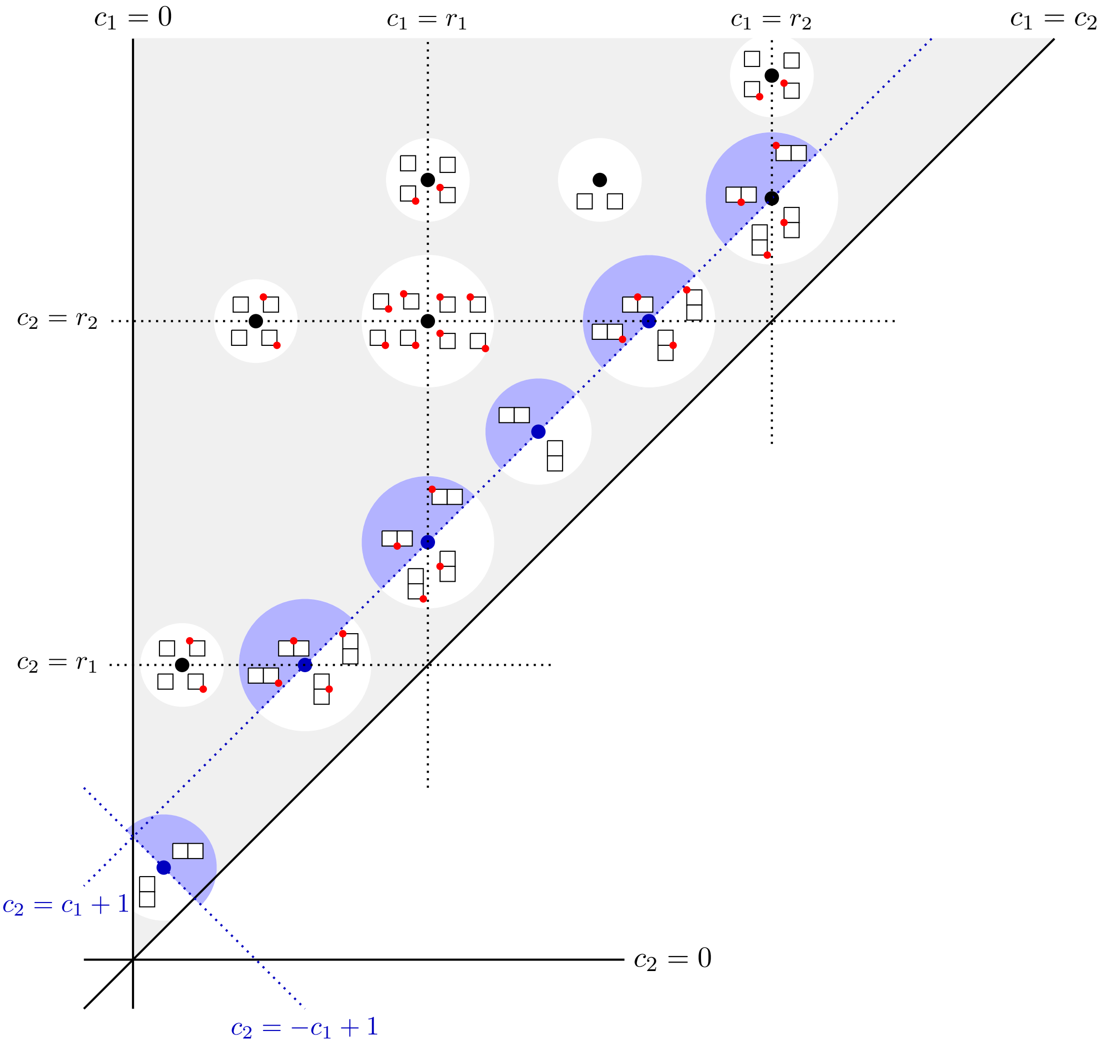


FIGURE 4.1. Calibrated representations of H_2 have regular central character. For each $(\mathbf{c}, J)$ the corresponding configuration of boxes κ is displayed in the local region of chambers corresponding to the elements of $\mathcal{F}^{(\mathbf{c},J)}$; only the boxes on positive diagonals are shown, since they determine κ when $\mathbf{c}$ is regular. The local regions marked in blue are those that factor through the Temperley–Lieb quotient.

Proof. Let $v \in H_k^{(z,\mathbf{c},J)}$ be such that $W_i v = q^{c+i-1}$ for $i \in \{1, \dots, k\}$. Then $Z v_w = z v_w$ where

$$\begin{aligned} z &= t^{-(c+k-1)} + \cdots + t^{-(c+1)} + t^{-c} + t^{c} + t^{c+1} + \cdots + t^{c+k-1} \\ &= \left(t^{c+\frac{k-1}{2}} + t^{-\left(c+\frac{k-1}{2}\right)}\right)\left(t^{-\frac{k-1}{2}} + \cdots + t^{\frac{k-1}{2}}\right) = \left(t^{\frac{\theta}{2}} + t^{-\frac{\theta}{2}}\right)\frac{t^{\frac{k}{2}} - t^{-\frac{k}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = [\![t^{\theta}]\!]\,[k]. \end{aligned}$$

Since Z is a central element of H_k^{ext} and $H_k^{(z,\mathbf{c},J)}$ is a simple H_k^{ext}-module, Schur's lemma implies that if $v \in H_k^{(z,\mathbf{c},J)}$ then $Zv = zv$. ☐

5. SCHUR–WEYL DUALITY BETWEEN TL_k^{ext} AND $U_q\mathfrak{gl}_2$

In this section we show that the Schur–Weyl duality studied in [3] provides calibrated irreducible representations of the two boundary Temperley–Lieb algebra. We classify these representations using the combinatorial classification of irreducible calibrated TL_k^{ext} modules obtained in Theorem 4.1. We follow the combinatorics of [2, § 4] and [3, § 5]. Similar constructions hold for replacing $\mathfrak{gl}_2$ with $\mathfrak{sl}_2$.

The irreducible finite dimensional representations $L(\lambda)$ of $U_q(\mathfrak{gl}_2)$ are indexed by $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ with $\lambda_1 \geq \lambda_2$. The dimension of $L(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2 + 1$. By the Clebsch–Gordan formula or the Littlewood–Richardson rule (see [13, (5.16)])

$$L(a,0) \otimes L(b,0) = L(a+b,0) \oplus L(a+b-1,1) \oplus \cdots \oplus L(a+1,b-1) \oplus L(a,b),$$

and

$$L(\lambda_1,\lambda_2) \otimes L(1,0) = \begin{cases} L(\lambda_1+1,\lambda_2) \oplus L(\lambda_1,\lambda_2+1), & \text{if } \lambda_1 > \lambda_2, \\ L(\lambda_1+1,\lambda_2), & \text{if } \lambda_1 = \lambda_2, \end{cases}$$

for $a, b \in \mathbb{Z}_{\geq 0}$ with $a \geq b$ and $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ with $\lambda_1 \geq \lambda_2$.

Now, fix $a, b \in \mathbb{Z}_{\geq 0}$ with $a \geq b$ and fix the simple $U_q\mathfrak{gl}_2$-modules

$$M = L(a,0), \qquad N = L(b,0) \qquad V = L(1,0). \tag{5.1}$$

We identify $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ with a left-justified arrangement of boxes with λ_i boxes in the ith row. As in [3, (5.28)], with a and b fixed as in (5.1), the *shifted content* of a box in row i and column j of (λ_1, λ_2) as

$$\widetilde{c}(\mathrm{box}) = j - i - \tfrac{1}{2}(a+b-2) \tag{5.2}$$

i.e. the shifted content is its diagonal number, where the box in the upper left corner has shifted content $-\frac{1}{2}(a+b-2)$.

For $j \in \mathbb{Z}_{\geq -1}$ let $\mathcal{P}^{(j)}$ be an index set for the irreducible $U_q\mathfrak{gl}_2$-modules that appear in $M \otimes N \otimes V^{\otimes j}$. Following [3, § 5.4], the associated Bratteli diagram is the (ranked) graph with

(v) vertices on level j labeled by the partitions in $\mathcal{P}^{(j)}$, where

$$\mathcal{P}^{(-1)} = (a,0), \quad \text{and} \quad \mathcal{P}^{(0)} = \{(a+b-j, j) \mid j = 0, 1, \ldots, b\}$$

and

$$\mathcal{P}^{(j)} = \left\{ (a+b+j-\ell, \ell) \,\middle|\, 0 \leq \ell \leq \tfrac{1}{2}(j+a+b) \right\}, \text{ for } j \geq 1;$$

(e) an edge $(a,0) \longrightarrow \mu$ for each $\mu \in \mathcal{P}^{(0)}$; and for each $j \geq 0$, $\mu \in \mathcal{P}^{(j)}$ and $\lambda \in \mathcal{P}^{(j+1)}$, there is

$$\text{an edge} \quad \mu \to \lambda \quad \text{if } \lambda \text{ is obtained from } \mu \text{ by adding a box.}$$

The case when $a = 6$ and $b = 3$ is illustrated in Figure 5.1.

Assume $q \in \mathbb{C}^\times$ and $a > b+2$ so that the generality condition $(a+1) - (b+1) \notin \{0, \pm 1, \pm 2\}$ of [3, Theorem 5.5] is satisfied. Define

$$r_1 = \frac{1}{2}(a-b) \qquad \text{and} \qquad r_2 = \frac{1}{2}(a+b+2), \tag{5.3}$$

and let H_k^{ext} be the extended two boundary Hecke algebra with parameters $t_0^{\frac{1}{2}}$, $t_k^{\frac{1}{2}}$, and $t^{\frac{1}{2}}$ given by

$$t^{\frac{1}{2}} = q, \quad t_0 = -t^{r_2 - r_1} = -q^{(b+1)}, \quad \text{and} \quad t_k = -t^{r_2 + r_1} = -q^{2(a+1)}, \tag{5.4}$$

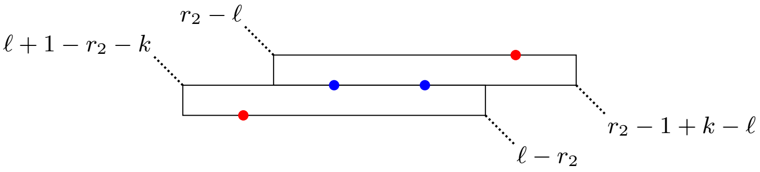
so that $-t_k^{\frac{1}{2}} t_0^{-\frac{1}{2}} = -t^{r_1}$ and $t_k^{\frac{1}{2}} t_0^{\frac{1}{2}} = -t^{r_2}$ as in [3, (3.5), (5.21)]. By [3, Theorem 5.4 and (5.21)] there are commuting actions of $U_q\mathfrak{gl}_2$ and H_k^{ext} on $M \otimes N \otimes V^{\otimes k}$, where the H_k^{ext} action is given via R-matrices for the quantum group $U_q\mathfrak{gl}_2$.

It might seem that the conditions in (5.3) and (5.4) are restrictive. However, from the point of view of obtaining irreducible calibrated TL_k-modules by Schur–Weyl duality this does not pose any restriction. By the construction of irreducible calibrated H_k-modeules in [3, Theorem 3.5], the structure of the irreducible module depends only on the skew local region $(\mathbf{c}, J)$. Theorem 4.1 determines which of these are TL_k-modules. The following theorem shows that each of the possibilities (the various choices of J) does appear as one of the modules $B^{(a+b+k-\ell,\ell)}$ of the following theorem. The conditions (5.3) and (5.4) only specify which quantum group one should use to produce the desired TL_k-module. The conditions on the TL_k-parameters that are required for Theorem 5.1 are equivalent to t_0, t_k, t are invertible and $t_k \neq t_0$ and $t_k \neq t_0 t^{\pm 1}$ and $t_k \neq t^{\pm 2}$ and t^2 is not a root of unity.

Theorem 5.1. *Let* $a, b \in \mathbb{Z}_{\geq 0}$ *with* $a > b + 2$. *Let* $q \in \mathbb{C}^\times$ *not a root of unity and let* H_k^{ext} *be the two boundary Hecke algebra with parameters* $t_0^{\frac{1}{2}}$, $t_k^{\frac{1}{2}}$ *and* $t^{\frac{1}{2}}$ *as in* (5.4). *Let* $U_q\mathfrak{gl}_2$ *be the Drinfeld-Jimbo quantum group corresponding to* $\mathfrak{gl}_2$ *and let* M, N *and* V *be the simple* $U_q\mathfrak{gl}_2$*-modules given in* (5.1). *Then the* H_k^{ext} *action factors through* TL_k^{ext} *and, as* $(U_q\mathfrak{gl}_2, TL_k^{\mathrm{ext}})$*-bimodules (taking both commuting actions as left actions),*

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B_k^\lambda \qquad \text{with} \quad B_k^{(a+b+k-\ell,\ell)} \cong H^{(z,\mathbf{c},J)},$$

where $z = (-1)^k q^{(a+b-\ell)(a+b-\ell-1)+\ell(\ell-3)-a(a-1)-b(b-1)-k(a+b-2)}$ *and* $(\mathbf{c}, J)$ *is the local region corresponding to the configuration* κ *of* $2k$ *boxes*



(5.5)

that has k *boxes in each row, the shifted content of the leftmost box in the first row is* $r_2 - \ell$, *the shifted content of the leftmost box in the second row is* $\ell + 1 - r_2 - k$. *Between the rows there are blue markers in diagonals with shifted content* $\pm r_1$ *and there are red markers in diagonals with shifted content* $\pm r_2$, *as pictured. (These markers are the same as in* [3, Examples 5.7 and 5.8], *with colors red and blue used to highlight which of diagonals* $\pm r_1$ *and* $\pm r_2$ *they mark.) Explicitly,* $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ *is the sequence of*

$$\textit{absolute values of} \quad c,\ c+1,\ \cdots,\ c+k-1, \qquad \textit{where } c = \frac{1}{2}(a+b) - \ell + 1,$$

arranged in increasing order; and J *is the union of*

$$J_1 = \begin{cases} \emptyset, & \textit{if } a \geq b \geq \ell, \\ \{\varepsilon_{\ell-b}\}, & \textit{if } a \geq \ell > b, \\ \{\varepsilon_{a-b}\}, & \textit{if } \ell > a > b, \end{cases}$$

and

$$J_2 = \begin{cases} \emptyset, & \textit{if } \frac{1}{2}(a+b+2) > \ell, \\ \{\varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \ldots, \varepsilon_{2\ell-a-b} - \varepsilon_{2\ell-a-b-1}\}, & \textit{if } \ell \geq \frac{1}{2}(a+b+2) \textit{ and } a+b \textit{ even}, \\ \{\varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \ldots, \varepsilon_{2\ell-a-b} - \varepsilon_{2\ell-a-b-1}\}, & \textit{if } \ell \geq \frac{1}{2}(a+b+2) \textit{ and } a+b \textit{ odd}. \end{cases}$$

Proof. Fix $\lambda = (a+b+k-\ell, \ell) \in \mathcal{P}^{(k)}$. The sum of the contents of the boxes in λ is

$$\begin{aligned}\sum_{\mathrm{box}\,\in\,\lambda} c(\mathrm{box}) &= (0+1+\ldots+(a+b+k-\ell-1)) + (-1+0+\cdots+\ell-2) \\ &= \frac{1}{2}(a+b+k-\ell-1)(a+b+k-\ell) + \frac{1}{2}\ell(\ell-3).\end{aligned}$$

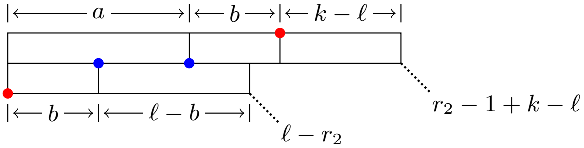
By [3, Theorem 5.5 and (5.35)], $\mathcal{B}_k^\lambda \cong H_k^{(z,\mathbf{c},J)}$ where

$$z = (-1)^k q^{2c_0}, \quad \text{where} \quad c_0 = -\frac{1}{2}(k(a+b-2) + a(a-1) + b(b-1)) + \sum_{\mathrm{box}\,\in\,\lambda} c(\mathrm{box}),$$

and $\mathbf{c}$ and J and the corresponding configuration κ of $2k$ boxes are determined as follows.

Place markers at the NW corner of the boxes at positions $(1, a+b+1)$, $(2, a+1)$, $(2, b+1)$, and $(3, 1)$ so that these markers are in the diagonals with shifted contents $\pm r_1$ and $\pm r_2$.

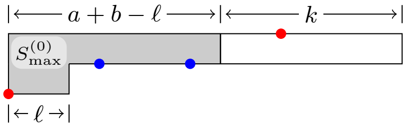
$$\lambda = (a+b+k-\ell, \ell) =$$



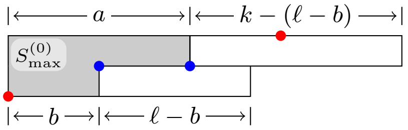
Following [3, (5.27)], let

$$S^{(0)}_{\max} = \begin{cases}(a+b-\ell, \ell), & \text{if } a \geq b \geq \ell, \\ (a, b), & \text{if } a \geq \ell \geq b\end{cases}$$

(since $a \geq b$ we are in the left case of [3, (5.15)] with $c = d = 1$ so that $\mu^c = \min(\ell, b)$ and $S^{(0)}_{\max} = \mathring{\mu} = (a+b-\mu^c, \mu^c)$):



if $a \geq b \geq \ell$



.

if $a \geq \ell > b$ or $\ell > a \geq b$

By [3, (5.35)], the corresponding configuration of boxes is $\kappa = \mathrm{rot}(\lambda/S^{(0)}_{\max}) \cup \lambda/S^{(0)}_{\max}$, as pictured above in (5.5).

To determine $(\mathbf{c}, J)$, use the conditions $(\kappa 1)$–$(\kappa 4)$ of [3, § 3.1] which specify the relation between κ and $(\mathbf{c}, J)$. First index the boxes of κ with $-k, \ldots, -1, 1, \ldots, k$ by diagonals, left to right, and NW to SE along diagonals. The sequence $\mathbf{c} = (c_1, \ldots, c_k)$ with $0 \leq c_1 \leq c_2 \leq \cdots \leq c_k$ is the sequence of the absolute values of the shifted contents of boxes in the first row of κ. Next, the set J is determined as follows.

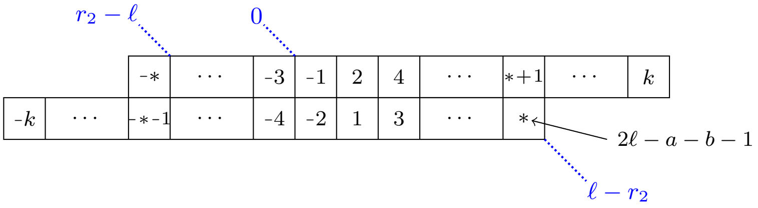
(1) By $(\kappa 4)$, the set J contains ε_i if $i > 0$ and box_i is NW of the marker in the diagonal with shifted content r_1 or r_2 in κ. This occurs on diagonal r_1 whenever $\ell > b$ (marked in blue),

$$\varepsilon_{\ell-b} \in J \text{ if } a \geq \ell > b \qquad \text{and} \qquad \varepsilon_{a-b} \in J \text{ if } \ell > a \geq b;$$

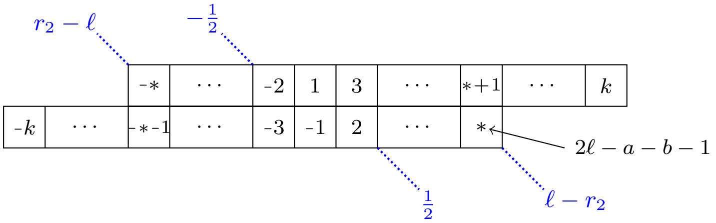
and J contains no roots of the form ε_j when $a \geq b \geq \ell$.

(2) By $(\kappa 3)$, the set J contains $\varepsilon_j - \varepsilon_i$ if $j > i > 0$ and box_i and box_j are in the same column of κ (so that box_i and box_j are in adjacent diagonals and box_j is NW of box_i). This occurs exactly when $0 \geq r_2 - \ell = \frac{1}{2}(a+b+2) - \ell$. If $\ell \geq \frac{1}{2}(a+b+2)$ and

$a+b$ is even then the boxes indexed $1, 3, \dots, 1+2(\ell-\frac{1}{2}(a+b+2)) = 2\ell-(a+b+1)$ are in the second row directly below boxes of index $2, 4, \dots, 2\ell-a-b$. If $\ell \geq \frac{1}{2}(a+b+2)$ and $a+b$ is odd then boxes $2, 4, \dots, 2(\ell-\frac{1}{2}(a+b+1))$, directly below boxes of index $3, 5, \dots, 2\ell-a-b$:



if $a+b$ is even,



if $a+b$ is odd.

So J contains

$$\begin{aligned} &\varepsilon_2-\varepsilon_1, \varepsilon_4-\varepsilon_3, \dots, \varepsilon_{2\ell-a-b}-\varepsilon_{2\ell-a-b-1} && \text{if } \ell \geq \frac{1}{2}(a+b+2) \text{ and } a+b \text{ is even, or} \\ &\varepsilon_3-\varepsilon_2, \varepsilon_5-\varepsilon_4, \dots, \varepsilon_{2\ell-a-b}-\varepsilon_{2\ell-a-b-1} && \text{if } \ell \geq \frac{1}{2}(a+b+2) \text{ and } a+b \text{ is odd.} \end{aligned}$$

(3) Also by $(\kappa 3)$, the set J contains $\varepsilon_j+\varepsilon_i$ if $j > i > 0$, and box_j is directly above box_{-i}, which does not occur.

In this way $\mathbf{c}$ and J are determined from κ. Since all of these $H_k^{(z,\mathbf{c},J)}$ satisfy the conditions of Theorem 4.1, it follows that the H_k^{ext}-action on $M \otimes N \otimes V^{\otimes k}$ factors through TL_k^{ext}. ☐

Remark 5.2. The dimension of $B_k^{(a+b+k-\ell,\ell)}$ is the number of paths in the Bratteli diagram from a shape on level 0 to the shape $\lambda = (a+b+k-\ell, \ell)$ on level k. Summing over the shapes on level 0 for which there is a path to λ gives

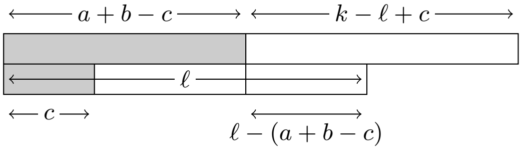
$$\dim\left(B^{(a+b+k-\ell,\ell)}\right) = \sum_{c=\max(0,\ell-k)}^{\min(b,\ell)} f^{\lambda/(a+b-c,c)},$$

where $f^{\lambda/\mu}$ is the number of standard tableaux of skew shape λ/μ. If $\ell \leq a+b-c$ then the second row of $\lambda/(a+b-\ell,\ell)$ does not overlap the first row and thus

$$f^{\lambda/(a+b-c,c)} = \binom{k}{\ell-c} \qquad \text{if } \ell \leq a+b-c.$$

Since $c \leq \min(b,\ell)$, the case $\ell > a+b-c$ can occur only when $\ell > a \geq b$, in which case

$$(a+b+k-\ell,\ell)/(a+b-c,c) =$$



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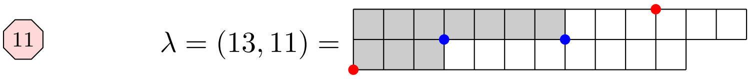
so that

$$f^{(a+b+k-\ell,\ell)/(a+b-c,c)} = \sum_{j=\ell-(a+b-c)}^{k+\ell-c} f^{(k-j,j)} = \sum_{j=\ell-(a+b-c)}^{\min(k-(\ell-c),\ell-c)} \binom{k}{j} - \binom{k}{j-1}$$
$$= \binom{k}{\ell-c} - \binom{k}{\ell-(a+b-c)-1}.$$

Namely, the first equality comes from the Pieri formula and the expansion of a skew Schur function by Littlewood–Richardson coefficients (see [13, (5.16)] for the Pieri formula and [13, (5.2) and (5.3)] for Littlewood–Richardson coefficients) and the second equality comes from the number of standard tableaux of a two row shape as given, for example, in [7, Theorem 2.8.5 and Lemma 2.8.4].

The following examples reference the node label styles in Figure 5.1.

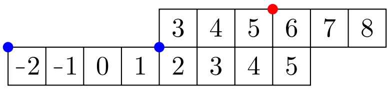
Example 5.3. Let $a = 7$ and $b = 3$. The markers are in the diagonals with shifted contents $\pm r_1$ and $\pm r_2$, where $r_1 = 2$ and $r_2 = 6$. An example where $\ell > a \geq b$. Let $\ell = 11$ and $k = 14$, then



$\lambda = (13, 11) =$ with $S^{(0)}_{\max} = (7, 3)$.

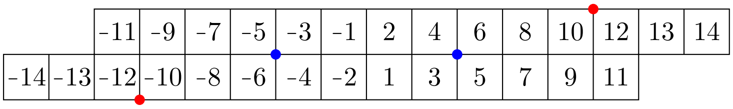
The boxes of $\lambda/S^{(0)}_{\max}$ have

shifted contents:



				3	4	5	6	7	8
-2	-1	0	1	2	3	4	5		

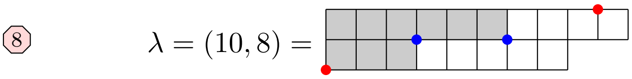
Then $\mathbf{c}$ is the rearrangement of the absolute values of $(-2, -1, 0, 1, 2, 3, 3, 4, 4, 5, 5, 6, 7, 8)$ into increasing order and $J = \{\varepsilon_4, \varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \varepsilon_8 - \varepsilon_7, \varepsilon_{10} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{11}\}$. The configuration of boxes κ corresponding to $(\mathbf{c}, J)$ has indexing of boxes



		-11	-9	-7	-5	-3	-1	2	4	6	8	10	12	13	14
-14	-13	-12	-10	-8	-6	-4	-2	1	3	5	7	9	11		

Example 5.4. Let $a = 6$ and $b = 3$ to take advantage of the setting and notation of Figure 5.1. The markers are in the diagonals with shifted contents $\pm r_1$ and $\pm r_2$, where $r_1 = \frac{3}{2}$ and $r_2 = \frac{11}{2}$.

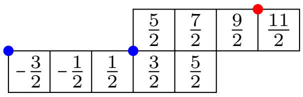
(1) An example where $\ell > a \geq b$. Let $\ell = 8$ and $k = 9$, then



$\lambda = (10, 8) =$ with $S^{(0)}_{\max} = (6, 3)$.

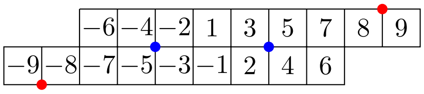
The boxes of $\lambda/S^{(0)}_{\max}$ have

shifted contents:



			$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$
$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$		

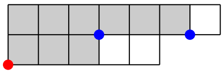
Then $\mathbf{c}$ is the rearrangement of the absolute values of $(-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2})$ into increasing order and $J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \varepsilon_7 - \varepsilon_6\}$. The configuration of boxes κ corresponding to $(\mathbf{c}, J)$ has indexing of boxes



with $$P(\mathbf{c}) = \left\{ \begin{array}{l} \varepsilon_3, \varepsilon_9, \varepsilon_2 + \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_3 - \varepsilon_1 \\ \varepsilon_5 - \varepsilon_4, \varepsilon_5 - \varepsilon_3, \varepsilon_6 - \varepsilon_4, \varepsilon_6 - \varepsilon_3, \\ \varepsilon_7 - \varepsilon_6, \varepsilon_7 - \varepsilon_5, \varepsilon_8 - \varepsilon_7, \varepsilon_9 - \varepsilon_8 \end{array} \right\}.$$

(2) An example with $a \geq \ell > b$: Let $k = 3$ and $\ell = 5$, so that $a + b + k - \ell = 7$.

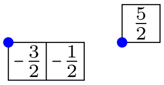
5 $\lambda = (7,5) =$



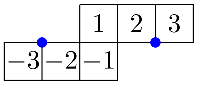
with $S^{(0)}_{\max} = (6,3)$.

The boxes of $\lambda / S^{(0)}_{\max}$ have

shifted contents:



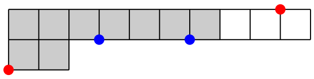
Then $\mathbf{c}$ is the rearrangement of the absolute values of $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2})$ in increasing order and $J = \{\varepsilon_2\}$. The configuration of boxes κ corresponding to $(\mathbf{c}, J)$ is



with $P(\mathbf{c}) = \{\varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$

(3) An example with $a \geq b \geq \ell$: Let $k = 3$ and $\ell = 2$, so that $a + b + k - \ell = 10$. Then

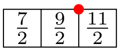
2 $\lambda = (10,2) =$



with $S^{(0)}_{\max} = (7,2)$.

The boxes of $\lambda / S^{(0)}_{\max}$ have

shifted contents:



.

Then $\mathbf{c}$ is the rearrangement of the absolute values of $(\frac{7}{2}, \frac{9}{2}, \frac{11}{2})$ in increasing order and $J = \emptyset$. The configuration of boxes κ corresponding to $(\mathbf{c}, J)$ is

with $P(\mathbf{c}) = \{\varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$.

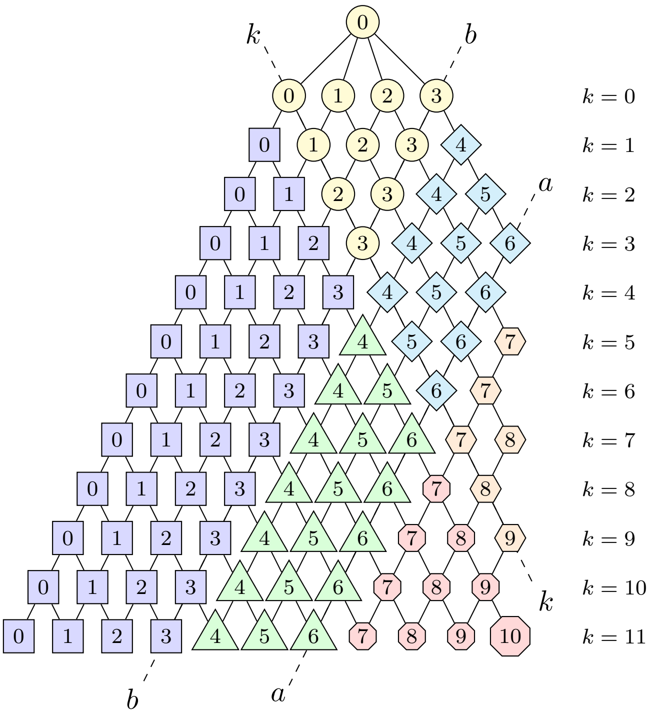
(4) In the case that $k = 1$ then, as $U_q\mathfrak{gl}_2$-modules, $L(6,0) \otimes L(3,0) \otimes L(1,0)$ is isomorphic to

$$\begin{aligned} L(6+3+1-0,0) \oplus L(6+3+1-1,1)^{\oplus 2} \oplus L(6+3+1-2,2)^{\oplus 2} \\ \oplus L(6+3+1-3,3)^{\oplus 2} \oplus L(6+3+1-3,3) \end{aligned}$$

and the dimensions of the corresponding TL^{ext}_k-modules are

$$\dim\left(B_1^{(6+3+1-0,0)}\right) = 1, \qquad \dim\left(B_1^{(6+3+1-1,1)}\right) = 2, \qquad \dim\left(B_1^{(6+3+1-2,2)}\right) = 2,$$
$$\dim\left(B_1^{(6+3+1-3,3)}\right) = 2, \qquad \dim\left(B_1^{(6+3+1-4,4)}\right) = 1.$$

These modules correspond to the line in Figure 5.1 indexed by $k = 1$.



$$\dim\Big(\text{◯}\,\ell\Big) = \sum_{c=\ell-k}^{\ell}\binom{k}{\ell-c} = \sum_{i=0}^{k}\binom{k}{i} = 2^k \qquad 0 \le \ell-k,\quad a \ge b \ge \ell$$

$$\dim\Big(\text{□}\,\ell\Big) = \sum_{c=0}^{\ell}\binom{k}{\ell-c} = \sum_{i=0}^{\ell}\binom{k}{i} \qquad 0 > \ell-k,\quad a \ge b \ge \ell$$

$$\dim\Big(\text{◇}\,\ell\Big) = \sum_{c=\ell-k}^{b}\binom{k}{\ell-c} = \sum_{i=0}^{b+k-\ell}\binom{k}{i} \qquad 0 \le \ell-k,\quad a \ge \ell > b$$

$$\dim\Big(\text{△}\,\ell\Big) = \sum_{c=0}^{b}\binom{k}{\ell-c} = \sum_{i=\ell-b}^{\ell}\binom{k}{i} \qquad 0 > \ell-k,\quad a \ge \ell > b$$

$$\dim\Big(\text{⬡}\,\ell\Big) = \sum_{c=\ell-k}^{b}\binom{k}{\ell-c} - \binom{k}{\ell-(a+b-c)-1} \qquad 0 \le \ell-k,\quad \ell > a \ge b$$

$$\dim\Big(\text{⯃}\,\ell\Big) = \sum_{c=0}^{b}\binom{k}{\ell-c} - \binom{k}{\ell-(a+b-c)-1} \qquad 0 > \ell-k,\quad \ell > a \ge b$$

Figure 5.1. The Temperley–Lieb Bratteli diagram for $a = 6$ and $b = 3$, levels 0–11. Partitions $\lambda = (a+b+k-\ell, \ell)$ are labeled by ℓ. The dimensions of the module indexed by $\lambda = (a+b+k-\ell, \ell)$ is equal to the number of downward-moving paths from the top vertex 0 to the vertex labeled ℓ on level k. Combinatorial formulas for these dimensions are determined in Remark 5.2, and depend on the regions in the diagram, delineated visually by style of nodes.

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