Annals of Representation Theory

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Volume 1, issue 4 (2024), p. 517-537

https://doi.org/10.5802/art.17

Communicated by Radha Kessar.

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Annals of Representation Theory is published by the Norwegian University of Science and Technology and a member of the Centre Mersenne for Open Scientific Publishing

e-ISSN: 2704-2081

CENTRE MERSENNE

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The Broué invariant of a p-permutation equivalence

Robert Boltje *

Abstract. A perfect isometry I (introduced by Broué) between two blocks B and C is a frequent phenomenon in the block theory of finite groups. It maps an irreducible character ψ of C to $\pm$ an irreducible character of B. Broué proved that the ratio of the codegrees of ψ and $I(\psi)$ is a rational number with p-value zero and that its class in $\mathbb{F}_p$ is independent of ψ. We call this element the Broué invariant of I. The goal of this paper is to show that if I comes from a p-permutation equivalence or a splendid Rickard equivalence between B and C then, up to a sign, the Broué invariant of I is determined by local data of B and C and therefore, up to a sign, is independent of the p-permutation equivalence or splendid Rickard equivalence. Apart from results on p-permutation equivalences, our proof requires new results on extended tensor products and bisets that are also proved in this paper. As application of the theorem on the Broué invariant we show that various refinements of the Alperin–McKay Conjecture, introduced by Isaacs–Navarro, Navarro, and Turull are consequences of p-permutation equivalences or splendid Rickard equivalences over a sufficiently large complete discrete valuation ring or over $\mathbb{Z}_p$, depending on the refinement.

1. Introduction

Throughout this introduction, we fix finite groups G and H and a complete discrete valuation ring $\mathcal{O}$ containing a root of unity ζ of order $\exp(G\times H)$, the exponent of $G\times H$. We assume that the field of fractions $\mathbb{K}$ of $\mathcal{O}$ has characteristic 0 and that its residue field $F=\mathcal{O}/J(\mathcal{O})$ has prime characteristic p. We denote by $a\mapsto \bar{a}$ the canonical epimorphisms $\mathcal{O}\to F$ and $\mathcal{O}X\to FX$ for any finite group X.

Further, we fix primitive central idempotents b of $\mathcal{O}G$ and c of $\mathcal{O}H$. We denote by $B:=\mathcal{O}Gb$ and $C:=\mathcal{O}Hc$ the corresponding block algebras. By $R(\mathbb{K}Gb,\mathbb{K}Hc)$ we denote the Grothendieck group of finitely generated $(\mathbb{K}Gb,\mathbb{K}Hc)$-bimodules and always view it via the usual category isomorphism ${}_{\mathbb{K}G}\mathsf{mod}_{\mathbb{K}H}\cong {}_{\mathbb{K}[G\times H]}\mathsf{mod}$ as subgroup of the Grothendieck group $R(\mathbb{K}[G\times H])$ of finitely generated left $\mathbb{K}[G\times H]$-modules. As usual, we also identify the latter with the group of virtual $\mathbb{K}$-characters of $G\times H$.

Manuscript received 2024-02-22, revised 2024-07-30 and accepted 2024-09-06.

Keywords. Blocks of finite groups, perfect isometry, splendid Rickard equivalence, p-permutation equivalence, bisets.

2020 *Mathematics Subject Classification.* 19A22, 20C15, 20C20.

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In [6], Broué introduced the notion of a *perfect isometry* between B and C as follows. A virtual character $\mu \in R(\mathbb{K}[G \times H])$ is called *perfect* if it satisfies the following two conditions:

(i) For any $(g,h) \in G \times H$ one has $\mu(g,h)/|C_G(g)| \in \mathcal{O}$ and $\mu(g,h)/|C_H(h)| \in \mathcal{O}$.
(ii) If $(g,h) \in G \times H$ satisfies $\mu(g,h) \neq 0$ then g is a p'-element if and only if h is a p'-element.

Note that one has a group isomorphism $R(\mathbb{K}Gb, \mathbb{K}Hc) \xrightarrow{\sim} \mathrm{Hom}(R(\mathbb{K}Hc), R(\mathbb{K}Gb))$, $\mu \mapsto I_\mu$, induced by the functor $M \otimes_{\mathbb{K}Hc} -\colon {}_{\mathbb{K}Hc}\mathsf{mod} \to {}_{\mathbb{K}Gb}\mathsf{mod}$ for any finitely generated $(\mathbb{K}Gb, \mathbb{K}Hc)$-bimodule M. A *perfect isometry* between B and C is an isomorphism $I = I_\mu\colon R(\mathbb{K}Hc) \to R(\mathbb{K}Gb)$ induced by a perfect element $\mu \in R(\mathbb{K}Gb, \mathbb{K}Hc)$, which respects the Schur inner products on $R(\mathbb{K}G)$ and $R(\mathbb{K}H)$. Thus, I is a "bijection with signs" between $\mathrm{Irr}(\mathbb{K}Hc)$ and $\mathrm{Irr}(\mathbb{K}Gb)$: I determines a bijection $\alpha\colon \mathrm{Irr}(\mathbb{K}Hc) \xrightarrow{\sim} \mathrm{Irr}(\mathbb{K}Gb)$ and signs $\varepsilon_\psi \in \{\pm 1\}$, $\psi \in \mathrm{Irr}(\mathbb{K}Hc)$, such that $I(\psi) = \varepsilon_\psi \alpha(\psi)$, for all $\psi \in \mathrm{Irr}(\mathbb{K}Hc)$. If $I = I_\mu$ is a perfect isometry one also calls μ a *perfect isometry*.

In [6, Lemme 1.6], it is shown that if μ is a perfect isometry between B and C then the rational numbers

$$\frac{|G|/I_\mu(\psi)(1)}{|H|/\psi(1)}\,, \quad \psi \in \mathrm{Irr}(\mathbb{K}Hc)\,, \tag{1.1}$$

are units in the localization $\mathbb{Z}_{(p)}$ and their residue classes in $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} = \mathbb{F}_p$ are equal, independent of ψ. We will denote this element in $\mathbb{F}_p^\times$, which is uniquely determined by μ, by $\beta(\mu)$ and will call it the *Broué invariant* of μ.

By [6, Proposition 1.2], an element $\mu \in R(\mathbb{K}Gb, \mathbb{K}Hc)$ is perfect if it belongs to the $\mathbb{Z}$-span of characters of indecomposable p-permutation (B,C)-bimodules M (i.e., direct summands of permutation $\mathcal{O}[G \times H]$-modules when viewed as $\mathcal{O}[G \times H]$-module) which have a *twisted diagonal* vertices, i.e., vertices of the form $\Delta(P,\phi,Q) := \{(\phi(y), y) \mid y \in Q\}$, where $P \leq G$ and $Q \leq H$ are p-subgroups and $\phi\colon Q \xrightarrow{\sim} P$ is an isomorphism. We denote the free abelian group on the set of isomorphism classes $[M]$ of such indecomposable modules M by $T^\Delta(B,C)$. In [2], a *p-permutation equivalence* between B and C was defined to be an element $\gamma \in T^\Delta(B,C)$ with the property that $\gamma \cdot_H \gamma^\circ = [B]$, where $-^\circ\colon T^\Delta(B,C) \to T^\Delta(C,B)$ is induced by taking $\mathcal{O}$-duals and $- \cdot_H -\colon T^\Delta(B,C) \times T^\Delta(C,B) \to T^\Delta(B,B)$ is induced by $- \otimes_{\mathcal{O}H} -$. It follows that also $\gamma^\circ \cdot_G \gamma = [C] \in T^\Delta(C,C)$, see [2, Theorem 12.3]. Using the canonical map

$$\kappa\colon T^\Delta(B,C) \to R(\mathbb{K}Gb, \mathbb{K}Hc)$$

induced by $\mathbb{K} \otimes_{\mathcal{O}} -$, every p-permutation equivalence yields a perfect isometry I_μ with $\mu := \kappa(\gamma)$, see [2, Proposition 9.9]. We set $\beta(\gamma) := \beta(\kappa(\gamma))$ and call $\beta(\gamma)$ again the *Broué invariant* of the p-permutation equivalence γ.

If $\gamma \in T^\Delta(B,C)$ is a p-permutation equivalence then, by [2, Theorems 14.1 and 14.3], there exists an indecomposable p-permutation (B,C)-bimodule M with vertex of the form $\Delta(D,\phi,E)$, where D is a defect group of B, E is a defect group of C, and $\phi\colon E \xrightarrow{\sim} D$ is an isomorphism, and there exists a sign $\varepsilon \in \{\pm 1\}$ such that

$$\gamma = \varepsilon \cdot [M] + \sum_{i=1}^{r} n_i \cdot [M_i]$$

with integers $n_1, \ldots, n_r$ and indecomposable p-permutation (B,C)-bimodules $M_1, \ldots, M_r$ with vertices that are properly contained in $\Delta(D,\phi,E)$. Thus, M and $\varepsilon = \varepsilon(\gamma)$ are uniquely determined (up to isomorphism in the case of M) by γ. They are called the *maximal module* and the *sign* of γ.

In the main result of this paper we show that $\varepsilon(\gamma) \cdot \beta(\gamma)$ is determined by the "most local" data of B and C, and is independent of γ. Let (D, e) be a maximal B-Brauer pair. Thus, D is a defect group of B and e is a block idempotent of $\mathcal{O}C_G(D)$ with $\mathrm{br}_D(b)\bar{e} \neq 0$, where $\mathrm{br}_D\colon (\mathcal{O}G)^D \to FC_G(D)$, $\sum_{g\in G} \alpha_g g \mapsto \sum_{g \in C_G(D)} \overline{\alpha_g} g$, is the *Brauer homomorphism*, an $\mathcal{O}$-algebra homomorphism, and where $(\mathcal{O}G)^D$ denotes the set of D-fixed points of $\mathcal{O}G$ under D-conjugation. The block algebra $FC_G(D)e$ has defect group $Z(D)$ (see [12, Corollary 6.3.10]) and, up to isomorphism, it has a unique simple module V (see [12, Proposition 6.6.5]). We set

$$b(B) := \frac{[C_G(D) : Z(D)]}{\dim_F(V)} \,. \tag{1.2}$$

Since any two maximal B-Brauer pairs are G-conjugate, the rational number $b(B)$ does not depend on the choices of (D, e). Since the block algebra $\mathcal{O}C_G(D)e$ has the central defect group $Z(D)$, its image under the natural epimorphism $\mathcal{O}C_G(D) \to \mathcal{O}[C_G(D)/Z(D)]$ is a block B_* of defect zero (see [12, Proposition 6.6.5]) whose unique irreducible character ζ satisfies $\zeta(1) = \dim_F(V)$ and also has defect zero. Therefore, $b(B)$ is the *codegree* of the irreducible character ζ of $C_G(D)/Z(D)$ and an integer which is not divisible by p. We denote by

$$\beta(B) := \overline{b(B)} \in \mathbb{F}_p^\times$$

its residue class in $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. It is an invariant of the block algebra B.

Theorem 1.1. *Let $B = \mathcal{O}Gb$ and $C = \mathcal{O}Hc$ be block algebras as above and let $\gamma \in T^\Delta(B, C)$ be a p-permutation equivalence between B and C. Then*

$$\beta(\gamma) = \varepsilon(\gamma) \cdot \frac{\beta(B)}{\beta(C)} \,. \tag{1.3}$$

In particular, up to a sign, $\beta(\gamma)$ is independent of γ.

Remark 1.2.

(a) Let $X_\bullet$ be a *splendid Rickard equivalence* between B and C. For our purposes this is (see [16], where this concept was introduced first) a bounded chain complex $X_\bullet$ of finitely generated p-permutation (B, C)-bimodules, whose indecomposable direct summands have twisted diagonal vertices, such that

$$X_\bullet \otimes_{\mathcal{O}H} X_\bullet^\circ \simeq B \quad \text{and} \quad X_\bullet^\circ \otimes_{\mathcal{O}G} X_\bullet \simeq C$$

where $X_\bullet^\circ$ denotes the $\mathcal{O}$-dual chain complex of $X_\bullet$, $\simeq$ means homotopy equivalence of chain complexes of (B, B)-bimodules (resp. (C, C)-bimodules), and B (resp. C) denotes the chain complex with B (resp. C) placed in degree 0 and being the only non-zero term. Note that our definition for the purpose of this paper is more general than the original one (see [16]) and also later definitions (see for instance [12, Definition 9.7.5]). For $X_\bullet$ as above, the element

$$\gamma := \sum_{n\in\mathbb{Z}} (-1)^n [X_n] \in T^\Delta(B, C)$$

is a p-permutation equivalence (see [2, Theorem 15.2]) and the element

$$\mu := \kappa(\gamma) = \sum_{n\in\mathbb{Z}} (-1)^n \kappa([X_n]) \in R(\mathbb{K}Gb, \mathbb{K}Hc)$$

is a perfect isometry (see [2, Proposition 9.9]). This way one can define the Broué invariant of $X_\bullet$ as $\beta(X_\bullet) := \beta(\gamma) = \beta(\mu)$ and the statement of Theorem 1.1 applies also to $\beta(X_\bullet)$.

(b) With the notation of Theorem 1.1 and the preceding paragraph, let M_* be the unique indecomposable $\mathcal{O}$-torsion-free (B_*, C_*)-bimodule (up to isomorphism). Since B_* and C_* are block algebras of defect 0, M_* is a p-permutation bimodule, has twisted diagonal vertex, and induces a Morita equivalence between B_* and C_*. The chain complex consisting of the only non-zero term M_* in degree zero is therefore a splendid Rickard equivalence and $\gamma_* := [M_*] \in T^\Delta(B_*, C_*)$ is a p-permutation equivalence between B_* and C_*. Equation (1.3) can now also be interpreted as $\beta(\gamma) = \epsilon(\gamma) \cdot \beta([M_*])$.

Of particular interest is the situation where C is the Brauer correspondent of B with respect to Brauer's first main theorem.

Corollary 1.3. *Suppose that D is a defect group of $B = \mathcal{O}Gb$, that $H = N_G(D)$, and that the block idempotent c of $\mathcal{O}H$ is the Brauer correspondent of b with respect to Brauer's first main theorem, i.e., $\bar{c} = \mathrm{br}_D(b)$. Then the Broué invariant of any p-permutation equivalence and any splendid Rickard equivalence between B and C is equal to* 1 *or* -1*. In particular, if there exists a p-permutation equivalence (resp. splendid Rickard equivalence) between B and C then there also exists one with Broué invariant equal to* 1.

Proof. In the situation of the corollary, one can choose $D = E$ so that $C_G(D) = C_H(E)$. Moreover, one can choose $f = e$ to obtain that $b(B) = b(C)$. The first statement follows now from Theorem 1.1. The last statement follows from replacing γ with $-\gamma$ or shifting $X_\bullet$ by one degree. ☐

Next we apply Corollary 1.3 in order to relate p-permutation equivalences and splendid Rickard equivalences to various refinements of the Alperin–McKay conjecture. Suppose that b, D, H, and c are as in Corollary 1.3. Let $\mathrm{Irr}_0(KGb)$ denote the set of characters of irreducible $\mathbb{K}Gb$-modules of height zero. Then the Alperin–McKay conjecture, see [1, Conjecture 3], states that

$$|\mathrm{Irr}_0(\mathbb{K}Gb)| = |\mathrm{Irr}_0(\mathbb{K}Hc)| \ . \tag{1.4}$$

We first recall two refinements of this conjecture, introduced by Isaacs and Navarro in [10] and Navarro in [14]. For $\chi \in \mathrm{Irr}(\mathbb{K}Gb)$, set $r(\chi) := (|G|/\chi(1))_{p'}$, the p'-part of the *codegree* of χ. For $r \in \{1, \ldots, p-1\}$, let $\mathrm{Irr}_0(\mathbb{K}Gb, r)$ denote the set of all $\chi \in \mathrm{Irr}_0(\mathbb{K}Gb)$ with $r(\chi) \equiv \pm r \mod p$. In the same way we define $\mathrm{Irr}_0(\mathbb{K}Hc, r)$. [10, Conjecture B] states that

$$|\mathrm{Irr}_0(\mathbb{K}Gb, r)| = |\mathrm{Irr}_0(\mathbb{K}Hc, r)| \tag{1.5}$$

for every $r \in \{1, \ldots, p-1\}$. Isaacs and Navarro also considered Galois actions on characters. Let $\mathbb{Q}_p \subseteq L \subseteq \mathbb{K}$ be an intermediate field and consider the set $\mathrm{Irr}_0(\mathbb{K}Gb, L)$ consisting of those $\chi \in \mathrm{Irr}_0(\mathbb{K}Gb, r)$ with $\mathbb{Q}_p(\chi) = L$. Here, $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(g) \mid g \in G)$. Similarly, define $\mathrm{Irr}_0(\mathbb{K}Hc, L)$. Clearly, $\mathrm{Irr}_0(\mathbb{K}Gb, L)$ and $\mathrm{Irr}_0(\mathbb{K}Hc, L)$ are empty unless $L \subseteq \mathbb{Q}(\zeta)$, where $\zeta \in \mathbb{K}^\times$ has order $\exp(G)$. A slightly stronger version of [14, Conjecture B] states that

$$|\mathrm{Irr}_0(\mathbb{K}Gb, L)| = |\mathrm{Irr}_0(\mathbb{K}Hc, L)| \tag{1.6}$$

for every intermediate field $\mathbb{Q}_p \subseteq L \subseteq \mathbb{Q}_p(\zeta)$. Combining the refined Alperin–McKay conjectures in (1.5) and (1.6), let $\mathrm{Irr}_0(\mathbb{K}Gb, r, L)$ be the set of all $\chi \in \mathrm{Irr}_0(\mathbb{K}Gb)$ with $r(\chi) \equiv \pm r \mod p$ and $\mathbb{Q}_p(\chi) = L$. Then one can consider the conjecture

$$|\mathrm{Irr}_0(\mathbb{K}Gb, r, L)| = |\mathrm{Irr}_0(\mathbb{K}Hc, r, L)| \tag{1.7}$$

for every $r \in \{1, \ldots, p-1\}$ and every intermediate field $\mathbb{Q}_p \subseteq L \subseteq \mathbb{Q}_p(\zeta)$. Finally, Turull, see [18], suggested a further refinement involving endomorphism rings of simple modules. For $\chi \in \mathrm{Irr}(\mathbb{K}Gb)$, let $h(\chi) \in \mathbb{Q}/\mathbb{Z}$ be defined as follows: Let V be the unique irreducible $\mathbb{Q}_pG$-module with the property that χ is a constituent of the character of V and let $D := \mathrm{End}_{\mathbb{Q}_pG}(V)$, a central simple $\mathbb{Q}_p(\chi)$-division algebra. Then let $h(\chi) \in \mathbb{Q}/\mathbb{Z}$ be the Hasse-invariant of D, see [15, Section 14] for a definition. For $h \in \mathbb{Q}/\mathbb{Z}$, define $\mathrm{Irr}_0(\mathbb{K}Gb, r, L, h)$ as the set of those $\chi \in \mathrm{Irr}_0(\mathbb{K}Gb, r, L)$ with $h(\chi) = h$. Turull's conjecture then states that

$$|\mathrm{Irr}_0(\mathbb{K}Gb, r, L, h)| = |\mathrm{Irr}_0(\mathbb{K}Hc, r, L, h)| \tag{1.8}$$

for all $r \in \{1, \ldots, p-1\}$, $\mathbb{Q}_p \subseteq L \subseteq \mathbb{Q}(\zeta)$, and $h \in \mathbb{Q}/\mathbb{Z}$. Note that $h(\chi)$ is represented by an element in $\{0, 1/p, 2/p, \ldots, (p-1)/p\}$ if p is odd, and by 0 or 1/2 if $p = 2$, for any $\chi \in \mathrm{Irr}(\mathbb{K}G)$, see [20, 19].

Corollary 1.3 allows us now to make a connection between p-permutation equivalences and splendid Rickard equivalences and the refinements (1.5), (1.7), (1.8) of the Alperin–McKay conjecture. Note that the definitions of p-permutation equivalences and splendid Rickard equivalences extend in the obvious way to arbitrary complete discrete valuation rings in place of $\mathcal{O}$, in particular to $\mathbb{Z}_p$, as used in the statement of the next theorem. They also extend to sums of blocks over arbitrary complete discrete valuation rings, as used in the proof of the next theorem.

Theorem 1.4. *Let b be a block idempotent of $\mathcal{O}G$ and let $\widetilde{b}$ be the unique block idempotent of $\mathbb{Z}_pG$ with $b\widetilde{b} \neq 0$. Let D be a defect group of b and set $H := N_G(D)$. Further, let c be the block idempotent of $\mathcal{O}H$ which corresponds to b under Brauer's First Main Theorem, i.e., $\mathrm{br}_D(b) = \bar{c}$, and let $\widetilde{c}$ be the unique block idempotent of $\mathbb{Z}_pH$ with $c\widetilde{c} \neq 0$. Finally, let $\zeta \in \mathbb{K}$ be a root of unity of order $\exp(G)$ and set $\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)$.*

(a) *If there exists a p-permutation equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ then (1.5) holds. In other words, there exists a bijection*

$$\alpha\colon \mathrm{Irr}_0(\mathbb{K}Gb) \xrightarrow{\sim} \mathrm{Irr}_0(\mathbb{K}Hc)$$

such that $r(\chi) \equiv \pm r(\alpha(\chi)) \mod p$, for all $\chi \in \mathrm{Irr}_0(\mathbb{K}Gb)$.

(b) *If there exists a p-permutation equivalence between $\mathbb{Z}_pG\widetilde{b}$ and $\mathbb{Z}_pH\widetilde{c}$ then (1.7) holds. Moreover, the Γ-stabilizers Γ_b and Γ_c of b and c, respectively, coincide and there exists a Γ_b-equivariant bijection*

$$\alpha\colon \mathrm{Irr}_0(\mathbb{K}Gb) \xrightarrow{\sim} \mathrm{Irr}_0(\mathbb{K}Hc)$$

such that $r(\chi) \equiv \pm r(\alpha(\chi)) \mod p$, for all $\chi \in \mathrm{Irr}_0(\mathbb{K}Gb)$.

(c) *If there exists a splendid Rickard equivalence between $\mathbb{Z}_pG\widetilde{b}$ and $\mathbb{Z}_pH\widetilde{c}$ then (1.8) holds. Moreover, there exists a Γ_b-equivariant bijection*

$$\alpha\colon \mathrm{Irr}_0(\mathbb{K}Gb) \xrightarrow{\sim} \mathrm{Irr}_0(\mathbb{K}Hc)$$

such that $r(\chi) \equiv \pm r(\alpha(\chi)) \mod p$ and $h(\chi) = h(\alpha(\chi))$, for all $\chi \in \mathrm{Irr}_0(\mathbb{K}Gb)$.

Remark 1.5.

(a) Splendid Rickard equivalences and p-permutation equivalences between block algebras of $\mathcal{O}G$ with abelian defect groups and their Brauer correspondents are expected to exist according to Broué's *abelian* defect group conjecture, but they are known not to exist for general defect groups. Therefore, Theorem 1.4 cannot be a general tool for proving the (refinements of the) Alperin–McKay Conjecture.

(b) For blocks with cyclic defect groups, Isaacs and Navarro proved in [10] that (1.5) and a weaker version of (1.7) hold. Turull proved in [17] that for the same blocks a weaker version of (1.8) holds, which takes Schur indices over $\mathbb{Q}_p$ into account. More precisely, it was shown that there exists a bijection between $\mathrm{Irr}_0(\mathcal{O}Gb)$ and $\mathrm{Irr}_0(\mathcal{O}Hc)$ preserving the invariants $\mathbb{Q}_p(\chi)$, and the order of $h(\chi)$ in $\mathbb{Q}/\mathbb{Z}$. Turull proved in [18] that a slightly weaker version of (1.8) holds for arbitrary blocks when G is p-solvable. In Turull's result the invariant $h(\chi) \in \mathbb{Q}/\mathbb{Z}$ is replaced by its order in $\mathbb{Q}/\mathbb{Z}$, which is also the Schur index of χ over $\mathbb{Q}_p$.

(c) The hypothesis in Theorem 1.4 (c) is satisfied in the following cases: If $\mathcal{O}Gb$ has cyclic defect groups (see [11]), if $\mathcal{O}Gb$ has a Klein four group as defect group (see [7]), if $G = \mathrm{GL}(2, p^n)$ or $G = \mathrm{SL}(2, p^n)$ (see [9]), if G is an alternating group and $\mathcal{O}Gb$ has abelian defect groups (see [8]), and for some blocks of p-nilpotent groups (see [3]). Thus, by Theorem 1.4 (c), the refined version (1.8) of the Alperin–McKay Conjecture holds in all these cases.

The paper is arranged as follows. Theorem 1.1 is proved in Section 4. The proof uses properties of p-permutation equivalences and the notion of extended tensor products of bimodules, a construction first introduced by Bouc in [5]. In Section 3 we recall this construction, prove that it can be realized as a biset operation, and prove Theorem 3.5, a formula for the extended tensor product of induced modules, which is used in the proof of Theorem 1.1. Section 3 uses the language of bisets which we introduce in Section 2 following [4]. There, we also define the notion of an extended tensor product for bisets, and prove a formula for bisets, analogous to Theorem 3.5 for modules. Finally, Theorem 1.4 is proved as an application of Theorem 1.1 in Section 5.

2. Bisets and extended tensor products

In the first part of this section we recall from [4, Chapter 2] the notions, notations, and results related to bisets, that we will need in Section 3. In the second part we introduce the notion of *extended tensor products* for bisets (see Section 2.6) and prove results about this construction.

Throughout this section, G, H, K, L, I and J denote finite groups.

2.1 (G-sets, (G,H)-bisets, tensor products and external products)**.**

(a) We denote by ${}_G\mathsf{set}$ the category of finite left G-sets and by ${}_G\mathsf{set}_H$ the category of finite (G,H)-bisets. Recall that a (G,H)-biset is a set U endowed with a left G-action and a right H-action that commute with each other. We will often view, without further notice, a (G,H)-biset U as left $G \times H$-set via $(g,h)u = guh^{-1}$ for $g \in G$, $h \in H$, and $u \in U$, and vice-versa. This defines an obvious isomorphism of categories

$$ {}_G\mathsf{set}_H \cong {}_{G\times H}\mathsf{set}\,. $$

(b) One has a functor

$$ - \otimes_H -\colon {}_G\mathsf{set}_H \times {}_H\mathsf{set}_K \to {}_G\mathsf{set}_K\,, $$

where, for $U \in {}_G\mathsf{set}_H$ and $V \in {}_H\mathsf{set}_K$, one defines $U \otimes_H V$ as the set of H-orbits of $U \times V$ with respect to the H-action given by $h \cdot (u,v) := (uh^{-1}, hv)$. The H-orbit of (u,v) is denoted by $u \otimes v \in U \otimes_H V$ (or $u \otimes_H v$ for clarity). Thus, $uh \otimes v = u \otimes hv$ for $h \in H$ and $(u,v) \in U \times V$. The set $U \otimes_H V$ has a well-defined (G,K)-biset structure given by $g(u \otimes v)k := (gu) \otimes (vk)$. If $\phi\colon U \to U'$ and $\psi\colon V \to V'$

are morphisms of (G,H)-bisets and (H,K)-bisets, respectively, then $\phi \otimes_H \psi$ is defined by $(\phi \otimes_H \psi)(u \otimes v) := \phi(u) \otimes \psi(v)$. Note that in [4, Definition 2.3.11] this construction is denoted by $U \times_H V$ and called the *composition* of U and V. Since we will use fiber products later (see 2.7) with conflicting notation, we chose here the notation $U \otimes_H V$ and call it the *tensor product* of U and V.

(c) Choosing $K = \{1\}$ in (b), fixing $U \in {}_G\mathsf{set}_H$, and using the obvious category isomorphisms ${}_H\mathsf{set}_{\{1\}} \cong {}_H\mathsf{set}$ and ${}_G\mathsf{set}_{\{1\}} \cong {}_G\mathsf{set}$, we obtain a functor

$$U \otimes_H -\colon {}_H\mathsf{set} \to {}_G\mathsf{set}\,.$$

(d) Finally, one has a functor

$$- \times -\colon {}_G\mathsf{set} \times {}_H\mathsf{set} \to {}_{G\times H}\mathsf{set}$$

which maps an object (U,V) of ${}_G\mathsf{set} \times {}_H\mathsf{set}$ to $U \times V$ endowed with the $G \times H$-action $(g,h)(u,v) := (gu, hv)$, for $(g,h) \in G \times H$ and $(u,v) \in U \times V$.

Note that the disjoint union $U \coprod U'$ of two finite sets provides a coproduct in the categories ${}_G\mathsf{set}$ and ${}_G\mathsf{set}_H$. The functors defined in Subsection 2.1 satisfy the following properties and compatibilities.

Lemma 2.2.

(a) For $U, U' \in {}_G\mathsf{set}_H$ and $V, V' \in {}_H\mathsf{set}_K$ one has isomorphisms

$$\left(U \coprod U'\right) \otimes_H V \cong (U \otimes_H V) \coprod (U' \otimes_H V)$$
$$\text{and} \quad U \otimes_H \left(V \coprod V'\right) \cong (U \otimes_H V) \coprod (U \otimes_H V')$$

in ${}_G\mathsf{set}_K$, which are natural in U, U', V, and V'.

(b) For $U \in {}_G\mathsf{set}_H$, $V \in {}_H\mathsf{set}_K$, and $W \in {}_K\mathsf{set}_L$ (resp. $W \in {}_K\mathsf{set}$) one has an isomorphism

$$(U \otimes_H V) \otimes_K W \cong U \otimes_H (V \otimes_K W)$$

in ${}_G\mathsf{set}_L$ (resp. in ${}_G\mathsf{set}$), which is natural in U, V and W. It maps $(u \otimes v) \otimes w$ to $u \otimes (v \otimes w)$.

(c) For $U \in {}_G\mathsf{set}_H$, $V \in {}_K\mathsf{set}_L$, $R \in {}_H\mathsf{set}_I$, and $S \in {}_L\mathsf{set}_J$ one has an isomorphism

$$(U \otimes_H R) \times (V \otimes_L S) \cong (U \times V) \otimes_{H \times L} (R \times S)$$

in ${}_{G\times K}\mathsf{set}_{I\times J}$, which is natural in U, V, R, and S, and which maps $(u \otimes r, v \otimes s)$ to $(u,v) \otimes (r,s)$.

(d) For $U \in {}_G\mathsf{set}_H$ one has isomorphisms

$$G \otimes_G U \cong U \quad \text{and} \quad U \otimes_H H \cong U$$

in ${}_G\mathsf{set}_H$, which are natural in U and are given by $g \otimes u \to gu$ and $u \otimes h \to uh$. Here G is viewed as (G,G)-biset and H is viewed as (H,H)-biset via left and right multiplication.

The bisets defined in the following example are called *elementary* bisets.

Example 2.3.

(a) For $H \leq G$, Res^G_H (resp. Ind^G_H) denotes the (H,G)-biset (resp. (G,H)-biset) G endowed with left and right multiplication. The resulting functors

$$\mathrm{Res}^G_H := \mathrm{Res}^G_H \otimes_G -\colon {}_G\mathsf{set} \to {}_H\mathsf{set} \quad \text{and} \quad \mathrm{Ind}^G_H := \mathrm{Ind}^G_H \otimes_H -\colon {}_H\mathsf{set} \to {}_G\mathsf{set}$$

are denoted by the same symbols and are called *restriction* from G to H and *induction* from H to G. Note that $\mathrm{Res}^G_H \cong (H\times G)/\Delta(H)$ in ${}_H\mathsf{set}_G \cong {}_{H\times G}\mathsf{set}$ and $\mathrm{Ind}^G_H \cong (G\times H)/\Delta(H)$ in ${}_G\mathsf{set}_H \cong {}_{G\times H}\mathsf{set}$, where $\Delta(H) := \{(h,h) \in h \in H\}$ is the stabilizer of $1\in G$.

(b) For $N \trianglelefteq G$, $\mathrm{Inf}^G_{G/N}$ (resp. $\mathrm{Def}^G_{G/N}$) denotes the $(G, G/N)$-biset (resp. $(G/N, G)$-biset) G/N endowed with left and right multiplication, after using the natural epimorphism $\pi\colon G\to G/N$. The resulting functors

$$\mathrm{Inf}^G_{G/N} := \mathrm{Inf}^G_{G/N}\otimes_{G/N} -\colon {}_{G/N}\mathsf{set}\to {}_G\mathsf{set} \quad\text{and}\quad \mathrm{Def}^G_{G/N} := \mathrm{Def}^G_{G/N}\otimes_{G} -\colon {}_{G}\mathsf{set}\to {}_{G/N}\mathsf{set}$$

are called *inflation* from G/N to G (resp. *deflation* from G to G/N). Note that $\mathrm{Inf}^G_{G/N} \cong (G\times(G/N))/\Delta(G,\pi)$ in ${}_G\mathsf{set}_{G/N}$ and $\mathrm{Def}^G_{G/N} \cong ((G/N)\times G)/\Delta(\pi,G)$ in ${}_{G/N}\mathsf{set}_G$, where

$$\Delta(G,\pi) := \{(g, gN)\mid g\in G\} \quad\text{and}\quad \Delta(\pi,G) := \{(gN, g)\mid g\in G\}$$

are the stabilizers of $1 \in G/N$.

(c) If $\alpha\colon G \xrightarrow{\sim} G'$ is an isomorphism we denote by Iso_α the (G', G)-biset G' with G' acting by multiplication from the left and G acting via α and multiplication from the right. Note that $\mathrm{Iso}_\alpha \cong (G'\times G)/\Delta(\alpha, G)$ in ${}_{G'}\mathsf{set}_G$, with $\Delta(\alpha,G) = \{(\alpha(g), g)\mid g\in G\}$ being the stabilizer of $1\in G'$ under the corresponding $(G'\times G)$-action. Note that if also $\beta\colon G' \xrightarrow{\sim} G''$ is an isomorphism then $\mathrm{Iso}_{\beta\alpha} \cong \mathrm{Iso}_\beta \otimes_{G'} \mathrm{Iso}_\alpha$ in ${}_{G''}\mathsf{set}_G$.

If $\alpha = c_g\colon H\to gHg^{-1}$, $h\mapsto ghg^{-1}$, is the conjugation map for a subgroup $H\le G$ and $g\in G$, we set $\mathrm{Con}_g := \mathrm{Iso}_{c_g} \in {}_{gHg^{-1}}\mathsf{set}_H$ and ${}^gU := \mathrm{Con}_g(U)$, for $U\in {}_H\mathsf{set}$.

By Lemma 2.2 (d), the functors Res^G_H, $\mathrm{Inf}^G_{G/N}$, Iso_α, and Con_g above are naturally isomorphic to the functors that don't change the underlying set, but restrict the action along the group homomorphisms $H\to G$, $G\to G/N$, $\alpha^{-1}\colon G'\to G$ and $c_g^{-1}\colon gHg^{-1}\to H$, respectively.

2.4 (Subgroups of direct products)**.** Let $X\le G\times H$. We denote by $p_1\colon G\times H\to G$ and $p_2\colon G\times H\to H$ the projection maps. If one sets

$$k_1(X) := \{g\in G\mid (g,1)\in X\} \quad\text{and}\quad k_2(X) := \{h\in H\mid (1,h)\in X\}$$

then

$$k_1(X)\trianglelefteq p_1(X)\le G\,,\quad k_2(X)\trianglelefteq p_2(X)\le H\,,$$
$$\text{and}\quad k_1(X)\times k_2(X)\le X\le p_1(X)\times p_2(X)\,.$$

Moreover, if additionally $Y\le H\times K$, one sets

$$X*Y := \{(g,k)\in G\times K\mid \exists\, h\in H\colon (g,h)\in X \text{ and } (h,k)\in Y\}\,.$$

It is easy to see that $X*Y$ is a subgroup of $G\times K$ and that the construction $-*-$ is associative and monotonous with respect to inclusion in each argument. Moreover, one has

$$k_1(X)\le k_1(X*Y)\trianglelefteq p_1(X*Y)\le p_1(X)\,,\quad k_2(Y)\le k_2(X*Y)\trianglelefteq p_2(X*Y)\le p_2(Y)\,,$$
$$\text{and}\quad k_1(X)\times k_2(Y)\le X*Y\le p_1(X)\times p_2(Y)\,.$$

The following theorem is an explicit formula for the functor in Subsection 2.1 (b) applied to transitive bisets, see [4, Lemma 2.3.24].

Theorem 2.5. *Let G, H, K be finite groups and let $X \leq G \times H$ and $Y \leq H \times K$. Then*

$$(G \times H)/X \otimes_H (H \times K)/Y \cong \coprod_{h \in [p_2(X)\backslash H/p_1(Y)]} (G \times K)/\Big(X * {}^{(h,1)}Y\Big)$$

in ${}_G\mathsf{set}_K$, where $[p_2(X)\backslash H/p_1(Y)]$ denotes a set of representatives of the $(p_2(X), p_1(Y))$-double cosets in H. The isomorphism maps the element $(g,k)(X * {}^{(h,1)}Y)$ in the h-component of the right hand side to $(g,1)X \otimes (h,k)Y$.

2.6 (Extended tensor products for bisets)**.** We generalize the tensor product of bisets from Subsection 2.1 (b) as follows. Let $X \leq G \times H$, $Y \leq H \times K$, $U \in {}_X\mathsf{set}$, $V \in {}_Y\mathsf{set}$, and set

$$k(X,Y) := k_2(X) \cap k_1(Y) \leq H\,.$$

We may consider U and V via restriction as $U \in {}_{k_1(X)}\mathsf{set}_{k(X,Y)}$ and $V \in {}_{k(X,Y)}\mathsf{set}_{k_2(Y)}$ and form the tensor product $U \otimes_{k(X,Y)} V \in {}_{k_1(X)}\mathsf{set}_{k_2(Y)} \cong {}_{k_1(X)\times k_2(Y)}\mathsf{set}$. The action of $k_1(X) \times k_2(X)$ on $U \otimes_{k(X,Y)} V$ (as defined in Subsection 2.1) can be extended to an action of $X * Y$ as follows. Let $(g,k) \in X * Y$, $u \in U$, and $v \in V$. Choose $h \in H$ such that $(g,h) \in X$ and $(h,k) \in Y$, and set

$$(g,k)(u \otimes v) := ((g,h)u) \otimes ((h,k)v)\,.$$

This definition does not depend on the choice of h and defines a functor that we denote by

$$- \overset{X,Y}{\underset{k(X,Y)}{\otimes}} - \colon {}_X\mathsf{set} \times {}_Y\mathsf{set} \to {}_{X*Y}\mathsf{set} \tag{2.1}$$

or simply by $- \overset{X,Y}{\otimes} -$. We call it the *extended tensor product* (with respect to X and Y). Note that this construction coincides with the construction in Subsection 2.1 (b) when $X = G \times H$ and $Y = H \times K$ (and $X * Y = G \times K$). The extended tensor product functor is associative. More precisely, if also $Z \leq K \times L$ and $W \in {}_Z\mathsf{set}$, then $(u\otimes v)\otimes w \mapsto u\otimes(v\otimes w)$ defines an isomorphism between

$$\Big(U \overset{X,Y}{\otimes} V\Big) \overset{X*Y,Z}{\otimes} W \quad \text{and} \quad U \overset{X,Y*Z}{\otimes} \Big(V \overset{Y,Z}{\otimes} W\Big) \quad \text{in} \quad {}_{X*Y*Z}\mathsf{set}.$$

Moreover, the extended tensor product functor respects coproducts in each argument.

2.7 (The biset and functor $\mathrm{DefRes}^{X\times Y}_{X*Y}$)**.** For $X \leq G \times H$ and $Y \leq H \times K$, consider the pull-back $X \times_H Y$ of the two projection maps $p_2 \colon X \to H$ and $p_1 \colon Y \to H$. Thus,

$$X \times_H Y := \Big\{\Big((g,h),(\tilde{h},k)\Big) \in X \times Y \,\Big|\, h = \tilde{h}\Big\} \leq X \times Y\,.$$

Moreover, consider the surjective homomorphism $\nu \colon X \times_H Y \to X * Y$, $((g,h),(h,k)) \mapsto (g,k)$, with kernel $\{((1,h),(h,1)) \mid h \in k(X,Y)\}$ and the resulting isomorphism $\bar{\nu} \colon (X \times_H Y)/\ker(\nu) \xrightarrow{\sim} X * Y$. We define the $(X * Y, X \times Y)$-biset

$$\mathrm{DefRes}^{X\times Y}_{X*Y} := \mathrm{Iso}_{\bar{\nu}} \otimes_{(X\times_H Y)/\ker(\nu)} \mathrm{Def}^{X\times_H Y}_{(X\times_H Y)/\ker(\nu)} \otimes_{X\times_H Y} \mathrm{Res}^{X\times Y}_{X\times_H Y}$$

and use the same notation for the induced functor

$$\mathrm{DefRes}^{X\times Y}_{X*Y} := \mathrm{DefRes}^{X\times Y}_{X*Y} \otimes_{X\times Y} - \colon {}_{X\times Y}\mathsf{set} \to {}_{X*Y}\mathsf{set}\,.$$

It follows quickly from the explicit descriptions in Example 2.3 of the three factors of $\mathrm{DefRes}^{X\times Y}_{X*Y}$ and the formula in Theorem 2.5 that one has an isomorphism

$$\mathrm{DefRes}^{X\times Y}_{X*Y} \cong ((X * Y) \times (X \times Y))/\{(\nu(z), z) \mid z \in X \times_H Y\} \tag{2.2}$$

in ${}_{X*Y}\mathsf{set}_{X\times Y}$.

The following proposition shows that the extended tensor product functor $-\overset{X,Y}{\otimes}-$ can be regarded as a composition of a biset operation and the functor $-\times-\colon {}_X\mathsf{set}\times{}_Y\mathsf{set}\to{}_{X\times Y}\mathsf{set}$ from Subsection 2.1 (c) and (d).

Proposition 2.8. *Let $X\le G\times H$ and $Y\le H\times K$. The functor $-\overset{X,Y}{\otimes}-\colon {}_X\mathsf{set}\times{}_Y\mathsf{set}\to{}_{X*Y}\mathsf{set}$ in (2.1) is naturally isomorphic to the functor $\mathrm{DefRes}^{X\times Y}_{X*Y}\circ(-\times-)\colon {}_X\mathsf{set}\times{}_Y\mathsf{set}\to{}_{X*Y}\mathsf{set}$.*

Proof. Let $U\in{}_X\mathsf{set}$ and $V\in{}_Y\mathsf{set}$. Using the isomorphism (2.2), it suffices to show that one has an isomorphism

$$U\underset{k(X,Y)}{\overset{X,Y}{\otimes}}V\cong\frac{(X*Y)\times(X\times Y)}{\{(\nu(z),z)\mid z\in X\times_H Y\}}\otimes_{X\times Y}(U\times V)\tag{2.3}$$

of $(X*Y)$-sets which is natural in U and V. But this follows from the following statements whose straightforward but lenghty verification we leave to the reader. Mapping $u\otimes v$ to $\overline{1}\otimes(u,v)$ is a well-defined morphism ϕ of $(X*Y)$-sets from the left hand side in (2.3) to the right hand side, which is natural in U and V. Moreover, mapping $\overline{((g,k),(x,y))}\otimes(u,v)$ to $(g,h)x^{-1}u\otimes(h,k)y^{-1}v$, where $h\in H$ is chosen such that $(g,h)\in X$ and $(h,k)\in Y$, yields a well-defined function ψ from the right hand side of (2.3) to the left hand side, such that $\phi\circ\psi$ and $\psi\circ\phi$ are the respective identity maps. □

Lemma 2.9. *Let $X'\le X\le G\times H$ and $Y'\le Y\le H\times K$. Then*

$$\mathrm{DefRes}^{X\times Y}_{X*Y}\otimes_{X\times Y}\mathrm{Ind}^{X\times Y}_{X'\times Y'}\cong\coprod_{(x,y)}\mathrm{Ind}^{X*Y}_{{}^xX'*{}^yY'}\otimes_{{}^xX'*{}^yY'}\mathrm{DefRes}^{{}^xX'\times{}^yY'}_{{}^xX'*{}^yY'}\otimes_{{}^xX'\times{}^yY'}\mathrm{Con}_{(x,y)}$$

as $(X*Y,X'\times Y')$-bisets, where (x,y) runs through a set of representatives of the $(X\times_H Y,X'\times Y')$-double cosets of $X\times Y$.

Proof. By the definition of $\mathrm{DefRes}^{X\times Y}_{X*Y}$ we have (omitting the indices of tensor products)

$$\mathrm{DefRes}^{X\times Y}_{X*Y}\otimes\mathrm{Ind}^{X\times Y}_{X'\times Y'}=\mathrm{Iso}_{\bar\nu}\otimes\mathrm{Def}^{X\times_H Y}_{(X\times_H Y)/\ker(\nu)}\otimes\mathrm{Res}^{X\times Y}_{X\times_H Y}\otimes\mathrm{Ind}^{X\times Y}_{X'\times Y'}\,.\tag{2.4}$$

Using the commutation rule for Res and Ind, see [4, 1.1.3.2.c], we find that the right hand side of (2.4) is the coproduct of the $(X*Y,X'\times Y')$-bisets

$$L_{(x,y)}:=\mathrm{Iso}_{\bar\nu}\otimes\mathrm{Def}^{X\times_H Y}_{(X\times_H Y)/\ker(\nu)}\otimes\mathrm{Ind}^{X\times_H Y}_{{}^xX'\times_H{}^yY'}\otimes\mathrm{Res}^{{}^xX'\times{}^yY'}_{{}^xX'\times_H{}^yY'}\otimes\mathrm{Con}_{(x,y)}\,,\tag{2.5}$$

where (x,y) runs through a set of representatives of the $(X\times_H Y,X'\times Y')$-double cosets of $X\times Y$. Here we used that $(X\times_H Y)\cap({}^xX'\times{}^yY')={}^xX'\times_H{}^yY'$. Next we use the commutation rule for Def and Ind, see [4, 1.1.3.2.e], to obtain

$$\begin{aligned}L_{(x,y)}\cong\mathrm{Iso}_{\bar\nu}\otimes\mathrm{Ind}^{(X\times_H Y)/\ker(\nu)}_{({}^xX'\times_H{}^yY')\ker(\nu)/\ker(\nu)}\otimes\mathrm{Iso}_\gamma\otimes\mathrm{Def}^{{}^xX'\times_H{}^yY'}_{({}^xX'\times_H{}^yY')/\ker(\nu')}\\\otimes\mathrm{Res}^{{}^xX'\times{}^yY'}_{{}^xX\times_H{}^yY'}\otimes\mathrm{Con}_{(x,y)}\,,\end{aligned}$$

where $\nu'\colon{}^xX'\times_H{}^yY'\to{}^xX'*{}^yY'$ is the epimorphism analogous to ν in Subsection 2.7 and $\gamma\colon({}^xX'\times_H{}^yY')/\ker(\nu')\xrightarrow{\sim}({}^xX'\times_H{}^yY')\ker(\nu)/\ker(\nu)$ is the canonical isomorphism, noting that $\ker(\nu')=({}^xX'\times_H{}^yY')\cap\ker(\nu)$. Finally, the commutation of the two left most factors Iso and Ind, see [4, 1.1.3.2.a], yields

$$L_{(x,y)}\cong\mathrm{Ind}^{X*Y}_{{}^xX'*{}^yY'}\otimes\mathrm{Iso}_\delta\otimes\mathrm{Iso}_\gamma\otimes\mathrm{Def}^{{}^xX'\times_H{}^yY'}_{({}^xX'\times_H{}^yY')/\ker(\nu')}\otimes\mathrm{Res}^{{}^xX'\times{}^yY'}_{{}^xX\times_H{}^yY'}\otimes\mathrm{Con}_{(x,y)}\,,$$

with an isomorphism $\delta\colon ({}^{x}X' \times_H {}^{y}Y')\ker(\nu)/\ker(\nu) \xrightarrow{\sim} {}^{x}X' * {}^{y}Y'$ with the property that $\delta \circ \gamma = \overline{\nu'}$. Substituting the definition of $\mathrm{DefRes}^{{}^{x}X' \times {}^{y}Y'}_{{}^{x}X' * {}^{y}Y'}$, the proof is now complete. □

We can now generalize the formula in Theorem 2.5.

Theorem 2.10. *Let $X' \leq X \leq G \times H$, $Y' \leq Y \leq H \times K$, $U \in {}_{X'}\mathsf{set}$ and $V \in {}_{Y'}\mathsf{set}$. Then one has an isomorphism of $X * Y$-sets,*

$$\mathrm{Ind}_{X'}^{X}(U) \overset{X,Y}{\otimes} \mathrm{Ind}_{Y'}^{Y}(V) \cong \coprod_{(x,y)} \mathrm{Ind}_{{}^{x}X' * {}^{y}Y'}^{X*Y}\left(({}^{x}U) \overset{{}^{x}X', {}^{y}Y'}{\otimes} ({}^{y}V) \right), \tag{2.6}$$

where (x, y) runs through a set of representatives of the $(X \times_H Y, X' \times Y')$-double cosets of $X \times Y$. The isomorphism is induced by the maps $u \otimes v \mapsto (x \otimes u) \otimes (y \otimes v)$ from

$$({}^{x}U) \overset{{}^{x}X', {}^{y}Y'}{\otimes} ({}^{y}V) \quad \text{to} \quad \mathrm{Ind}_{X'}^{X}(U) \overset{X,Y}{\otimes} \mathrm{Ind}_{Y'}^{Y}(V).$$

Here we view ${}^{x}U$ as the set U with the left ${}^{x}X'$-action via the isomorphism $c_{x^{-1}}$ (see the paragraph after Example 2.3(c)*). It is natural in U and V, providing a natural isomorphism of functors ${}_{X'}\mathsf{set} \times {}_{Y'}\mathsf{set} \to {}_{X*Y}\mathsf{set}$.*

Proof. By Proposition 2.8, the left hand side of (2.6) is isomorphic to

$$\mathrm{DefRes}_{X*Y}^{X \times Y} \otimes_{X \times Y} \left(\mathrm{Ind}_{X'}^{X}(U) \times \mathrm{Ind}_{Y'}^{Y}(V) \right),$$

with

$$\mathrm{Ind}_{X'}^{X}(U) \times \mathrm{Ind}_{Y'}^{Y}(V) \cong \mathrm{Ind}_{X' \times Y'}^{X \times Y}(U \times V)$$

by Lemma 2.2 (c). Lemma 2.9 now yields an isomorphism as in (2.6). Using the isomorphisms from Proposition 2.8, Lemma 2.2 (c), and the proof of Lemma 2.9, one obtains the indicated map in the theorem. □

3. Bimodules and extended tensor products

In this section we recall the construction of extended tensor products of modules for group algebras (which is analogous to the construction in 2.6 for bisets). It was first introduced by Bouc in [5]. A list of properties of this construction can be found in [2, Section 6]. It turns out that this construction for modules can again be viewed as a "biset operation", see Proposition 3.4. This allows to derive Theorem 3.5 for modules over group algebras in analogy to Theorem 2.10 for sets with group actions. Theorem 3.5 will be used in the proof of Theorem 1.1 in Section 4.

Throughout this section, G, H, K, and L denote finite groups and $\Bbbk$ denotes a commutative ring. As with bisets, without further notice we view a $(\Bbbk G, \Bbbk H)$-bimodule as a left $\Bbbk[G \times H]$-module via the obvious category isomorphism ${}_{\Bbbk G}\mathsf{mod}_{\Bbbk H} \cong {}_{\Bbbk[G \times H]}\mathsf{mod}$.

3.1. Let $X \leq G \times H$, $Y \leq H \times K$, $M \in {}_{\Bbbk X}\mathsf{mod}$ and $N \in {}_{\Bbbk Y}\mathsf{mod}$. After restriction, M can be viewed as $(\Bbbk k_1(X), \Bbbk k(X,Y))$-bimodule and N can be viewed as $(\Bbbk k(X,Y), \Bbbk k_2(Y))$-bimodule, so that one obtains in the usual way a $(\Bbbk k_1(X), \Bbbk k_2(Y))$-bimodule $M \otimes_{\Bbbk k(X,Y)} N$. The corresponding $\Bbbk[k_1(X) \times k_2(Y)]$-module structure can be extended to $\Bbbk[X * Y]$ by setting $(g,k)(m \otimes n) := (g,h)m \otimes (h,k)n$, for $(g,k) \in X * Y$ and $h \in H$ such that $(g,h) \in X$ and $(h,k) \in Y$. We call this $\Bbbk[X * Y]$-module the *extended tensor product* of M and N. This defines a functor

$$- \underset{\Bbbk k(X,Y)}{\overset{X,Y}{\otimes}} - \colon {}_{\Bbbk X}\mathsf{mod} \times {}_{\Bbbk Y}\mathsf{mod} \to {}_{\Bbbk[X*Y]}\mathsf{mod}$$

which we sometimes simply denote by $-\overset{X,Y}{\otimes}-$. This functor is associative and respects direct sums in each argument.

The extended tensor product behaves well under scalar extension. The proof of the following Lemma is straightforward and left to the reader.

Lemma 3.2. *Let $X \leq G \times H$ and $Y \leq H \times K$, and let $M \in {}_{\Bbbk X}\mathsf{mod}$ and $N \in {}_{\Bbbk Y}\mathsf{mod}$.*

(a) If N is $\Bbbk$-projective and M is projective as right $\Bbbk k_2(X)$-module then

$$M \underset{\Bbbk k(X,Y)}{\overset{X,Y}{\otimes}} N$$

is $\Bbbk$-projective.

(b) Assume that $\Bbbk \to \Bbbk'$ is a homomorphism of commutative rings. One has an isomorphism

$$\Bbbk' \otimes_{\Bbbk} \left(M \underset{\Bbbk k(X,Y)}{\overset{X,Y}{\otimes}} N \right) \xrightarrow{\sim} (\Bbbk' \otimes_{\Bbbk} M) \underset{\Bbbk' k(X,Y)}{\overset{X,Y}{\otimes}} (\Bbbk' \otimes_{\Bbbk} N)$$

which is functorial in M and N.

Note that one has obvious *linearization functors* $\Bbbk - : {}_G\mathsf{set}_H \to {}_{\Bbbk G}\mathsf{mod}_{\Bbbk H}$ and $\Bbbk - : {}_G\mathsf{set} \to {}_{\Bbbk G}\mathsf{mod}$, were $\Bbbk U$ denotes the free $\Bbbk$-module with basis U, for any finite set U. Moreover, one has a functor

$$- \cdot_H - : {}_G\mathsf{set}_H \times {}_{\Bbbk H}\mathsf{mod} \to {}_{\Bbbk G}\mathsf{mod}\,, \quad (U, M) \mapsto \Bbbk U \otimes_{\Bbbk H} M\,.$$

We will sometimes just write $U \cdot M$ instead of $U \cdot_H M$ to simplify the notation. In the following lemma we compile a list of basic properties of this functor, whose straightforward verification we leave to the reader. Note that all isomorphisms in the following Lemma are natural in every variable. Recall that for $M \in {}_{\Bbbk G}\mathsf{mod}_{\Bbbk H}$ (resp. $M \in {}_{\Bbbk G}\mathsf{mod}$) and $N \in {}_{\Bbbk K}\mathsf{mod}_{\Bbbk L}$ (resp. $N \in {}_{\Bbbk K}\mathsf{mod}$) we may view $M \otimes_{\Bbbk} N$ as $(\Bbbk[G \times K], \Bbbk[H \times L])$-bimodule (resp. left $\Bbbk[G \times K]$-module), often referred to as the *external* product structure.

Lemma 3.3.

(a) For $U \in {}_G\mathsf{set}_H$, $V \in {}_H\mathsf{set}_K$ one has $\Bbbk(U \otimes_H V) \cong \Bbbk U \otimes_{\Bbbk H} \Bbbk V$ in ${}_{\Bbbk G}\mathsf{mod}_{\Bbbk K}$.

(b) For $U \in {}_G\mathsf{set}_H$, $V \in {}_K\mathsf{set}_L$, one has $\Bbbk U \otimes_{\Bbbk} \Bbbk V \cong \Bbbk(U \times V)$ in ${}_{\Bbbk[G\times K]}\mathsf{mod}_{\Bbbk[H \times L]}$.

(c) For $U, U' \in {}_G\mathsf{set}_H$ and $M \in {}_{\Bbbk H}\mathsf{mod}$ one has $(U \coprod U') \cdot_H M \cong (U \cdot_H M) \oplus (U' \cdot_H M)$ in ${}_{\Bbbk G}\mathsf{mod}$.

(d) For $U \in {}_G\mathsf{set}_H$ and $M, M' \in {}_{\Bbbk H}\mathsf{mod}$ one has $U \cdot_H (M \oplus M') \cong (U \cdot_H M) \oplus (U \cdot_H M')$ in ${}_{\Bbbk G}\mathsf{mod}$.

(e) For $U \in {}_G\mathsf{set}_H$, $V \in {}_H\mathsf{set}_K$, $M \in {}_{\Bbbk K}\mathsf{mod}$ one has $(U \otimes_H V) \cdot_K M \cong U \cdot_H (V \cdot_K M)$ in ${}_{\Bbbk G}\mathsf{mod}$.

(f) For $U \in {}_G\mathsf{set}_H$, $V \in {}_K\mathsf{set}_L$, $M \in {}_{\Bbbk H}\mathsf{mod}$, and $N \in {}_{\Bbbk L}\mathsf{mod}$ one has an isomorphism

$$(U \cdot_H M) \otimes_{\Bbbk} (V \cdot_L N) \cong (U \times V) \cdot_{H \times L} (M \otimes_{\Bbbk} N)$$

in ${}_{\Bbbk[G \times K]}\mathsf{mod}$.

Proposition 3.4. *Let $X \leq G \times H$ and $Y \leq H \times K$. The functors $-\overset{X,Y}{\otimes}-$ and*

$$\mathrm{DefRes}^{X\times Y}_{X*Y} \cdot_{X \times Y} (- \otimes_{\Bbbk} -)$$

from ${}_{\Bbbk X}\mathsf{mod} \times {}_{\Bbbk Y}\mathsf{mod}$ to ${}_{\Bbbk[X*Y]}\mathsf{mod}$ are naturally isomorphic.

Proof. This mirrors the proof of Proposition 2.8. Let $M \in {}_{\Bbbk X}\mathsf{mod}$ and $N \in {}_{\Bbbk Y}\mathsf{mod}$, then by the isomorphism (2.2) it suffices to show that one has an isomorphism

$$M \overset{X,Y}{\underset{\Bbbk k(X,Y)}{\otimes}} N \cong \frac{(X * Y) \times (X \times Y)}{\{(\nu(z), z) \mid z \in X \times_H Y\}} \cdot_{X \times Y} (M \otimes_{\Bbbk} N)$$

of $\Bbbk[(X*Y)\times(X\times Y)]$-modules. Mapping $m\otimes n$ to $\overline{1}\otimes(m\otimes n)$ defines such an isomorphism with inverse analogous to the inverse in the proof of Proposition 2.8. □

Theorem 3.5. *Let $X' \leq X \leq G \times H$, $Y' \leq Y \leq H \times K$, $M \in {}_{\Bbbk X'}\mathsf{mod}$, and $N \in {}_{\Bbbk Y'}\mathsf{mod}$. Then one has an isomorphism*

$$\mathrm{Ind}_{X'}^{X}(M) \overset{X,Y}{\otimes} \mathrm{Ind}_{Y'}^{Y}(N) \;\cong\; \bigoplus_{(x,y)} \mathrm{Ind}_{{}^{x}X' * {}^{y}Y'}^{X*Y} \left(({}^{x}M) \overset{{}^{x}X', {}^{y}Y'}{\otimes} ({}^{y}N) \right), \tag{3.1}$$

of $\Bbbk[X * Y]$-modules, where (x, y) runs through a set of representatives of the $(X \times_H Y, X' \times Y')$-double cosets of $X \times Y$. The isomorphism is induced by the maps $m \otimes n \mapsto (x \otimes m) \otimes (y \otimes n)$ from

$$({}^{x}M) \overset{{}^{x}X', {}^{y}Y'}{\otimes} ({}^{y}N) \quad \text{to} \quad \mathrm{Ind}_{X'}^{X}(M) \overset{X,Y}{\otimes} \mathrm{Ind}_{Y'}^{Y}(N).$$

Here we view ${}^{x}M$ as the $\Bbbk$-module M endowed with the $\Bbbk[{}^{x}X']$-module structure using the conjugation map $c_{x^{-1}}$. It is natural in M and N, providing a natural isomorphism of functors ${}_{\Bbbk X'}\mathsf{mod} \times {}_{\Bbbk Y'}\mathsf{mod} \to {}_{\Bbbk[X*Y]}\mathsf{mod}$.

Proof. Let $M \in {}_{\Bbbk X'}\mathsf{mod}$ and $N \in {}_{\Bbbk Y'}\mathsf{mod}$. Then

$$\mathrm{Ind}_{X'}^{X}(M) \overset{X,Y}{\otimes} \mathrm{Ind}_{Y'}^{Y}(N) \;\cong\; ((\mathrm{Ind}_{X'}^{X}) \cdot_{X'} M) \overset{X,Y}{\otimes} ((\mathrm{Ind}_{Y'}^{Y}) \cdot_{Y'} N)$$

and, by Proposition 3.4 and Lemma 3.3 (f) and (e), the latter is isomorphic to

$$\begin{aligned}
&\mathrm{DefRes}_{X*Y}^{X\times Y} \cdot_{X\times Y} \left(\left(\mathrm{Ind}_{X'}^{X} \cdot_{X'} M\right) \otimes_{\Bbbk} \left(\mathrm{Ind}_{Y'}^{Y} \cdot_{Y'} N\right) \right) \\
&\qquad \cong \mathrm{DefRes}_{X*Y}^{X\times Y} \cdot_{X\times Y} \left(\left(\mathrm{Ind}_{X'}^{X} \times \mathrm{Ind}_{Y'}^{Y}\right) \cdot_{X'\times Y'} (M \otimes_{\Bbbk} N) \right) \\
&\qquad \cong \mathrm{DefRes}_{X*Y}^{X\times Y} \cdot_{X\times Y} \left(\mathrm{Ind}_{X'\times Y'}^{X\times Y} \cdot_{X'\times Y'} (M \otimes_{\Bbbk} N) \right) \\
&\qquad \cong \left(\mathrm{DefRes}_{X*Y}^{X\times Y} \otimes_{X\times Y} \mathrm{Ind}_{X'\times Y'}^{X\times Y} \right) \cdot_{X'\times Y'} (M \otimes_{\Bbbk} N)\;.
\end{aligned}$$

Applying Lemma 2.9, Lemma 3.3 (c) and (e), and Proposition 3.4, the latter becomes isomorphic to the right hand side of (3.1), since ${}^{(x,y)}(M \otimes_{\Bbbk} N) \cong {}^{x}M \otimes_{\Bbbk} {}^{y}N$. □

As a special case of the above theorem with $X = G\times H$ and $Y = H\times K$ we recover Bouc's formula from [5]. In fact if h runs through a set of representatives of the $(p_2(X'), p_1(Y'))$-double cosets of H then $((1, 1), (h, 1))$ runs through a set of representatives of the $((G \times H) \times_H (H \times K), (X' \times Y'))$-double cosets of $(G \times H) \times (H \times K)$. After renaming X' and Y' as X and Y we obtain the following formulation.

Corollary 3.6. *Let $X \leq G \times H$, $Y \leq H \times K$, $M \in {}_{\Bbbk X}\mathsf{mod}$, and $N \in {}_{\Bbbk Y}\mathsf{mod}$. Then one has an isomorphism*

$$\mathrm{Ind}_{X}^{G\times H}(M) \otimes_{\Bbbk H} \mathrm{Ind}_{Y}^{H\times K}(N) \cong \bigoplus_{h \in [p_2(X)\backslash H/p_1(Y)]} \mathrm{Ind}_{X * {}^{(h,1)}Y}^{G\times K} \left(M \overset{X, {}^{(h,1)}Y}{\otimes} {}^{(h,1)}N \right)$$

of $(\Bbbk G, \Bbbk K)$-bimodules.

4. Proof of Theorem 1.1

Before we start with the proof of Theorem 1.1 we need some preparation. Let $(\mathbb{K}, \mathcal{O}, F)$ be a p-modular system and let X be a finite group. First we formulate for convenient reference the following well-known lemma.

Lemma 4.1. *Let $a \in Z(\mathcal{O}X)$ be a block idempotent, $A := \mathcal{O}Xa$ the corresponding block algebra, and let $M \in {}_A\mathsf{mod}$ be indecomposable with vertex P.*

(a) P is contained in a defect group of A, and if M is $\mathcal{O}$-free then $\mathrm{rk}_{\mathcal{O}}(M)$ is divisible by $[X : P]_p$.

(b) Let $M' \in {}_{\mathcal{O}N_X(P)}\mathsf{mod}$ be the Green correspondent of M and let (P, d) be a Brauer pair such that $dM' \neq \{0\}$. Then dM' is an indecomposable $\mathcal{O}N_X(P, d)d$-module with vertex P and $M' \cong \mathrm{Ind}_{N_X(P,d)}^{N_X(P)}(dM')$.

Proof.

(a) See [13, Theorems 5.1.9 (i) and 4.7.5].

(b) Let d' be the block idempotent of $\mathcal{O}N_X(P)$ to which M' belongs. Since P is normal in $N_X(P)$, d' is contained in $\mathcal{O}C_X(P)$ (see [12, Theorem 6.2.6 (ii)]) and $dd' = d$, since $\{0\} \neq dM = dd'M$ implies $dd' \neq 0$. Let $I := N_X(P, d)$ denote the stabilizer of (P, d). Then dM' is an $\mathcal{O}Id$-module. The $(\mathcal{O}Id, \mathcal{O}N_X(P)d')$-bimodule $d\mathcal{O}N_X(P)d' = d\mathcal{O}N_X(P)$ and the $(\mathcal{O}N_X(P)d', \mathcal{O}Id)$-bimodule $d'\mathcal{O}N_X(P)d = \mathcal{O}N_X(P)d$ induce mutually inverse Morita equivalences between ${}_{\mathcal{O}N_X(P)d'}\mathsf{mod}$ and ${}_{\mathcal{O}Id}\mathsf{mod}$ (see [12, Theorem 6.2.6 (iii)]). Moreover, these functors are naturally isomorphic to

$$d \cdot \mathrm{Res}_I^{N_X(P)} \quad \text{and} \quad \mathrm{Ind}_I^{N_X(P)},$$

respectively. Since M' is indecomposable, so is its image $dM' = d \cdot \mathrm{Res}_I^{N_X(P)}(M')$ under the Morita equivalence. Moreover, if Q is a vertex of M' then $M' = \mathrm{Ind}_I^{N_X(P)}(dM')$ implies that $P \leq Q$ and $dM' \mid \mathrm{Res}_I^{N_X(P)}(M')$ implies that $Q \leq P$. $\square$

4.2. We recall some facts about p-permutation-modules and the Brauer construction (see [2, Section 3] for more details). $M \in {}_{\mathcal{O}X}\mathsf{mod}$ (resp. $M \in {}_{FX}\mathsf{mod}$) is called a *$p$-permutation* module if it is isomorphic to a direct summand of a permutation module. We denote the Grothendieck group of the category of p-permutation $\mathcal{O}X$-modules with respect to split exact sequences by $T(\mathcal{O}X)$. If $a \in Z(\mathcal{O}X)$, we similarly define $T(\mathcal{O}Xa)$, $T(FX)$, and $T(FX\bar{a})$. The functor $F \otimes_{\mathcal{O}} -$ induces an isomorphism $T(\mathcal{O}Xa) \xrightarrow{\sim} T(FX\bar{a})$, $[M] \mapsto [\overline{M}]$, preserving indecomposablility and vertices. The *Brauer construction* with respect to a p-subgroup $P \leq X$ is a functor $-(P)\colon {}_{\mathcal{O}X}\mathsf{mod} \to {}_{FN_X(P)}\mathsf{mod}$ that takes p-permutation $\mathcal{O}Xa$-modules to p-permutation $FN_X(P)\mathrm{br}_P(a)$-modules and defines a group homomorphism $-(P)\colon T(\mathcal{O}Xa) \to T(FN_G(P)\mathrm{br}_X(a))$. If M is an indecomposable p-permutation $\mathcal{O}X$-module with vertex P, then $M(P)$ and the Green correspondent M' of M are related via $\overline{M'} \cong M(P)$. For $\omega \in T(\mathcal{O}X)$ and a Brauer pair (P, e) of $\mathcal{O}X$, we write (as in [2]) $\bar{\omega}(P, e) \in T(F[N_X(P, e)]\bar{e})$ for the element obtained by first applying $-(P)$ and then multiplying with the idempotent $\bar{e}$, and by $\omega(P, e)$ we denote the corresponding element in $T(\mathcal{O}[N_X(P, e)]e)$. We call (P, e) an *ω-Brauer pair* if $\omega(P, e) \neq 0$ in $T(\mathcal{O}[N_X(P, e)]e)$.

For the proof of Theorem 1.1 and the rest of this section, we fix again finite groups G and H, and assume that $\mathcal{O}$ has a root of unity of order $\exp(G \times H)$ as in Section 1. Furthermore, we fix block idempotents $b \in Z(\mathcal{O}G)$ and $c \in Z(\mathcal{O}H)$, and a p-permutation equivalence $\gamma \in T^{\Delta}(B,C)$ between the block algebras $B := \mathcal{O}Gb$ and $C := \mathcal{O}Hc$. Finally, we fix a maximal γ-Brauer pair (viewing γ as an element in $T(\mathcal{O}[G\times H])$). By [2, Remark 10.2 and Theorem 10.11] it is of the form $(\Delta(D,\phi,E), e\otimes f^*)$, for a maximal B-Brauer pair (D,e) and a maximal C-Brauer pair (E,f). Thus, D is a defect group of B and E is a defect group of C. Here, we write $-^*\colon \mathcal{O}X \to \mathcal{O}X$ for the map defined by $x \mapsto x^{-1}$, for $x \in X$. Note that this makes sense, since $C_{G\times H}(\Delta(D,\phi,E)) = C_G(D)\times C_H(E)$. Finally, we set $I := N_G(D,e)$ and $J := N_H(E,f)$.

Lemma 4.3. *Let $M \in {}_{\mathcal{O}G}\mathsf{mod}_{\mathcal{O}H}$ be indecomposable with vertex $X \le \Delta(D,\phi,E)$, and let $L \in {}_{\mathcal{O}H}\mathsf{mod}$ be indecomposable with vertex $Y \le E$. If $X < \Delta(D,\phi,E)$ or $Y < E$ then every indecomposable direct summand of $M\otimes_{\mathcal{O}H} L \in {}_{\mathcal{O}G}\mathsf{mod}$ has a vertex strictly contained in D. If additionally M and L are $\mathcal{O}$-free then also $M\otimes_{\mathcal{O}H} L$ is $\mathcal{O}$-free and its rank is divisible by $p\cdot[G:D]_p$.*

Proof. By Corollary 3.6, each indecomposable direct summand N of $M\otimes_{\mathcal{O}H} L$ satisfies

$$N \mid \mathrm{Ind}_{X*{}^hY}^{G}\left(\mathrm{Res}_X^{G\times H}(M) \overset{X,{}^hY}{\underset{\mathcal{O}k(X,{}^hY)}{\otimes}} \mathrm{Res}_{{}^hY}^{H}(L)\right)$$

for some $h \in H$, since $M \mid \mathrm{Ind}_X^{G\times H}(\mathrm{Res}_X^{G\times H}(M))$ and $L \mid \mathrm{Ind}_Y^H(\mathrm{Res}_Y^H(L))$. It is straightforward to verify that if $X < \Delta(D,\phi,E)$ or $Y < E$ then $X * {}^hY < D$, so that N has a vertex properly contained in D. If M and L are $\mathcal{O}$-free then so is

$$\mathrm{Res}_X^{G\times H}(M) \overset{X,{}^hY}{\otimes} \mathrm{Res}_{{}^hY}^{H}(L),$$

since $k(X,{}^hY) = \{1\}$. Thus, N is $\mathcal{O}$-free of $\mathcal{O}$-rank divisible by $p\cdot[G:D]_p$ (see Lemma 4.1 (a)). The result now follows. □

We will now prove Theorem 1.1 in four steps.

Proof of Theorem 1.1.

Step 1. Let $\psi \in \mathrm{Irr}(\mathbb{K}Hc)$ be an irreducible character of height zero. Then

$$\psi(1)_p = [H:E]_p \quad \text{and} \quad \psi(1)_{p'} = [H:E]_p^{-1}\cdot\psi(1)\,. \tag{4.1}$$

Let L be an $\mathcal{O}Hc$-lattice with character ψ. Then L is indecomposable and Lemma 4.1 (a) implies that E is a vertex of L. Let L' be the Green correspondent of L. Then $\mathrm{Ind}_{N_H(E)}^H(L') \cong L\oplus\widetilde{L}$ for some $\mathcal{O}H$-lattice $\widetilde{L}$ whose rank is divisible by $p\cdot[H:E]_p$ by Lemma 4.1 (a). Thus, with (4.1) we have

$$\psi(1)_{p'} = [H:E]_p^{-1}\cdot\mathrm{rk}_{\mathcal{O}}(L) \equiv [H:E]_p^{-1}\cdot\mathrm{rk}_{\mathcal{O}}\left(\mathrm{Ind}_{N_H(E)}^H(L')\right) \mod p\,. \tag{4.2}$$

Let $c' \in Z(\mathcal{O}N_H(E))$ be the block idempotent corresponding to c via Brauer's first main theorem, i.e., $\mathrm{br}_E(c) = \overline{c'}$. By [13, Corollary 5.3.11], c' acts as the identity on L'. Since (E,f) is a C-Brauer pair, $\overline{fc'} = \bar{f}\mathrm{br}_E(c) \neq 0$ and therefore $fc' \neq 0$. Moreover, c' is the sum of the distinct $N_H(E)$-conjugates of f (see [12, Theorem 6.2.6 (iii)]). Thus, $c'L \neq \{0\}$ implies $fL' \neq \{0\}$. Now Lemma 4.1 (b) implies that $L' = \mathrm{Ind}_J^{N_H(E)}(L'')$ for the indecomposable $\mathcal{O}Jf$-module $L'' := fL'$. With (4.2) we obtain

$$\psi(1)_{p'} \equiv [H:E]_p^{-1}\cdot\mathrm{rk}_{\mathcal{O}}\left(\mathrm{Ind}_J^H(L'')\right) = \frac{[H:J]}{[H:E]_p}\cdot\mathrm{rk}_{\mathcal{O}}(L'') = \frac{[H:J]_{p'}}{[J:E]_p}\cdot\mathrm{rk}_{\mathcal{O}}(L'') \mod p\,.$$

Since the left hand side of this congruence is not divisible by p, we have $\mathrm{rk}_{\mathcal{O}}(L'')_p = [J:E]_p$ and

$$\psi(1)_{p'} \equiv [H:J]_{p'} \cdot \mathrm{rk}_{\mathcal{O}}(L'')_{p'} \mod p\,. \tag{4.3}$$

STEP 2. Next, let $\mu := \kappa(\gamma) \in R(\mathbb{K}Gb, \mathbb{K}Hc)$ be as in the introduction and set

$$\chi := I_\mu(\psi) = \mu \underset{H}{\cdot} \psi\,.$$

Since μ is a perfect isometry (see [2, Proposition 9.9]), we have $\chi \in \pm\mathrm{Irr}(\mathbb{K}Gb)$. The goal of this step is to determine the congruence class of $\chi(1)_{p'}$ modulo p in terms of local data. First recall that the perfect isometry μ preserves heights (see [6, Lemme 1.6]) so that

$$\chi(1)_p = [G:D]_p \quad \text{and} \quad \chi(1)_{p'} = [G:D]_p^{-1} \cdot \chi(1) = [G:D]_p^{-1} \cdot \left(\mu \underset{H}{\cdot} \psi\right)(1)\,. \tag{4.4}$$

By [2, Theorems 14.1, 14.3, and 10.11] we can write

$$\gamma = \varepsilon \cdot [M] + \sum_{i=1}^{r} n_i \cdot [M_i]\,,$$

with $\varepsilon := \varepsilon(\gamma) \in \{\pm 1\}$ the sign of γ, integers $n_1, \ldots, n_r$, an indecomposable (B,C)-bimodule M with vertex $\Delta(D,\phi,E)$ and indecomposable (B,C)-bimodules M_i, $i = 1, \ldots, r$, each of which has a vertex strictly contained in $\Delta(D,\phi,E)$. By Lemma 3.2 (a), $M \otimes_{\mathcal{O}H} L$ and $M_i \otimes_{\mathcal{O}H} L$ are $\mathcal{O}$-free with

$$\left(\mu \underset{H}{\cdot} \psi\right)(1) = \varepsilon \cdot \mathrm{rk}_{\mathcal{O}}(M \otimes_{\mathcal{O}H} L) + \sum_{i=1}^{r} n_i \cdot \mathrm{rk}_{\mathcal{O}}(M_i \otimes_{\mathcal{O}H} L)\,. \tag{4.5}$$

Moreover, since $M \otimes_{\mathcal{O}H} L$ and $M_i \otimes_{\mathcal{O}H} L$ are $\mathcal{O}Gb$-lattices, their $\mathcal{O}$-ranks are divisible by $[G:D]_p$ (see Lemma 4.1), and by Lemma 4.3, the $\mathcal{O}$-rank of $M_i \otimes_{\mathcal{O}H} L$, $i = 1, \ldots, r$, is divisible by $p \cdot [G:D]_p$. Therefore,

$$[G:D]_p^{-1} \cdot \mathrm{rk}_{\mathcal{O}}\left(M_i \otimes_{\mathcal{O}H} L\right) \equiv 0 \mod p$$

for all $i = 1, \ldots, r$. Together with (4.4) and (4.5) this implies

$$\chi(1)_{p'} \equiv \varepsilon \cdot [G:D]_p^{-1} \cdot \mathrm{rk}_{\mathcal{O}}(M \otimes_{\mathcal{O}H} L) \mod p\,. \tag{4.6}$$

With $L'' \in {}_{\mathcal{O}Jf}\mathsf{mod}$ and $L' \cong \mathrm{Ind}_J^{N_H(E)}(L'')$ as in Step 1, we can write $\mathrm{Ind}_{N_H(E)}^H(L') \cong L \oplus \widetilde{L}$, with each indecomposable direct summand of $\widetilde{L}$ having a vertex Q strictly contained in E. By Lemma 4.3, $p \cdot [G:D]_p$ divides $\mathrm{rk}_{\mathcal{O}}(M \otimes_{\mathcal{O}} \widetilde{L})$. Thus, with (4.6) we obtain

$$\chi(1)_{p'} \equiv \varepsilon \cdot [G:D]_p^{-1} \cdot \mathrm{rk}_{\mathcal{O}}\left(M \otimes_{\mathcal{O}H} \mathrm{Ind}_J^H(L'')\right) \mod p\,. \tag{4.7}$$

Set $Y' := N_{G\times H}(\Delta(D,\phi,E))$ and $Y'' := N_{G\times H}(\Delta(D,\phi,E), e\otimes f^*) \le Y'$ and let $M' \in {}_{\mathcal{O}Y'}\mathsf{mod}$ be the Green correspondent of M. Then $F \otimes_{\mathcal{O}} M' = M(\Delta(D,\phi,E))$, since M is a p-permutation module with vertex $\Delta(D,\phi,E)$. Since $M_1, \ldots, M_r$ have vertices strictly contained in $\Delta(D,\phi,E)$, we have $0 \ne (e\otimes f^*)\cdot\gamma(\Delta(D,\phi,E)) = [(e\otimes f^*)M']$, and Lemma 4.1 (b) implies that $M' \cong \mathrm{Ind}_{Y''}^{Y'}(M'')$ for the indecomposable $\mathcal{O}Y''(e\otimes f^*)$-module $M'' := (e\otimes f^*)M'$ with vertex $\Delta(D,\phi,E)$. We have $\mathrm{Ind}_{Y'}^{G\times H}(M') \cong M \oplus \widetilde{M}$ for some $\mathcal{O}[G\times H]$-lattice $\widetilde{M}$, each of whose indecomposable direct summands have a vertex strictly contained in $\Delta(D,\phi,E)$. Lemma 4.3 implies that $p\cdot[G:D]_p$ divides $\mathrm{rk}_{\mathcal{O}}(\widetilde{M} \otimes_{\mathcal{O}H} \mathrm{Ind}_J^H(L''))$ and with (4.7) we obtain

$$\chi(1)_{p'} \equiv \varepsilon \cdot [G:D]_p^{-1} \cdot \mathrm{rk}_{\mathcal{O}}\left(\mathrm{Ind}_{Y''}^{G\times H}(M'') \otimes_{\mathcal{O}H} \mathrm{Ind}_J^H(L'')\right) \mod p\,. \tag{4.8}$$

By [2, Proposition 11.1], we have

$$p_1(Y'') = I\,, \quad p_2(Y'') = J\,, \quad k_1(Y'') = C_G(D)\,, \quad \text{and} \quad k_2(Y'') = C_H(E)\,. \tag{4.9}$$

Since $p_2(Y'') = J$, Corollary 3.6 (with $K = \{1\}$) implies that

$$\mathrm{Ind}_{Y''}^{G\times H}(\widetilde{M}) \otimes_{\mathcal{O}H} \mathrm{Ind}_J^H(L'') \cong \bigoplus_{h\in[J\backslash H/J]} \mathrm{Ind}_{Y''*{}^hJ}^G \left(M'' \overset{Y'',{}^hJ}{\otimes} {}^hL''\right)\,. \tag{4.10}$$

We study the direct summands in (4.10).

Case (i): Suppose $h \in H$ but $h \notin N_H(E)$. Let $S \in {}_{\mathcal{O}\,{}^hE}\mathsf{mod}$ be a source of ${}^hL''$. Since M'' has trivial source, we have

$$M'' \overset{Y'',{}^hJ}{\otimes} {}^hL'' \;\Big|\; \mathrm{Ind}_{\Delta(D,\phi,E)}^{Y''}(\mathcal{O}) \overset{Y'',{}^hJ}{\otimes} \mathrm{Ind}_{{}^hE}^{{}^hJ}(S)\,.$$

Moreover, Theorem 3.5 (with $K = \{1\}$) and Lemma 3.2 (a) imply that the latter module is a direct sum of $\mathcal{O}[Y'' * {}^hJ]$-lattices that are induced from subgroups of the form ${}^{(g,h_1)}\Delta(D,\phi,E) * {}^{h_2h}E$ with $(g,h_1) \in Y''$ and $h_2 \in {}^hJ$. Since $p_2(Y'') = J \le N_H(E)$, we obtain $p_2({}^{(g,h_1)}\Delta(D,\phi,E)) = {}^{h_1}E = E$. Since ${}^{h_2h}E = {}^hE$ and since $E \cap {}^hE < E$ by the choice of h, the group ${}^{(g,h_1)}\Delta(D,\phi,E) * {}^{h_2h}E$ is properly contained in D. Thus, by Lemma 4.1 (a), we obtain

$$[G:D]_p^{-1}\cdot \mathrm{rk}_{\mathcal{O}}\left(\mathrm{Ind}_{Y''*{}^hJ}^G\left(M'' \overset{Y'',{}^hJ}{\otimes} {}^hL''\right)\right) \equiv 0 \mod p$$

in this case.

Case (ii): Suppose $h \in N_H(E)$ but $h \notin J$. We claim that in this case $M'' \overset{Y'',{}^hJ}{\otimes} {}^hL'' = \{0\}$. In fact, since $k_2(Y'') = C_H(E) \le {}^hJ$ (see (4.9)), we have $k(Y'', {}^hJ) = C_H(E)$. Thus, by the definition of $- \overset{Y'',{}^hJ}{\otimes} -$, we have

$$\mathrm{Res}_{C_G(D)}^{Y''*{}^hJ}\left(M'' \overset{Y'',{}^hJ}{\otimes} {}^hL''\right) = \mathrm{Res}_{C_G(D)\times C_H(E)}^{Y''}(M'') \otimes_{\mathcal{O}C_H(E)} \mathrm{Res}_{C_H(E)}^{{}^hJ}({}^hL'')\,.$$

Since the block idempotent f of $\mathcal{O}C_H(E)$ acts as the identity on

$$\mathrm{Res}_{C_G(D)\times C_H(E)}^{Y''}(M'') = eM'f$$

from the right, since the block idempotent ${}^hf$ acts as the identity on

$$\mathrm{Res}_{C_H(Q)}^{{}^hJ}\left({}^hL''\right) = {}^hf\cdot L'$$

from the left, and since $f\cdot {}^hf = 0$, the claim is proved.

With the conclusions for Case (i) and Case (ii), (4.8) and (4.10) imply that

$$\chi(1)_{p'} \equiv \varepsilon\cdot [G:D]_p^{-1}\cdot \mathrm{rk}_{\mathcal{O}}\left(\mathrm{Ind}_{Y''*J}^G\left(M'' \overset{Y'',J}{\otimes} L''\right)\right) \mod p\,,$$

with $Y'' * J = I$, since $p_1(Y'') = I$ and $p_2(Y'') = J$ (see (4.9)). Thus,

$$\chi(1)_{p'} \equiv \varepsilon\cdot\frac{[G:I]}{[G:D]_p}\cdot \mathrm{rk}_{\mathcal{O}}\left(M'' \overset{Y'',J}{\otimes} L''\right) = \varepsilon\cdot\frac{[G:I]_{p'}}{[I:D]_p}\cdot \mathrm{rk}_{\mathcal{O}}\left(M'' \overset{Y'',J}{\otimes} L''\right) \mod p\,.$$

Since the left hand side of this congruence is not divisible by p, we have $\mathrm{rk}_{\mathcal{O}}(M'' \overset{Y'',J}{\otimes} L'')_p = [I:D]_p$ and

$$\chi(1)_{p'} \equiv \varepsilon \cdot [G:I]_{p'} \cdot \mathrm{rk}_{\mathcal{O}} \left(M'' \overset{Y'',J}{\otimes} L'' \right)_{p'} \mod p\,. \tag{4.11}$$

Step 3. Let V and W be the unique simple modules of $FC_G(D)e$ and $FC_H(E)f$, respectively, as defined in the paragraph preceding Theorem 1.1. We claim that

$$\frac{\mathrm{rk}_{\mathcal{O}} \left(M'' \overset{Y'',J}{\otimes} L'' \right)}{\mathrm{rk}_{\mathcal{O}}(L'')} = \frac{\dim_F(V)}{\dim_F(W)}\,. \tag{4.12}$$

By [2, Proposition 14.4] with $S = C_G(D)$ and $T = C_H(E)$, the $(\mathcal{O}C_G(D)e, \mathcal{O}C_H(E)f)$-bimodule $M''' := \mathrm{Res}^{Y''}_{C_G(D)\times C_H(E)}(M'')$ induces a Morita equivalence between the block algebras $\mathcal{O}C_G(D)e$ and $\mathcal{O}C_H(E)f$. Therefore, the $(FC_G(D)\bar{e}, FC_H(E)\bar{f})$-bimodule $\overline{M'''} := F \otimes_{\mathcal{O}} M'''$ induces a Morita equivalence between the block algebras $FC_G(D)\bar{e}$ and $FC_H(E)\bar{f}$. This implies that

$$\overline{M'''} \otimes_{FC_H(E)} W \cong V\,,$$

since V and W are the unique simple modules in these block algebras. Moreover, the multiplicity of W as composition factor in $\overline{L''} := F \otimes_{\mathcal{O}} L''$ is equal to the multiplicity of V as composition factor in $\overline{M'''} \otimes_{FC_H(E)} \overline{L''}$. Thus,

$$\frac{\dim_F \left(\overline{M'''} \otimes_{FC_H(E)} \overline{L''} \right)}{\dim_F(\overline{L''})} = \frac{\dim_F(V)}{\dim_F(W)}\,.$$

Since $\dim_F(\overline{M'''} \otimes_{FC_H(E)} \overline{L''}) = \mathrm{rk}_{\mathcal{O}}(M'' \overset{Y'',J}{\otimes} L'')$ and $\dim_F(\overline{L''}) = \mathrm{rk}_{\mathcal{O}}(L'')$ by Lemma 3.2 (b), Equation (4.12) holds.

Step 4. Since we have an isomorphism $\phi\colon E \xrightarrow{\sim} D$, we obtain $|Z(D)| = |Z(E)|$. Moroever, since the fractions in (1.1) and $b(B)$ and $b(C)$ are units in $\mathbb{Z}_{(p)}$, it suffices to show that

$$\big(|G| \cdot \psi(1) \cdot |C_H(E)| \cdot \dim_F(V)\big)_{p'} \equiv \varepsilon \cdot \big(|H| \cdot \chi(1) \cdot |C_G(D)| \cdot \dim_F(W)\big)_{p'} \mod p\,. \tag{4.13}$$

Using (4.3) and (4.12), the left hand side of (4.13) is congruent to

$$\begin{aligned} &|G|_{p'} \cdot [H:J]_{p'} \cdot \mathrm{rk}_{\mathcal{O}}(L'')_{p'} \cdot |C_H(E)|_{p'} \cdot \dim_F(V)_{p'} \\ = &|G|_{p'} \cdot [H:J]_{p'} \cdot |C_H(E)|_{p'} \cdot \mathrm{rk}_{\mathcal{O}} \left(M'' \overset{Y'',J}{\otimes} L'' \right)_{p'} \cdot \dim_F(W)_{p'} \end{aligned} \tag{4.14}$$

modulo p. Using (4.11), the right hand side of (4.13) is congruent to

$$\varepsilon \cdot |H|_{p'} \cdot \varepsilon \cdot [G:I]_{p'} \cdot \mathrm{rk}_{\mathcal{O}} \left(M'' \overset{Y'',J}{\otimes} L'' \right)_{p'} \cdot |C_G(D)|_{p'} \cdot \dim_F(W)_{p'} \tag{4.15}$$

modulo p. But since $[I : C_G(D)] = [J : C_H(E)]$ by [2, Proposition 11.1], the integers in (4.14) and (4.15) are equal. This proves the congruence in (4.13) and completes the proof of Theorem 1.1. □

5. Proof of Theorem 1.4

Proof of Theorem 1.4(a). Let $\gamma \in T^{\Delta}(B, C)$ be a p-permutation equivalence between B and C. Then $\mu := \kappa_{G\times H}(\gamma) \in R(\mathbb{K}Gb, \mathbb{K}Hc)$ is a perfect isometry. Let $\alpha\colon \mathrm{Irr}(\mathbb{K}Hc) \xrightarrow{\sim} \mathrm{Irr}(\mathbb{K}Gb)$ be the bijection induced by the perfect isometry I_μ. Since the quotient in (1.1) is a unit in $\mathbb{Z}_{(p)}$, the bijection α preserves heights. Moreover, since $\beta(\mu) \in \{\pm 1\}$ by Corollary 1.3, we obtain $r(\alpha(\chi)) \equiv \pm r(\chi) \mod p$ for all $\chi \in \mathrm{Irr}(\mathbb{K}Hc)$. □

Proof of Theorem 1.4(b). Block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ have coefficients in $\mathbb{Z}_p[\zeta']$, where ζ' is the $\exp(G\times H)_p^{\mathrm{th}}$ power of ζ. Moreover, the natural map between $\mathrm{Gal}(\mathbb{Q}_p(\zeta')/\mathbb{Q}_p)$ and the Galois group of the residue field of $\mathbb{Q}_p(\zeta')$ over $\mathbb{F}_p$ is an isomorphism. Therefore, [3, Theorem 4.2] implies that $\Gamma_b = \Gamma_c$.

Let $\widetilde{\gamma} \in T^{\Delta}(\mathbb{Z}_pG\widetilde{b}, \mathbb{Z}_pH\widetilde{c})$ with $\widetilde{\gamma} \cdot_H \widetilde{\gamma}^\circ = [\mathbb{Z}_pG\widetilde{b}] \in T^{\Delta}(\mathbb{Z}_pG\widetilde{b}, \mathbb{Z}_pG\widetilde{b})$ and $\widetilde{\gamma}^\circ \circ_G \widetilde{\gamma} = [\mathbb{Z}_pH\widetilde{c}] \in T^{\Delta}(\mathbb{Z}_pH\widetilde{c}, \mathbb{Z}_pH\widetilde{c})$ and let $\gamma \in T^{\Delta}(\mathcal{O}G\widetilde{b}, \mathcal{O}H\widetilde{c})$ be the image of $\widetilde{\gamma}$ under the natural map induced by scalar extension from $\mathbb{Z}_p$ to $\mathcal{O}$. Then $\gamma \cdot_H \gamma^\circ = [\mathcal{O}G\widetilde{b}]$ and $\gamma^\circ \cdot_G \gamma = [\mathcal{O}H\widetilde{c}]$. So γ is a p-permutation equivalence between $\mathcal{O}G\widetilde{b}$ and $\mathcal{O}H\widetilde{c}$.

Set $\mu := \kappa_{G\times H}(\gamma)$ and let

$$\alpha\colon \mathrm{Irr}(\mathbb{K}H\widetilde{c}) \xrightarrow{\sim} \mathrm{Irr}(\mathbb{K}G\widetilde{b}) \tag{5.1}$$

be the bijection induced by the perfect isometry $I_\mu = \mu \cdot_H -\colon R(\mathbb{K}H\widetilde{c}) \to R(\mathbb{K}G\widetilde{b})$. Since the character μ has values in $\mathbb{Q}_p$, the isomorphism I_μ and then also the bijection α in (5.1) are Γ-equivariant.

Let $b = b_1, \cdots, b_n$ (resp. $c = c_1, \ldots, c_n$) denote the elements of the Γ-orbit of b (resp. c). Note that they have the same length, since $\Gamma_b = \Gamma_c$. Then $\widetilde{b} = b_1 + \cdots + b_n$ and $\widetilde{c} = c_1 + \cdots + c_n$. By [2, Theorem 10.10] there exists $i \in \{1, \ldots, n\}$ such that $\gamma_c := b_i \cdot \gamma \cdot c$ is a p-permutation equivalence between $\mathcal{O}Gb_i$ and $\mathcal{O}Hc$ and the resulting perfect isometry $I_c\colon R(\mathbb{K}Hc) \xrightarrow{\sim} R(\mathbb{K}Gb_i)$ is the restriction of I_μ. Thus, the bijection α in (5.1) restricts to a bijection

$$\alpha_c\colon \mathrm{Irr}(\mathbb{K}Hc) \xrightarrow{\sim} \mathrm{Irr}(\mathbb{K}Gb_i)\,. \tag{5.2}$$

Note that $\Gamma_{b_i} = \Gamma_b = \Gamma_c$, since Γ is abelian. Moreover, since α_c is the restriction of the Γ-equivariant bijection α in (5.1), the bijection α_c is Γ_b-invariant. Further, since $\beta(\gamma_c) = \pm 1$ by Corollary 1.3, we have $r(\alpha_c(\psi)) \equiv \pm r(\psi) \mod p$ for all $\psi \in \mathrm{Irr}(\mathbb{K}Hc)$.

Let $\sigma \in \Gamma$ be such that $\sigma(b_i) = b$. Note that the bijection $\sigma\colon \mathrm{Irr}(\mathbb{K}Gb_i) \to \mathrm{Irr}(\mathbb{K}Gb)$, $\chi \mapsto {}^{\sigma}\chi$ is also Γ_b-equivariant, since Γ is abelian, and satisfies $r({}^{\sigma}\chi) = r(\chi)$ for all $\chi \in \mathrm{Irr}(\mathbb{K}Gb_i)$. Therefore, the composition of α_c in (5.2) and σ yields a bijection $\mathrm{Irr}(\mathbb{K}Hc) \xrightarrow{\sim} \mathrm{Irr}(\mathbb{K}Gb)$ with the desired properties. □

In order to prove Part (c) of Theorem 1.4, we need two more results. The first one is known to specialists, but does not seem to occur in this formulation in the literature. It's proof follows from the results in [15, Sections 12–14], especially from Theorem 14.5.

Theorem 5.1. *Two finite-dimensional $\mathbb{Q}_p$-division algebras D_1 and D_2 are isomorphic as $\mathbb{Q}_p$-algebras if and only if their centers are isomorphic as $\mathbb{Q}_p$-algebras and their Hasse invariants coincide.*

For the second result we need to introduce some additional notation. Let A and B be algebras over a field k. For a left A-module M we denote by $M^\circ := \mathrm{Hom}_k(M, k)$ the k-dual of A, viewed as right A-module. By $K(A)$ we denote the homotopy category of bounded chain complexes of finitely generated A-modules. We identify the category of (A,B)-bimodules with the category of left $A\otimes_k B^\circ$-modules in the usual way, where B° denotes the opposite k-algebra of B. We denote by $K(A,B)$ the homotopy category of bounded chain complexes of finitely generated (A,B)-bimodules. If M is an (A,B)-bimodule then M° is a (B,A)-bimodule. If X_* is a chain complex of (A,B)-bimodules then X_*° is a chain complex of (B,A)-bimodules. For any integer i and any (A,B)-bimodule M we write $M[i]$ for the chain complex with term M in degree i and terms $\{0\}$ in all other degrees. Recall that for a semisimple k-algebra A and any bounded chain complex X_* of finitely generated A-modules, one has $X_*\cong H(X_*)$ in $K(A)$, where $H(X_*)\in K(A)$ is the $\mathbb{Z}$-graded A-module consisting of the homology of X_* with trivial boundary maps.

Proposition 5.2. *Let k be a field, let A and B be semisimple k-algebras, let X_* be a bounded chain complex of finitely generated (A,B)-bimodules satisfying $X_*\otimes_B X_*^\circ\cong A[0]$ in $K(A,A)$ and $X_*^\circ\otimes_A X_*\cong B[0]$ in $K(B,B)$. If W is a simple B-module then $X_*\otimes_B W\cong V[i]$ in $K(A)$ for a simple A-module V and an integer i. Moreover,* $\mathrm{End}_A(V)\cong\mathrm{End}_B(W)$ *as k-algebras.*

Proof. Note that $X_*\otimes_B -\colon K(B)\to K(A)$ is an equivalence. Moreover, for any simple B-module W, we have isomorphisms

$$\begin{aligned}\mathrm{End}_B(W)&\cong\mathrm{Hom}_{K(B)}(W[0],W[0])\cong\mathrm{Hom}_{K(A)}\left(X_*\otimes_B W[0],X_*\otimes_B W[0]\right)\\&\cong\mathrm{Hom}_{K(A)}\left(H(X_*\otimes_B W[0]),H(X_*\otimes_B W[0])\right)\cong\prod_{i\in\mathbb{Z}}\mathrm{End}_A\left(H_i(X_*\otimes_B W[0])\right)\end{aligned}$$

of k-algebras. Since $\mathrm{End}_B(W)$ is a division algebra over k, there exists a unique $i\in\mathbb{Z}$ such that $\mathrm{End}_A(H_i(X_*\otimes_B W[0]))$ is a division algebra isomorphic to $\mathrm{End}_B(W)$ and $H_j(X_*\otimes_B W[0])=0$ for all $j\in\mathbb{Z}$ with $j\neq i$. Thus, $X_*\otimes_B W[0]\cong H(X_*\otimes_B W[0])\cong H_i(X_*\otimes_B W[0])\cong V[i]$ in $K(A)$ for $V:=H_i(X_*\otimes_B W[0])\in {}_A\mathsf{mod}$. Since $\mathrm{End}_A(V)$ is a division algebra, V is a simple A-module and the result follows. □

Proof of Theorem 1.4 (c). Let X_* be a splendid Rickard equivalence between $\mathbb{Z}_pG\widetilde{b}$ and $\mathbb{Z}_pH\widetilde{c}$, set $\widetilde{\gamma}:=\sum_{i\in\mathbb{Z}}(-1)^i[X_i]\in T^\Delta(\mathbb{Z}_pG\widetilde{b},\mathbb{Z}_pH\widetilde{c})$. Then $\widetilde{\gamma}$ is a p-permutation equivalence between $\mathbb{Z}_pG\widetilde{b}$ and $\mathbb{Z}_pH\widetilde{c}$. Thus we can use all the steps and notations in the proof of Theorem 1.4 (b) and obtain a Γ_b-equivariant bijection $\alpha_c\colon\mathrm{Irr}(\mathbb{K}Hc)\xrightarrow{\sim}\mathrm{Irr}(\mathbb{K}Gb_i)$ as in (5.2) which satisfies $r(\alpha(\psi))\equiv\pm r(\psi)\mod p$ for all $\psi\in\mathrm{Irr}(\mathbb{K}Hc)$. We claim that also $h(\psi)=h(\alpha_c(\psi))$ for all $\psi\in\mathrm{Irr}(\mathbb{K}Hc)$. In fact, let W be the unique irreducible $\mathbb{Q}_pH\widetilde{c}$-module such that ψ is a constituent of the character of W. By Proposition 5.2, there exists $i\in\mathbb{Z}$ and an irreducible $\mathbb{Q}_pG\widetilde{b}$-module V such that $(\mathbb{Q}_p\otimes_{\mathbb{Z}_p}X)\otimes_{\mathbb{Q}_pH\widetilde{c}}W$ is homotopy equivalent to $V[i]$. Moreover, by the definition of α_c, the irreducible character $\alpha_c(\psi)$ is a constituent of the character of $\mathbb{K}\otimes_{\mathbb{Q}_p}V$. Thus, $h(\psi)$ is the Hasse invariant of the $\mathbb{Q}_p$-division algebra $\mathrm{End}_{\mathbb{Q}_pH}(W)$ and $h(\alpha_c(\psi))$ is the Hasse invariant of the $\mathbb{Q}_p$-division algebra $\mathrm{End}_{\mathbb{Q}_pG}(V)$. By Proposition 5.2, these two $\mathbb{Q}_p$-division algebras are isomorphic. Now, Theorem 5.1 impllies the claim.

Let $\sigma\in\Gamma$ be such that $\sigma(b_i)=b$. As in the proof of Theorem 1.4 (b) we consider the composition of the bijection $\alpha_c\colon\mathrm{Irr}(\mathbb{K}Hc)\xrightarrow{\sim}\mathrm{Irr}(\mathbb{K}Gb_i)$ and the bijection $\sigma\colon\mathrm{Irr}(\mathbb{K}Gb_i)\xrightarrow{\sim}\mathrm{Irr}(\mathbb{K}Gb)$. The only thing that still needs to be shown is that $h({}^\sigma\chi)=h(\chi)$, for any $\chi\in\mathrm{Irr}(\mathbb{K}G)$ and any $\sigma\in\Gamma$. But this is immediate, since χ and ${}^\sigma\chi$ determine the same irreducible $\mathbb{Q}_pG$-module whose character has χ and ${}^\sigma\chi$ as constituent. This finishes the proof of Theorem 1.4 (c). □

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