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Group algebras in which the socle of the center is an ideal

Sofia Brenner and Burkhard Külshammer*

Abstract. Let F be a field of characteristic $p > 0$. We study the structure of the finite groups G for which the socle of the center of FG is an ideal in FG and classify the finite p-groups G with this property. Moreover, we give an explicit description of the finite groups G for which the Reynolds ideal of FG is an ideal in FG.

1. Introduction

Let F be a field and consider the group algebra FG of a finite group G and its center ZFG. The question when the Jacobson radical of ZFG is an ideal in FG has been answered by Clarke [4], Koshitani [7] and Külshammer [9]. We now study the corresponding problem for the socle $\operatorname{soc}(ZFG)$ of ZFG as well as for the Reynolds ideal $R(FG)$ of FG. In a prequel to this paper [3], we have already given some approaches to these problems for general symmetric algebras. Now, our aim is to analyze the structure of the finite groups G for which $\operatorname{soc}(ZFG)$ or $R(FG)$ are ideals of FG in a group-theoretic manner. For the Reynolds ideal, we obtain the following characterization:

Theorem A. *Let F be a field of characteristic $p > 0$ and let G be a finite group. Then the Reynolds ideal $R(FG)$ is an ideal in FG if and only if G' is contained in the p-core $O_p(G)$ of G.*

As a consequence of this result, it follows that if $\operatorname{soc}(ZFG)$ is an ideal in FG, one has $G = P \rtimes H$ for a Sylow p-subgroup P of G and an abelian p'-group H. Based on this decomposition, we derive some fundamental results on the structure of finite groups G for which $\operatorname{soc}(ZFG)$ is an ideal in FG. Subsequently, we classify the finite p-groups G with this property:

Theorem B. *Let F be a field of characteristic $p > 0$ and let G be a finite p-group. Then $\operatorname{soc}(ZFG)$ is an ideal in FG if and only if*

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(i) G has nilpotency class at most two, that is, $G' \subseteq Z(G)$ holds, or
(ii) $p = 2$ and $G' \subseteq Y(G)Z(G)$ with $Y(G) = \langle fg^{-1} \colon \{f, g\}$ is a conjugacy class of length 2 of $G\rangle$.

In particular, G is metabelian.

Note that since the p-groups of nilpotency class at most two form a large subclass of the finite p-groups, the condition that $\mathrm{soc}(ZFG)$ is an ideal in FG is often satisfied. One implication of Theorem B generalizes to arbitrary finite groups:

Theorem C. *Let F be a field of characteristic $p > 0$ and let G be a finite group. Suppose that one of the following holds:*
(i) $G' \subseteq Z(O_p(G))$, or
(ii) $p = 2$ and $G' \subseteq Y(O_p(G))Z(O_p(G))$.

Then $\mathrm{soc}(ZFG)$ is an ideal in FG.

The above results are major ingredients for the proof of the main result of this paper, which is a decomposition of G into a central product:

Theorem D. *Let F be a field of characteristic $p > 0$. Suppose that G is a finite group for which $\mathrm{soc}(ZFG)$ is an ideal in FG and write $G = P \rtimes H$ for a Sylow p-subgroup P of G and an abelian p'-group H as before. Then G is the central product of the centralizer $C_P(H)$ and the p-residual group $O^p(G)$. Moreover, $\mathrm{soc}(ZFC_P(H))$ and $\mathrm{soc}(ZFO^p(G))$ are ideals in $FC_P(H)$ and $FO^p(G)$, respectively. Furthermore, we have*

$$\mathrm{soc}(ZFG) = (Z(P)G')^+ \cdot FG,$$

where $(Z(P)G')^+ \in FG$ denotes the sum of the elements in $Z(P)G'$.

This statement will allow us to restrict our investigation to the case $P = G'$. A detailed analysis of the structure of finite groups G for which $\mathrm{soc}(ZFG)$ is an ideal in FG, based on the above results, will be carried out in a sequel to this paper.

We proceed as follows: First, we introduce our notation (see Section 2) and study the general structure of the finite groups G for which $\mathrm{soc}(ZFG)$ or $R(FG)$ are ideals in FG (see Section 3). In Section 4, we classify the p-groups G for which $\mathrm{soc}(ZFG)$ is an ideal in FG for a field F of characteristic $p > 0$. In Section 5, we derive the decomposition of G given in Theorem D.

2. Notation

Let G be a finite group and p a prime number. As customary, let G', $Z(G)$ and $\Phi(G)$ denote the derived subgroup, the center and the Frattini subgroup of G, respectively. For elements $a, b \in G$, we define their commutator as $[a, b] = aba^{-1}b^{-1}$. We write $[g]$ for the conjugacy class of $g \in G$ and set $\mathrm{Cl}(G)$ to be the set of conjugacy classes of G. The nilpotency class of a nilpotent group G will be denoted by $c(G)$. Recall that every p-group is nilpotent. For subsets S and T of G, let $C_T(S)$ and $N_T(S)$ denote the centralizer and the normalizer of S in T, respectively. As customary, let $O_p(G)$, $O_{p'}(G)$ and $O_{p',p}(G)$ be the p-core, the p'-core and the p', p-core of G, respectively. By $O^p(G)$ and $O^{p'}(G)$, we denote the p-residual subgroup and the p'-residual subgroup of G, respectively. As customary, let g_p and $g_{p'}$ be the p-part and the p'-part of an element $g \in G$, respectively. The p'-section of g is given by all elements in G whose p'-part is conjugate to $g_{p'}$. We write $G = G_1 * G_2$ if G is the central product of subgroups G_1 and G_2, that is, we have $G = \langle G_1, G_2\rangle$ and $[G_1, G_2] = 1$.

For a field F and a finite-dimensional F-algebra A, we denote by $J(A)$ and $\operatorname{soc}(A)$ its Jacobson radical and (left) socle, the sum of all minimal left ideals of A, respectively. Both $J(A)$ and $\operatorname{soc}(A)$ are ideals in A. In this paper, an ideal I of A is always meant to be a two-sided ideal, and we denote it by $I \trianglelefteq A$. Additionally, we study the Reynolds ideal $R(A) := \operatorname{soc}(A) \cap Z(A)$ of A. Furthermore, let $K(A)$ denote the commutator space of A, that is, the F-subspace of A spanned by all elements of the form $ab - ba$ with $a, b \in A$.

In the following, we consider the group algebra FG of G over F. Recall that FG is a symmetric algebra with symmetrizing linear form

$$\lambda\colon FG \to F, \sum_{g \in G} a_g g \mapsto a_1. \tag{2.1}$$

For subsets S and T of FG, we write $\operatorname{lAnn}_T(S)$ and $\operatorname{rAnn}_T(S)$ for the left and the right annihilator of S in T, respectively, and $\operatorname{Ann}_T(S)$ if both subspaces coincide. For $H \subseteq G$, we set $H^+ := \sum_{h \in H} h \in FG$. It is well-known that the elements C^+ with $C \in \operatorname{Cl}(G)$ form an F-basis of the center ZFG of FG.

In this paper, we mainly study the Jacobson radical $J(ZFG)$ and the socle $\operatorname{soc}(ZFG)$ of the center of FG as well as the Reynolds ideal $R(FG)$. All three spaces are ideals in ZFG, but not necessarily in FG. Note that $J(ZFG) = J(FG) \cap ZFG$ holds (see [10, Theorem 1.10.8]) and that by [10, Theorem 1.10.22], we have $\operatorname{soc}(ZFG) = \operatorname{Ann}_{ZFG}(J(ZFG))$. Furthermore, observe that $J(ZFG)$, $\operatorname{soc}(ZFG)$ and $R(FG)$ are ideals in FG if and only if they are closed under multiplication with elements of FG since they are additively closed.

We recall the definition of the augmentation ideal

$$\omega(FG) = \left\{ \sum_{g \in G} a_g g \in FG\colon \sum_{g \in G} a_g = 0 \right\}.$$

An F-basis of $\omega(FG)$ is given by $\{1 - g\colon 1 \neq g \in G\}$. If F is a field of characteristic $p > 0$ and G is a p-group, then $J(FG)$ and $\omega(FG)$ coincide (see [10, Theorem 1.11.1]). For a normal subgroup N of G, we consider the canonical projection

$$\nu_N\colon FG \to F[G/N], \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g \cdot gN.$$

Its kernel is given by $\omega(FN) \cdot FG = FG \cdot \omega(FN)$ (see [10, Proposition 1.6.4]).

3. General properties

Let F be a field. In this part, we answer the question for which finite groups G the Reynolds ideal $R(FG)$ is an ideal in FG. Moreover, we derive structural results on finite groups G for which $\operatorname{soc}(ZFG)$ is an ideal in FG. In the next section, these will be applied in order to classify the finite groups of prime power order with this property.

Concerning the choice of the underlying field F, we note the following:

Remark 3.1.

(i) Assume that F is of characteristic zero or of positive characteristic not dividing $|G|$. By Maschke's theorem, the group algebra FG is semisimple. In particular, $J(FG) = J(ZFG) = 0$ follows, which yields $R(FG) = \operatorname{soc}(ZFG) = ZFG$. Since FG is unitary, $\operatorname{soc}(ZFG)$ is an ideal of FG if and only if $ZFG = FG$ holds, that is, if and only if G is abelian.

(ii) Let F be a field of characteristic $p > 0$ and let G be a finite group. Then $\mathrm{soc}(Z\mathbb{F}_pG)$ is an ideal in $\mathbb{F}_pG$ if and only if $\mathrm{soc}(ZFG)$ is an ideal in FG. A similar statement holds for the Reynolds ideal.

From now on until the end of this paper, we therefore assume that F is an algebraically closed field of characteristic $p > 0$.

This section is organized as follows: We first derive a criterion for $\mathrm{soc}(ZFG) \trianglelefteq FG$ (see Section 3.1) and answer the question when the Reynolds ideal of FG is an ideal in FG (see Section 3.2). In Section 3.3, we investigate p-blocks of FG. Subsequently, we find a basis for $J(ZFG)$ (see Section 3.4) and construct elements in $\mathrm{soc}(ZFG)$ arising from normal p-subgroups of G (see Section 3.5). In Section 3.6, we study the case that G' is contained in the center of a Sylow p-subgroup of G. We conclude this part by investigating the transition to quotient groups in Section 3.7 and studying central products in Section 3.8.

3.1. **Criterion for** $\mathrm{soc}(ZFG) \trianglelefteq FG$**.** Let G be a finite group. In this section, we derive an equivalent criterion for $\mathrm{soc}(ZFG) \trianglelefteq FG$.

Lemma 3.2. *We have $FG \cdot K(FG) = FG \cdot \omega(FG')$.*

Proof. As $FG/\omega(FG') \cdot FG$ is isomorphic to the commutative algebra $F[G/G']$, we have $K(FG) \subseteq \omega(FG') \cdot FG$ and hence $K(FG) \cdot FG \subseteq \omega(FG') \cdot FG$ follows. Now let $f\colon FG \to FG/K(FG) \cdot FG$ be the canonical projection map. For all $a, b \in G$, we have $f([a,b]) = f(a)f(b)f(a)^{-1}f(b)^{-1} = 1$ since $FG/K(FG) \cdot FG$ is a commutative algebra. For $g \in G'$, this yields $f(g) = 1$ and hence $f(g-1) = 0$. This shows $\omega(FG') \subseteq \mathrm{Ker}(f) = K(FG) \cdot FG$, which proves the claim. □

Lemma 3.3. *The socle $\mathrm{soc}(ZFG)$ is an ideal in FG if and only if $\mathrm{soc}(ZFG) \subseteq (G')^+ \cdot FG$ holds.*

Proof. By [9, Lemma 2.1], we have $\mathrm{soc}(ZFG) \trianglelefteq FG$ if and only if $K(FG) \cdot \mathrm{soc}(ZFG) = 0$ holds, which is equivalent to $FG \cdot K(FG) \cdot \mathrm{soc}(ZFG) = 0$. By Lemma 3.2, this is equivalent to $FG \cdot \omega(FG') \cdot \mathrm{soc}(ZFG) = 0$, that is, to $\mathrm{soc}(ZFG) \subseteq \mathrm{rAnn}_{FG}(\omega(FG')) = (G')^+ \cdot FG$ (see [11, Lemma 3.1.2]). □

3.2. **Reynolds ideal.** Let G be a finite group. In this section, we answer the question when the Reynolds ideal $R(FG)$ is an ideal in FG. Our main result is the following:

Theorem 3.4. *The following properties are equivalent:*

(i) $R(FG)$ is an ideal of FG.
(ii) $G' \subseteq O_p(G)$.
(iii) $G = P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H.

In this case, we have $R(FG) = O_p(G)^+ \cdot FG$.

Proof. Suppose that $R(FG)$ is an ideal in FG. Then FG is a basic F-algebra by [3, Lemma 2.2]. Since F is algebraically closed, this implies that $FG/J(FG)$ is commutative. By Lemma 3.2, we have $\omega(FG') \cdot FG = K(FG) \cdot FG \subseteq J(FG)$. Thus, for $g \in G'$, the element $g - 1$ is nilpotent. Hence there exists $n \in \mathbb{N}$ with $0 = (g-1)^{p^n} = g^{p^n} - 1$. This shows that G' is a p-group and hence contained in $O_p(G)$.

Now assume $G' \subseteq O_p(G)$ and let $P \in \mathrm{Syl}_p(G)$. Then $G' \subseteq P$ follows, so P is a normal subgroup of G and G/P is abelian. By the Schur-Zassenhaus theorem, P has a complement H in G. Moreover, H is isomorphic to G/P and thus abelian.

Finally suppose that $G = P \rtimes H$ holds, where $P \in \mathrm{Syl}_p(G)$ and H is an abelian p'-group. In particular, we have $P = O_p(G)$. We obtain $J(FG) = \omega(FP) \cdot FG$ and $\mathrm{soc}(FG) = \mathrm{Ann}_{FG}(J(FG)) = P^+ \cdot FG \subseteq (G')^+ \cdot FG \subseteq ZFG$, so that $R(FG) = P^+ \cdot FG$ is an ideal in FG. □

This proves Theorem A. Moreover, we obtain the following necessary condition for $\mathrm{soc}(ZFG) \trianglelefteq FG$:

Corollary 3.5. *If* $\mathrm{soc}(ZFG)$ *is an ideal of* FG*, we have* $G = P \rtimes H$ *with* $P \in \mathrm{Syl}_p(G)$ *and an abelian* p'*-group* H*.*

Proof. By [3, Lemma 1.3], $\mathrm{soc}(ZFG) \trianglelefteq FG$ implies $R(FG) \trianglelefteq FG$. With this, the claim follows from Theorem 3.4. □

Remark 3.6. Let $G = P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H.

(i) By [5, Theorem 5.3.5], we have $P = C_P(H)[P, H]$. Due to $[P, H] \subseteq G'$, this yields $G = HP = HC_P(H)[P, H] = HC_P(H)G'$. Note that $[G, H] = [P, H] = [[P, H], H] = [G', H] = [[G', H], H]$ holds by [5, Theorem 5.3.6] and that this is a normal subgroup of $PH = G$.

(ii) We have $O^p(G) = N$ for $N := H[G, H]$: Clearly, N is a normal subgroup of G. Since G/N is a p-group, we have $O^p(G) \subseteq N$. On the other hand, $G/O^p(G)$ is a p-group, which implies $H \subseteq O^p(G)$ and hence $N \subseteq O^p(G)$ as $O^p(G)$ is a normal subgroup of G. In particular, this implies $O^p(G)' \subseteq [G, H]$. On the other hand, we have $[G, H] = [[G', H], H] \subseteq [O^p(G), O^p(G)] = O^p(G)'$ by (i) and hence $O^p(G)' = [G, H] \in \mathrm{Syl}_p(O^p(G))$ follows.

(iii) Since $O_{p'}(G)$ is contained in the abelian group H and $[P, O_{p'}(G)] \subseteq P \cap O_{p'}(G) = 1$ holds, we have $O_{p'}(G) \subseteq Z(G)$. Hence [5, Theorem 6.3.3] implies $C_G(P) \subseteq O_{p'p}(G) = O_{p'}(G) \times P$, and we conclude that $C_G(P) = O_{p'}(G) \times Z(P)$ holds.

(iv) Since $R(FG)$ is spanned by the p'-section sums of G (see [8, Equation (39)]), every p'-section is of the form hP for some $h \in H$.

3.3. **Blocks and the** p'**-core.** Let G be an arbitrary finite group. In this section, we investigate the conditions $\mathrm{soc}(Z(B)) \trianglelefteq B$ and $R(B) \trianglelefteq B$ for a p-block B of FG.

Remark 3.7. Let $FG = B_1 \oplus \ldots \oplus B_n$ be the decomposition of FG into its p-blocks. Then we have

$$\mathrm{soc}(ZFG) = \mathrm{soc}(Z(B_1)) \oplus \ldots \oplus \mathrm{soc}(Z(B_n)).$$

In particular, $\mathrm{soc}(ZFG)$ is an ideal in FG if and only if $\mathrm{soc}(Z(B_i)) \trianglelefteq B_i$ holds for all $i \in \{1, \ldots, n\}$, and the analogous statement is true for the Reynolds ideal. Furthermore, it is known that the principal blocks of FG and $F\bar{G}$ are isomorphic for $\bar{G} := G/O_{p'}(G)$.

For the Reynolds ideal, we obtain the following result:

Lemma 3.8. *The following are equivalent:*

(i) There exists a block B *of* FG *for which* $R(B) \trianglelefteq B$ *holds.*

(ii) For the principal block B_0 *of* FG*, we have* $R(B_0) \trianglelefteq B_0$*.*

(iii) G' *is contained in* $O_{p'p}(G)$*.*

Proof. Assume that (i) holds. By [9, Proposition 4.1], this implies $B \cong B_0$ and hence (ii) holds. Now assume that (ii) holds. By [9, Remarks 2.2 and 3.1], every simple B_0-module is one-dimensional. Since the intersection of the kernels of the simple B_0-modules is given by $O_{p'p}(G)$ (see [1, Theorem 2]), we obtain $G' \subseteq O_{p'p}(G)$. Finally, assume that (iii) holds.

Then we have $\bar{G}' \subseteq O_p(\bar{G})$. Theorem 3.4 yields $R(F\bar{G}) \trianglelefteq F\bar{G}$, which implies $R(\bar{B}_0) \trianglelefteq \bar{B}_0$ by Remark 3.7. Since B_0 and $\bar{B}_0$ are isomorphic, we obtain $R(B_0) \trianglelefteq B_0$. □

Concerning the analogous problem for the socle of the center, we first observe the following:

Lemma 3.9. *The following are equivalent:*

(i) There exists a block B of FG for which $\operatorname{soc}(Z(B)) \trianglelefteq B$ holds.
(ii) For the principal block B_0 of FG, we have $\operatorname{soc}(Z(B_0)) \trianglelefteq B_0$.
(iii) For the principal block $\bar{B}_0$ of $F\bar{G}$, we have $\operatorname{soc}(Z(\bar{B}_0)) \trianglelefteq \bar{B}_0$.

Proof. As in the proof of Lemma 3.8, the equivalence of (i) and (ii) follows by [9, Proposition 4.1] and the equivalence of (ii) and (iii) follows from the fact that B_0 and $\bar{B}_0$ are isomorphic. □

This has the following important consequence:

Lemma 3.10. *We have $\operatorname{soc}(ZFG) \trianglelefteq FG$ if and only if $R(FG) \trianglelefteq FG$ and $\operatorname{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ hold.*

Proof. If $\operatorname{soc}(ZFG)$ is an ideal of FG, then $R(FG) \trianglelefteq FG$ holds by [3, Lemma 1.3] and $\operatorname{soc}(ZF\bar{G})$ is an ideal of $F\bar{G}$ by [3, Proposition 2.10]. For the latter, note that $F\bar{G} \cong FG/\operatorname{Ker}(\nu_{O_{p'}(G)})$ can be viewed as a quotient algebra of FG. Now let $R(FG)$ and $\operatorname{soc}(ZF\bar{G})$ be ideals in FG and $F\bar{G}$, respectively. By Remark 3.7, this yields $\operatorname{soc}(Z(\bar{B}_0)) \trianglelefteq \bar{B}_0$ and hence $\operatorname{soc}(Z(B_0)) \trianglelefteq B_0$ (see Lemma 3.9). Since $R(FG)$ is an ideal in FG, all blocks of FG are isomorphic to B_0 by [9, Proposition 4.1]. By Remark 3.7, we then obtain $\operatorname{soc}(ZFG) \trianglelefteq FG$. □

Remark 3.11. Assume that G is of the form $G = P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H. Then $\operatorname{soc}(ZFG) \trianglelefteq FG$ is equivalent to $\operatorname{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ (see Theorem 3.4 and Lemma 3.10). By going over to the quotient group $G/O_{p'}(G)$, we may therefore restrict our investigation to groups G with $O_{p'}(G) = 1$.

3.4. **Basis for** $J(ZFG)$**.** Let $G = P \rtimes H$ be a finite group with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H (see Theorem 3.4). The aim of this section is to determine an F-basis for $J(ZFG)$. In the given situation, the kernel of the canonical map $\nu_P \colon FG \to F[G/P]$ is given by $J(FG)$ (see [10, Corollary 1.11.11]). In the following, we distinguish two types of conjugacy classes:

Remark 3.12. Let $C \in \operatorname{Cl}(G)$. We obtain $|\bar{C}| = 1$ for the image $\bar{C} \in \operatorname{Cl}(G/P)$ of C in G/P since this group is abelian. Now two cases can occur:

- $|C|$ is divisible by p: Then $\nu_P(C^+) = |C| \cdot \bar{C}^+ = 0$ yields $C^+ \in \operatorname{Ker}(\nu_P) \cap ZFG = J(ZFG)$.
- $|C|$ is not divisible by p: In this case, $|P|$ divides $|C_G(g)|$ for any $g \in C$. This yields $P \subseteq C_G(g)$ and hence $C \subseteq C_G(P)$. As customary, we decompose $g = g_{p'} g_p$ into its p'-part and p-part. Note that $g_{p'} \in O_{p'}(G) \subseteq Z(G)$ and $g_p \in Z(P)$ hold by Remark 3.6. Due to $g_{p'} \in Z(G)$, we have $C = g_{p'}[g_p]$ and the element $C^+ - |C| \cdot g_{p'}$ is contained in $\operatorname{Ker}(\nu_P) \cap ZFG = J(ZFG)$.

Definition 3.13. For $C \in \operatorname{Cl}(G)$ with $C \not\subseteq O_{p'}(G)$, we set $b_C \coloneqq C^+$ if p divides $|C|$, and $b_C \coloneqq C^+ - |C| \cdot g_{p'}$ otherwise.

With this, we obtain the following basis for $J(ZFG)$:

Theorem 3.14. *An F-basis for $J(ZFG)$ is given by $B := \{b_C : C \in \mathrm{Cl}(G),\ C \nsubseteq O_{p'}(G)\}$.*

Proof. By Remark 3.12, we have $B \subseteq J(ZFG)$. Note that the elements in $B \cup O_{p'}(G)$ form an F-basis for ZFG. Since the algebra $FO_{p'}(G)$ is semisimple, $J(ZFG)$ is spanned by B. □

Remark 3.15. The decomposition $FG = \bigoplus_{h \in H} FhP$ gives rise to an H-grading of FG. Note that the basis of $J(ZFG)$ given in Theorem 3.14 consists of homogeneous elements with respect to this grading. In particular, $J(ZFG)$ is a H-graded subspace of FG. It follows that $\mathrm{soc}(ZFG) = \mathrm{Ann}_{ZFG}(J(ZFG))$ is a H-graded subspace of FG as well, that is, we have

$$\mathrm{soc}(ZFG) = \bigoplus_{h \in H} \left(\mathrm{soc}(ZFG) \cap FhP\right).$$

3.5. **Elements in** $\mathrm{soc}(ZFG)$**.** Let G be an arbitrary finite group. In this section, we study elements of $\mathrm{soc}(ZFG)$ which arise from certain normal p-subgroups of G. Using these, we show that G' has nilpotency class at most two if $\mathrm{soc}(ZFG)$ is an ideal in FG. Moreover, we derive a decomposition of G which will later be used to prove Theorem D.

Lemma 3.16. *Let N be a normal p-subgroup of G and set $M := \{x \in [N,G] : x^p \in [N,[N,G]]\}$. For $C \in \mathrm{Cl}(G)$ with $C \nsubseteq C_G(N)$, we have $\nu_M(C^+) = 0$ and hence $M^+ \cdot C^+ = 0$. In particular, this implies $\nu_N(C^+) = 0$ and $N^+ \cdot C^+ = 0$.*

Proof. Note that M is a normal subgroup of G. Let R be an orbit of the conjugation action of N on C and consider an element $r \in R$. Then $C \nsubseteq C_G(N)$ implies $N \nsubseteq C_G(r)$, which yields $|R| = |N : C_N(r)| \neq 1$. Set $X := \langle N, R\rangle = \langle N, r\rangle$.

First consider the case $[N,G] \subseteq Z(N)$. Then the map $f : N \to N,\ n \mapsto [n,r]$ is a group endomorphism with kernel $C_N(r)$. We set $S := \mathrm{Im}(f)$. Then we have $|R| = |N : C_N(r)| = |S|$, so in particular, $|S|$ is a nontrivial power of p. Let $\bar{G} := G/M$ and set $\bar{g} := gM \in \bar{G}$ for $g \in G$ (similarly for subsets of G). Note that $\bar{R}$ is an orbit of the conjugation action of $\bar{N}$ on $\bar{C}$. As before, we obtain $|\bar{R}| = |\bar{N} : C_{\bar{N}}(\bar{r})| = |\bar{S}| = |S : S \cap M|$. Since $S \subseteq [N,G]$ is a nontrivial p-group, $|S \cap M|$ is divisible by p. With this, we obtain

$$\nu_M(R^+) = \frac{|R|}{|\bar{R}|} \cdot \bar{R}^+ = |S \cap M| \cdot \bar{R}^+ = 0.$$

Now we consider the general case. Let $L := [N,[N,G]]$. We set $\widetilde{G} := G/L$ and write $\widetilde{g} := gL \in \widetilde{G}$ for $g \in G$ (similarly for subsets of G). Note that we have $[\widetilde{N},[\widetilde{N},\widetilde{G}]] = 1$ and hence $[\widetilde{N},\widetilde{G}] \subseteq Z(\widetilde{N})$. First assume $C_{\widetilde{N}}(\widetilde{r}) = \widetilde{N}$. For any $n \in N$, one then has $[n,r] \in L$, which implies $\nu_L(R^+) = |R| \cdot \widetilde{r} = 0$. Due to $L \subseteq M$, this yields $\nu_M(R^+) = 0$. Now assume $C_{\widetilde{N}}(\widetilde{r}) \subsetneq \widetilde{N}$. In particular, we have $\widetilde{C} \nsubseteq C_{\widetilde{G}}(\widetilde{N})$. The first part of the proof yields $\nu_{\widetilde{M}}(\widetilde{R}^+) = 0$, which implies

$$\nu_{\widetilde{M}}\left(\nu_L(R^+)\right) = \nu_{\widetilde{M}}\left(\frac{|R|}{\left|\widetilde{R}\right|} \cdot \widetilde{R}^+\right) = \frac{|R|}{\left|\widetilde{R}\right|} \cdot \nu_{\widetilde{M}}\left(\widetilde{R}^+\right) = 0.$$

Due to $\widetilde{G}/\widetilde{M} = (G/L)/(M/L) \cong G/M$, the map $\nu_{\widetilde{M}} \circ \nu_L$ can be identified with ν_M and hence $\nu_M(R^+) = 0$ follows. Since R was arbitrary, this yields $\nu_M(C^+) = 0$. In particular, we have $M^+ \cdot C^+ = 0$. □

Proposition 3.17. *Let N be a normal p-subgroup of G and set $M := \{x \in [N,G] : x^p \in [N,[N,G]]\}$ as in Lemma 3.16. Moreover, let K be a characteristic subgroup of $C_G(N)$ which satisfies $K^+ \in \operatorname{soc}(ZFC_G(N))$. Then we have $(MK)^+ \in \operatorname{soc}(ZFG)$. In particular, this applies to $K := O^{p'}(C_G(N))$.*

Proof. By Lemma 3.16, ZFG is the sum of the subspaces $ZFG \cap FC_G(N)$ and $ZFG \cap \operatorname{Ker}(\nu_M)$. Since $\operatorname{Ker}(\nu_M) = \omega(FM)FG = J(FM)FG \subseteq J(FG)$ holds (see [10, Proposition 1.6.4]), we have $ZFG \cap \operatorname{Ker}(\nu_M) \subseteq J(ZFG)$. Since $ZFG \cap FC_G(N)$ is contained in $ZFC_G(N)$, the space $J(ZFG \cap FC_G(N)) \subseteq J(ZFC_G(N))$ is annihilated by K^+. This proves that $(MK)^+$ annihilates $J(ZFG)$. Now let $K := O^{p'}(C_G(N))$. Since K^+ annihilates $J(FC_G(N)) = J(FK)FC_G(N)$ (see [10, Theorem 1.11.10]), we have $K^+ \in \operatorname{soc}(ZFC_G(N))$ as required. □

Now we return to the assumption that G is of the form $P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H as in Theorem 3.4.

Lemma 3.18. *Suppose that N is a normal p-subgroup of G. Then $(C_P(N)M)^+ \in \operatorname{soc}(ZFG)$ follows, where $M := \{x \in [N,G] : x^p \in [N,[N,G]]\}$ is defined as in Lemma 3.16. In particular, we have $(C_P(N)N)^+ \in \operatorname{soc}(ZFG)$. If $\operatorname{soc}(ZFG)$ is an ideal in FG, then $G' \subseteq C_P(N)M$ follows.*

Proof. Since $C_P(N)$ is a normal Sylow p-subgroup of $C_G(N)$, we have $O^{p'}(C_G(N)) = C_P(N)$. Proposition 3.17 then yields $(C_P(N)M)^+ \in \operatorname{soc}(ZFG)$. Since $C_P(N)N$ is a union of cosets of $C_P(N)M$, we obtain $(C_P(N)N)^+ \in \operatorname{soc}(ZFG)$. If $\operatorname{soc}(ZFG)$ is an ideal in FG, then $G' \subseteq C_P(N)M$ follows by Lemma 3.3. □

The following result will be particularly useful for our derivation on p-groups:

Corollary 3.19. *We have $(Z(P)G')^+ \cdot FG \subseteq \operatorname{soc}(ZFG) \subseteq O_p(Z(G))^+ \cdot FG$.*

Proof. By Lemma 3.18, we obtain $(Z(P)M)^+ \in \operatorname{soc}(ZFG)$ for $M = \{x \in [P,G] : x^p \in [P,[P,G]]\} \subseteq G'$. In particular, this implies $(Z(P)G')^+ \in \operatorname{soc}(ZFG)$. Since we have $(Z(P)G')^+ \cdot FG \subseteq (G')^+ \cdot FG \subseteq ZFG$, this implies $(Z(P)G')^+ \cdot FG \subseteq \operatorname{soc}(ZFG)$. Now for $z \in O_p(Z(G))$, the element $z-1$ is nilpotent and hence contained in $J(ZFG)$. For $x = \sum_{g \in G} a_g g \in \operatorname{soc}(ZFG)$, this yields $x \cdot (z-1) = 0$, which translates to $a_g = a_{gz}$ for all $g \in G$. Hence $x \in O_p(Z(G))^+ \cdot FG$ follows. □

Observe that the right inclusion in the preceding lemma holds for arbitrary finite groups. The next result is the central ingredient in the proof of Theorem D:

Proposition 3.20. *Suppose that $G' \subseteq C_P(N)N$ holds for every normal p-subgroup N of G. Then the following hold:*

- *(i) We have $[P,G'] \subseteq Z(G')$. In particular, this implies $G'' \subseteq Z(P)$ and that the nilpotency class of G' is at most two. Moreover, we obtain $\Phi(G') \subseteq Z(G')$.*
- *(ii) We have $P = C_P(H) * [P,H]$ and $G = C_P(H) * O^p(G)$.*

Proof.

(i) Let D be a critical subgroup of P (in the sense of [5, Theorem 5.3.11]). Then D is normal in G, and $Z(D)$ contains $\Phi(D)$, $C_P(D)$ and $[P,D]$. By assumption, we have $G' \subseteq DC_P(D) = D$. Hence we have

$$[P,G'] \subseteq [P,D] \subseteq Z(D) \subseteq C_G(G'),$$

which implies $[P, G'] \subseteq Z(G')$. With the 3-subgroups lemma, we obtain $[G'', P] = [[G', G'], P] = 1$, that is, $G'' \subseteq Z(P)$. Furthermore, for $x \in G'$, we have $x \in D$ and hence $x^p \in Z(D) \subseteq C_G(G')$, which implies $x^p \in Z(G')$.

(ii) By (i), we have $B := [C_P(H), [P, H]] \subseteq [P, G'] \subseteq Z(G')$. Furthermore, B is normal in $C_P(H)[P, H] = P$ and $PH = G$. Due to

$$[C_P(H), G] = [C_P(H), C_P(H)[P, H]H] = [C_P(H), C_P(H)[P, H]] \subseteq C_P(H)B,$$

the subgroup $N := C_P(H)B$ is normal in G. Moreover, we find $[N, H] = [C_P(H)B, H] = [B, H]$. By assumption, we have $G' \subseteq C_P(N)N$. By Remark 3.6, this yields

$$[P, H] = [G', H] \subseteq [C_P(N)N, H] \subseteq [N, H]\,[C_P(N), H]\,,$$

since for $c \in C_P(N)$, $n \in N$ and $h \in H$, we have $[cn, h] = c[n, h]c^{-1}[c, h] = [n, h][c, h]$. Hence $[P, H] \subseteq [B, H][C_P(N), H] \subseteq BC_P(N)$ follows, which yields

$$B = [C_P(H), [P, H]] \subseteq [C_P(H), BC_P(N)] = [C_P(H), B] \subseteq [P, B].$$

Hence $B = 1$ follows, which yields $P = C_P(H) * [P, H]$. By Remark 3.6, this implies $G = C_P(H) * H[P, H] = C_P(H) * O^p(G)$. ☐

By Lemma 3.18, the properties given in Proposition 3.20 hold whenever $\operatorname{soc}(ZFG)$ is an ideal in FG. We conclude this section with a result on p-groups, which is an immediate consequence of Lemma 3.18:

Lemma 3.21. *If G is a p-group satisfying $\operatorname{soc}(ZFG) \trianglelefteq FG$, then G is metabelian.*

Proof. Let A be a maximal abelian normal subgroup of G. Since $C_G(A) = A$ holds, Lemma 3.18 yields $G' \subseteq A$. In particular, G' is abelian. ☐

3.6. **Special case $G' \subseteq Z(P)$.** Let $G = P \rtimes H$ be a finite group with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H. In this section, we show that $\operatorname{soc}(ZFG)$ is an ideal in FG if $G' \subseteq Z(P)$ holds.

Lemma 3.22.

(i) Let $g \in G$ with $g_p \in Z(P)$. Then $[g] = [h] \cdot [g_p]$ holds for $h \in H \cap gP$.

(ii) For $u \in Z(P)$ and $h \in C_G(H)$, we have $h[u] \subseteq [hu]$.

(iii) Assume $[P, G] \subseteq Z(P)$. Let $h \in C_G(H)$ and write $[h] = U_h h$ with $U_h := \{[a, h] \colon a \in G\}$. Then U_h is a normal subgroup of G.

Proof.

(i) By Remark 3.6, gP is a p'-section of G. In particular, $[h]$ is the unique p'-conjugacy class contained in gP and hence $[g_{p'}] = [h]$ follows. Since H is abelian, we have $g_{p'} = uhu^{-1}$ for some $u \in P$. Due to $g_p \in Z(P)$, this yields $g = uhg_pu^{-1}$ and hence $[g] = [hg_p]$. We may therefore assume $g_{p'} = h$. For $x = p_xh_x$ with $p_x \in P$ and $h_x \in H$, we have $xgx^{-1} = p_xhp_x^{-1} \cdot h_xg_ph_x^{-1}$. This yields

$$[g] = \Big\{p_xhp_x^{-1} \colon p_x \in P\Big\} \cdot \Big\{h_xg_ph_x^{-1} \colon h_x \in H\Big\} = [h] \cdot [g_p].$$

(ii) Let $u' \in [u]$. Due to $u \in Z(P)$, there exists an element $h' \in H$ with $h'uh'^{-1} = u'$ (see Remark 3.6). Since h and h' commute, we obtain $hu' = h'huh'^{-1} \in [hu]$.

(iii) We have $U_h = \{[a, h] \colon a \in P\}$. As $[p_1p_2, h] = [p_1, h] \cdot [p_2, h]$ holds for all $p_1, p_2 \in P$, U_h is a subgroup of G'. Since the elements of P centralize $U_h \subseteq [P, G] \subseteq Z(P)$ and conjugation with elements of H permutes the elements $[a, h]$ with $a \in P$, it follows that U_h is normal in G. ☐

Corollary 3.23. *Let $g \in G$ with $g_p \in Z(P)$. For $y \in ZFG$ with $y \cdot [g_{p'}]^+ = 0$, we have $y \cdot [g]^+ = 0$.*

Proof. The group P acts on $[g]$ by conjugation with orbits of the form $[g_{p'}]u$ with $u \in P$ (see Lemma 3.22). In particular, $[g]$ is a disjoint union of sets of this form. Hence $y \cdot [g_{p'}]^+$ implies $y \cdot [g]^+ = 0$. □

Lemma 3.24. *Let $y = \sum_{g \in G} a_g g \in \operatorname{soc}(ZFG)$. For $h \in C_G(H)$ and $u \in Z(P)$, we have $a_{hu} = a_h$.*

Proof. We may assume $u \neq 1$. By Remark 3.6, $m := |[u]|$ is not divisible by p. Hence we have $b_{[u^{-1}]} = [u^{-1}]^+ - m \cdot 1$ (see Theorem 3.14) and the coefficient of h in $y \cdot b_{[u^{-1}]} = 0$ is given by

$$\sum_{u' \in [u]} a_{hu'} - m a_h = m\left(a_{hu} - a_h\right),$$

since the elements in $h[u]$ are conjugate by Lemma 3.22 (ii). Since p does not divide m, we obtain $a_{hu} = a_h$. □

Theorem 3.25. *If $G = C_G(H)Z(P)$ holds, then $\operatorname{soc}(ZFG) \subseteq Z(P)^+ \cdot FG$ follows. In particular, if we have $G' \subseteq Z(P)$, then $\operatorname{soc}(ZFG)$ is an ideal in FG.*

Proof. Consider an element $y = \sum_{g \in G} a_g g \in \operatorname{soc}(ZFG)$. Let $g \in G$ and write $g = cz$ with $c \in C_G(H)$ and $z \in Z(P)$. By Lemma 3.24, we have $a_g = a_{cz} = a_c$. Hence $y \in Z(P)^+ \cdot FG$ follows. If additionally $G' \subseteq Z(P)$ holds, then $\operatorname{soc}(ZFG) \subseteq Z(P)^+ \cdot FG \subseteq (G')^+ \cdot FG$ follows, so $\operatorname{soc}(ZFG)$ is an ideal in FG (see Lemma 3.3). □

This proves the first part of Theorem C. The next example shows that the condition $G' \subseteq Z(P)$ is not necessary for $\operatorname{soc}(ZFG) \trianglelefteq FG$.

Example 3.26. Let F be an algebraically closed field of characteristic $p = 3$ and consider the group $G = \texttt{SmallGroup}(216, 86)$ in GAP [12]. We have $G = G' \rtimes H$, where G' is the extraspecial group of order 27 and exponent three, and $H \cong C_8$ permutes the nontrivial elements of G'/G'' transitively and acts on $G'' = Z(G')$ by inversion. In particular, G' is nonabelian. For $h \in H$, we set $S_h := \operatorname{soc}(ZFG) \cap FhG'$. Due to the H-grading of FG introduced in Remark 3.15, it suffices to show $S_h = F(hG')^+$ for all $h \in H$. Clearly, we have $(hG')^+ \in S_h$. The derived subgroup G' decomposes into the G-conjugacy classes $\{1\}$, $G'' \backslash \{1\}$ and $G' \backslash G''$. For $1 \neq h \in H$, the coset hG' consists of a single conjugacy class for $\operatorname{ord}(h) = 8$ and of two conjugacy classes for $\operatorname{ord}(h) \in \{2, 4\}$. In the first case, we directly obtain $S_h = F(hG')^+$. In the latter case, we have $[h]^+ \cdot (G'')^+ = (hG')^+ \neq 0$, which implies $[h]^+ \notin \operatorname{soc}(ZFG)$ since $(G'')^+ \in J(ZFG)$ holds. Since $(hG')^+ - [h]^+ \notin \operatorname{soc}(ZFG)$ holds as well, $S_h = F(hG')^+$ follows. Moreover, this shows $(G'')^+ \notin \operatorname{soc}(ZFG)$ and hence $S_1 = F(G')^+$ follows as well. By Lemma 3.3, $\operatorname{soc}(ZFG)$ is an ideal of FG.

3.7. **Quotient groups.** Let G be a finite group of the form $P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H. We fix a normal subgroup $N \trianglelefteq G$ with quotient group $\bar{G} := G/N$. Our aim is to study the transition to the group algebra $F\bar{G}$. The image of an element $g \in G$ in $\bar{G}$ will be denoted by $\bar{g}$ (similarly for subsets of G). Note that $\bar{G}$ is of the form $\bar{P} \rtimes \bar{H}$ with $\bar{P} \in \operatorname{Syl}_p(\bar{G})$ and the abelian p'-group $\bar{H}$. In the following, we consider the canonical projection map

$$\nu_N \colon FG \to F\bar{G}, \ \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g \cdot gN,$$

together with its adjoint map $\nu_N^*\colon F\bar{G} \to FG$, which is defined by requiring $\lambda(\nu_N^*(x)y) = \bar{\lambda}(x\nu_N(y))$ for all $x \in F\bar{G}$ and $y \in FG$. Here, λ and $\bar{\lambda}$ denote the symmetrizing linear forms of FG and $F\bar{G}$ given in (2.1), respectively. It is easily verified that ν_N^* is given by

$$\nu_N^*\colon F\bar{G} \to FG, \quad \sum_{gN \in \bar{G}} a_{gN} \cdot gN \mapsto \sum_{g \in G} a_{gN} \cdot g.$$

Note that ν_N^* is a linear map with image $N^+ \cdot FG$ and that it is injective as ν_N is surjective.

Remark 3.27. For $a \in F\bar{G}$, it is easily seen that $a \in (\bar{G}')^+ \cdot F\bar{G}$ is equivalent to $\nu_N^*(a) \in (G')^+ \cdot FG$.

If $\mathrm{soc}(ZFG)$ is an ideal in FG, then $\mathrm{Ann}_{ZF\bar{G}}(\nu_N(J(ZFG)))$ is an ideal in $F\bar{G}$ by [3, Proposition 2.10]. For $C \in \mathrm{Cl}(G)$ with $C \not\subseteq O_{p'}(G)$, let b_C denote the associated element of $J(ZFG)$ (see Definition 3.13) and consider the basis $B \coloneqq \{b_C \colon C \in \mathrm{Cl}(G),\, C \not\subseteq O_{p'}(G)\}$ of $J(ZFG)$ (see Theorem 3.14). Clearly, $\nu_N(J(ZFG))$ is spanned by the images of the elements in B. We now derive a more convenient generating set.

Lemma 3.28. *Let $C \in \mathrm{Cl}(G)$ be a conjugacy class with $C \not\subseteq O_{p'}(G)$. We have $b_C \notin \mathrm{Ker}(\nu_N)$ if and only if $\bar{C} \not\subseteq O_{p'}(\bar{G})$ holds and $k \coloneqq |C|/|\bar{C}|$ is not divisible by p. In this case, the basis element $b_{\bar{C}}$ of $J(ZF\bar{G})$ corresponding to $\bar{C} \in \mathrm{Cl}(\bar{G})$ is well-defined and we have $\nu_N(b_C) = k \cdot b_{\bar{C}}$.*

Proof. Observe that $\bar{C}$ is indeed a conjugacy class of $\bar{G}$ and that $\nu_N(C^+) = k \cdot \bar{C}^+$ with $k \coloneqq |C|/|\bar{C}|$ holds. Suppose first that p divides $|C|$, so $b_C = C^+$ holds. Then $\nu_N(b_C) \neq 0$ is equivalent to $k \not\equiv 0 \pmod p$, and in this case we have $|\bar{C}| \equiv 0 \pmod p$. Since $O_{p'}(\bar{G}) \subseteq Z(\bar{G})$ holds, this implies $\bar{C} \not\subseteq O_{p'}(\bar{G})$. Moreover, we have $b_{\bar{C}} = \bar{C}^+$ and thus $\nu_N(b_C) = k \cdot b_{\bar{C}}$.

It remains to consider the case $C \subseteq C_G(P)$. There, we have $\bar{C} \subseteq C_{\bar{G}}(\bar{P})$. If $\bar{C} \not\subseteq O_{p'}(\bar{G})$ holds, then $b_{\bar{C}}$ is defined, and we have $b_C = C^+ - |C| \cdot g_{p'}$ and $b_{\bar{C}} = \bar{C}^+ - |\bar{C}| \cdot \bar{g}_{p'}$ for $g \in C$. This shows that $\nu_N(b_C) = k \cdot b_{\bar{C}}$ holds. If, in addition, $k \not\equiv 0 \pmod p$, then $\nu_N(b_C) \neq 0$ follows. Suppose conversely that $\nu_N(b_C) \neq 0$ holds. We write $C = g_{p'}D$ for $g_{p'} \in O_{p'}(G)$ and $D \in \mathrm{Cl}(G)$ with $D \subseteq Z(P)$ (see Remark 3.12). Assume that $\bar{C} \subseteq O_{p'}(\bar{G})$ holds. Then we have $\bar{D} = \bar{g}_{p'}^{-1}\bar{C} \subseteq O_{p'}(\bar{G})$ due to $\bar{g}_{p'} \in O_{p'}(\bar{G})$. As D consists of p-elements, we must have $\bar{D} = \{1\}$, which yields the contradiction $\nu_N(b_C) = \nu_N(g_{p'}D^+ - |D| \cdot g_{p'}) = 0$. This shows that $\bar{C} \not\subseteq O_{p'}(\bar{G})$ holds. Hence we have $\nu_N(b_C) = k \cdot b_{\bar{C}}$, so that $k \not\equiv 0 \pmod p$. ☐

Definition 3.29. Set $\mathrm{Cl}_{p',N}(G) \coloneqq \{C \in \mathrm{Cl}(G) \colon C \not\subseteq O_{p'}(G) \text{ and } b_C \notin \mathrm{Ker}(\nu_N)\}$ and let

$$\mathrm{Cl}^+_{p',N}(G) \coloneqq \{b_C \colon C \in \mathrm{Cl}_{p',N}(G)\}$$

be the set of corresponding basis elements of $J(ZFG)$ (see Definition 3.13). By $\overline{\mathrm{Cl}}_{p',N}(G) \subseteq \mathrm{Cl}(\bar{G})$, we denote the set of images of the conjugacy classes in $\mathrm{Cl}_{p',N}(G)$ and set

$$\overline{\mathrm{Cl}}^+_{p',N}(G) \coloneqq \left\{b_{\bar{C}} \colon \bar{C} \in \overline{\mathrm{Cl}}_{p',N}(G)\right\},$$

where $b_{\bar{C}}$ denotes the basis element of $J(ZF\bar{G})$ corresponding to $\bar{C}$.

If N is a p-group, the p'-conjugacy classes of length divisible by p in $\mathrm{Cl}_{p',N}(G)$ can be easily characterized:

Lemma 3.30. *Consider a normal p-subgroup N of G and let $C \not\subseteq C_G(P)$ be a p'-conjugacy class. Then we have $C \in \mathrm{Cl}_{p',N}(G)$ if and only if $C \subseteq C_G(N)$ holds.*

Proof. If $C \nsubseteq C_G(N)$ holds, we have $\nu_N(b_C) = \nu_N(C^+) = 0$ by Lemma 3.16, so $C \notin \mathrm{Cl}_{p',N}(G)$. Now let $h \in C \subseteq C_G(N)$. Since h is a p'-element, [5, Theorem 5.3.15] implies $C_{G/N}(hN) = C_G(h)N/N = C_G(h)/N$ and hence $|\bar{C}| = |G/N : C_{G/N}(hN)| = |G : C_G(h)| = |C|$. Thus we have $\nu_N(b_C) = \nu_N(C^+) = \bar{C}^+ \neq 0$, which yields $C \in \mathrm{Cl}_{p',N}(G)$. □

Now let N again be an arbitrary normal subgroup of G. We obtain the following necessary condition for $\mathrm{soc}(ZFG) \trianglelefteq FG$:

Theorem 3.31. *We have*

$$\mathrm{Ann}_{ZF\bar{G}}\left(\nu_N(J(ZFG))\right) = \mathrm{Ann}_{ZF\bar{G}}\left(\overline{\mathrm{Cl}}^+_{p',N}(G)\right) =: A.$$

If $\mathrm{soc}(ZFG)$ *is an ideal of* FG*, we have* $A \subseteq (\bar{G}')^+ \cdot F\bar{G}$*.*

Proof. Clearly, the elements $\nu_N(b_C)$ with $C \in \mathrm{Cl}_{p',N}(G)$ span $\nu_N(J(ZFG))$. For $C \in \mathrm{Cl}_{p',N}(G)$ and $y \in F\bar{G}$, we have $y \cdot \nu_N(b_C) = 0$ if and only if $y \cdot b_{\bar{C}} = 0$ holds (see Lemma 3.28). This implies $A = \mathrm{Ann}_{ZF\bar{G}}(\nu_N(J(ZFG)))$. Now assume that $\mathrm{soc}(ZFG)$ is an ideal in FG. By [3, Proposition 2.10], A is an ideal in $F\bar{G}$, so by [9, Lemma 2.1], we have $K(F\bar{G}) \cdot A = 0$. As in the proof of Lemma 3.3, this implies $A \subseteq (\bar{G}')^+ \cdot F\bar{G}$. □

As a first application, we give an alternative proof of the following special case of [3, Proposition 2.10]:

Corollary 3.32. *Let* $\mathrm{soc}(ZFG)$ *be an ideal of* FG*. Then* $\mathrm{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ *holds.*

Proof. Since $\overline{\mathrm{Cl}}^+_{p',N}(G)$ is a subset of $J(ZF\bar{G})$, Theorem 3.31 yields

$$\mathrm{soc}(ZF\bar{G}) = \mathrm{Ann}_{ZF\bar{G}}\, J(ZF\bar{G}) \subseteq \mathrm{Ann}_{ZF\bar{G}}\left(\overline{\mathrm{Cl}}^+_{p',N}(G)\right) \subseteq (\bar{G}')^+ \cdot F\bar{G}$$

and we obtain $\mathrm{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ by Lemma 3.3. □

3.8. **Central products.** Let G be a finite group. We consider the question when $\mathrm{soc}(ZFG)$ is an ideal of FG in case that $G = G_1 * G_2$ is a central product of two subgroups G_1 and G_2. Central products will play an important role throughout our investigation, for instance in the decomposition of G given in Theorem D.

Theorem 3.33. *Let* $G = G_1 * G_2$ *be the central product of* G_1 *and* G_2*. Then* $\mathrm{soc}(ZFG) \trianglelefteq FG$ *is equivalent to* $\mathrm{soc}(ZFG_i) \trianglelefteq FG_i$ *for* $i = 1, 2$*.*

Proof. First assume that $\mathrm{soc}(ZFG_i)$ is an ideal in FG_i for $i = 1, 2$. By [3, Proposition 1.9], this implies

$$\mathrm{soc}(Z(FG_1 \otimes_F FG_2)) \trianglelefteq FG_1 \otimes_F FG_2.$$

Since $F(G_1 \times G_2) \cong FG_1 \otimes_F FG_2$ holds, this yields $\mathrm{soc}(ZF(G_1 \times G_2)) \trianglelefteq F(G_1 \times G_2)$. The group G is isomorphic to a quotient group of $G_1 \times G_2$, so $\mathrm{soc}(ZFG)$ is an ideal in FG by Corollary 3.32.

Now assume conversely that $\mathrm{soc}(ZFG)$ is an ideal of FG. By Corollary 3.5, G is of the form $P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H. First suppose that $O_{p'}(G) = 1$ holds. Then $Z := G_1 \cap G_2 \subseteq Z(G) \subseteq C_G(P) = Z(P)$ is a p-group. We consider the canonical projection $\nu := \nu_{G_2} : FG \to F[G/G_2]$. By Theorem 3.31, we have

$$\mathrm{Ann}_{ZF[G/G_2]}\left(\nu(J(ZFG))\right) \subseteq \left([G/G_2]'\right)^+ \cdot F[G/G_2]. \tag{3.1}$$

Note that there is a canonical isomorphism $G_1/Z \cong G/G_2$. Furthermore, we have $ZFG_1 \subseteq ZFG$ and $\nu(ZFG_1) = \nu(ZFG)$, so also $\nu(J(ZFG_1)) = \nu(J(ZFG))$ holds. Hence we have

$$\mathrm{Ann}_{ZF[G_1/Z]}(\nu_1(J(ZFG_1))) \subseteq ([G_1/Z]')^+ \cdot F[G_1/Z],$$

where $\nu_1 \colon FG_1 \to F[G_1/Z]$ denotes the canonical projection. Let $x_1 \in \mathrm{soc}(ZFG_1)$ and observe that G_1' is a p-group. By Corollary 3.19, we have $x_1 \in Z^+ \cdot FG_1 = \nu_1^*(F[G_1/Z])$. Let $y_1 \in F[G_1/Z]$ with $x_1 = \nu_1^*(y_1)$. Then [3, Remark 2.9] yields

$$y_1 \in \mathrm{Ann}_{ZF[G_1/Z]}(\nu_1(J(ZFG_1))) \subseteq ([G_1/Z]')^+ \cdot F[G_1/Z].$$

By Remark 3.27, this yields $x_1 \in (G_1')^+ \cdot FG_1$ and hence $\mathrm{soc}(ZFG_1)$ is an ideal in FG_1 (see Lemma 3.3). By symmetry, we obtain $\mathrm{soc}(ZFG_2) \trianglelefteq FG_2$.

Now we consider the general case. For $\bar{G} := G/O_{p'}(G)$, we have $\bar{G} = \bar{G}_1 * \bar{G}_2$ with $\bar{G}_i := G_iO_{p'}(G)/O_{p'}(G)$ $(i = 1, 2)$. Note that $\bar{G}_i \cong G_i/O_{p'}(G) \cap G_i \cong G_i/O_{p'}(G_i)$ follows since $O_{p'}(G) \cap G_i = O_{p'}(G_i)$ holds. By the above, we obtain $\mathrm{soc}(ZF\bar{G}_i) \trianglelefteq F\bar{G}_i$. Since G' is a p-group, also G_1' and G_2' are p-groups. Lemma 3.10 then yields $\mathrm{soc}(ZFG_i) \trianglelefteq FG_i$ for $i = 1, 2$. □

Remark 3.34. For $G \cong G_1 \times G_2$, the statement of Theorem 3.33 is a special case of [3, Proposition 1.9].

4. Groups of prime power order

Let F be an algebraically closed field of characteristic $p > 0$. In this section, we classify the finite p-groups G for which $\mathrm{soc}(ZFG)$ is an ideal in FG (see Theorem B). Additionally, these results will be generalized to arbitrary finite groups (see Theorem C). First we prove that the property $\mathrm{soc}(ZFG) \trianglelefteq FG$ is preserved under isoclinism (see Section 4.1). Subsequently, we distinguish the cases $p \geq 3$ (see Section 4.2) and $p = 2$ (see Section 4.3).

4.1. Isoclinism.

Let G be a finite p-group. The aim of this section is to show that the property $\mathrm{soc}(ZFG) \trianglelefteq FG$ is invariant under isoclinism in the following sense: If Q is a finite p-group isoclinic to G, then $\mathrm{soc}(ZFQ) \trianglelefteq FQ$ holds precisely if we have $\mathrm{soc}(ZFG) \trianglelefteq FG$. The proof of this statement is based on some observations on the center of G and the transition to the quotient group $\bar{G} := G/Z(G)$.

Lemma 4.1.

(i) We have $\mathrm{soc}(ZFG) \subseteq Z(G)^+ \cdot FG$.

(ii) $\mathrm{soc}(ZFG)$ *is an ideal of* FG *if and only if* $\mathrm{soc}(ZFG) = (Z(G)G')^+ \cdot FG$ *holds.*

Proof. The first statement follows by Corollary 3.19. Now let $\mathrm{soc}(ZFG)$ be an ideal of FG. Lemma 3.3 then yields $\mathrm{soc}(ZFG) \subseteq (G')^+ \cdot FG$. Together with (i), this implies $\mathrm{soc}(ZFG) \subseteq (Z(G)G')^+ \cdot FG$, and by Corollary 3.19, we obtain equality. Conversely, $(Z(G)G')^+ \cdot FG$ is obviously an ideal in FG. □

In the given situation, we have

$$\mathrm{Cl}_{p'} := \mathrm{Cl}_{p',Z(G)}(G) = \left\{C \in \mathrm{Cl}(G) \colon C \not\subseteq Z(G),\ |C| = |\bar{C}|\right\}.$$

Note that the length of every conjugacy class in $\mathrm{Cl}_{p'}$ is a nontrivial power of p. Let $\overline{\mathrm{Cl}}_{p'} := \overline{\mathrm{Cl}}_{p',Z(G)}(G)$ be the set of images of the classes in $\mathrm{Cl}_{p'}$ and denote by $\overline{\mathrm{Cl}}_{p'}^+ :=$

$\overline{\mathrm{Cl}}^+_{p',Z(G)}(G)$ the corresponding class sums in $F\bar{G}$. In this situation, the implication given in Theorem 3.31 is an equivalence:

Lemma 4.2. *The socle* $\mathrm{soc}(ZFG)$ *is an ideal in* FG *if and only if* $\mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+_{p'}) \subseteq (\bar{G}')^+ \cdot F\bar{G}$ *holds.*

Proof. Consider the map $\nu^*_{Z(G)}\colon F\bar{G} \to FG$ introduced in Section 3.7. Lemma 4.1 yields

$$\mathrm{soc}(ZFG) \subseteq Z(G)^+ \cdot FG = \mathrm{Im}\, \nu^*_{Z(G)}.$$

By [3, Remark 2.9], we therefore obtain

$$\mathrm{soc}(ZFG) = \nu^*_{Z(G)}\left(\mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+_{p'})\right).$$

By Remark 3.27, we have $\mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+_{p'}) \subseteq (\bar{G}')^+ \cdot F\bar{G}$ if and only if $\mathrm{soc}(ZFG) \subseteq (G')^+ \cdot FG$ holds, which is equivalent to $\mathrm{soc}(ZFG) \trianglelefteq FG$ by Lemma 3.3. □

Now we proceed to the main result of this section. Recall that two finite p-groups G_1 and G_2 are isoclinic if there exist isomorphisms $\varphi\colon G_1' \to G_2'$ and $\beta\colon G_1/Z(G_1) \to G_2/Z(G_2)$ such that whenever $\beta(a_1Z(G_1)) = a_2Z(G_2)$ and $\beta(b_1Z(G_1)) = b_2Z(G_2)$ hold for $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$, then $\varphi([a_1, b_1]) = [a_2, b_2]$ follows. We set $\bar{G}_i := G_i/Z(G_i)$ and write $\mathrm{Cl}_{p',i}$ and $\overline{\mathrm{Cl}}_{p',i}$ to distinguish the sets $\mathrm{Cl}_{p'}$ and $\overline{\mathrm{Cl}}_{p'}$ for $i \in \{1, 2\}$.

Theorem 4.3. *Let* G_1 *and* G_2 *be finite isoclinic* p*-groups. Then* $\mathrm{soc}(ZFG_1) \trianglelefteq FG_1$ *is equivalent to* $\mathrm{soc}(ZFG_2) \trianglelefteq FG_2$.

Proof. Let $\varphi\colon G_1' \to G_2'$ and $\beta\colon \bar{G}_1 \to \bar{G}_2$ be the corresponding isomorphisms. We first show that $\overline{\mathrm{Cl}}_{p',1}$ and $\overline{\mathrm{Cl}}_{p',2}$ are in bijective correspondence under β. Let $C_1 \in \mathrm{Cl}_{p',1}$ and set $\bar{C}_1$ to be its image in $\bar{G}_1$. Then $\bar{C}_2 := \beta(\bar{C}_1)$ is a conjugacy class of $\bar{G}_2$. Consider a preimage $C_2 \in \mathrm{Cl}(G_2)$ of $\bar{C}_2$. Let $x_2 \in C_2$ and assume that $1 \neq [x_2, g_2] \in Z(G_2)$ holds for some $g_2 \in G_2$. Choose elements $x_1 \in C_1$ and $g_1 \in G_1$ with $\beta(x_1Z(G_1)) = x_2Z(G_2)$ and $\beta(g_1Z(G_1)) = g_2Z(G_2)$. We then obtain $\varphi([x_1, g_1]) = [x_2, g_2] \in Z(G_2)\backslash\{1\}$. Note that $\beta([x_1, g_1]Z(G_1)) = [x_2, g_2]Z(G_2) = Z(G_2)$ holds, so we have $1 \neq [x_1, g_1] \in Z(G_1)$. This implies $|\bar{C}_1| < |C_1|$, which is a contradiction to $C_1 \in \mathrm{Cl}_{p',1}$. Hence we obtain $\bar{C}_2 \in \overline{\mathrm{Cl}}_{p',2}$. The other implication follows by symmetry.

Extending β F-linearly gives rise to an F-algebra isomorphism $\widehat{\beta}\colon F\bar{G}_1 \to F\bar{G}_2$. By the above, we have $\widehat{\beta}(\overline{\mathrm{Cl}}^+_{p',1}) = \overline{\mathrm{Cl}}^+_{p',2}$. Now if $\mathrm{soc}(ZFG_1)$ is an ideal of FG_1, Lemma 4.2 implies $\mathrm{Ann}_{ZF\bar{G}_1}(\overline{\mathrm{Cl}}^+_{p',1}) \subseteq (\bar{G}_1')^+ \cdot F\bar{G}_1$. Applying the isomorphism $\widehat{\beta}$ yields $\mathrm{Ann}_{ZF\bar{G}_2}(\overline{\mathrm{Cl}}^+_{p',2}) \subseteq (\bar{G}_2')^+ \cdot F\bar{G}_2$. By Lemma 4.2, $\mathrm{soc}(ZFG_2)$ is an ideal in FG_2. The other implication follows by symmetry. □

4.2. **Odd characteristic.** In this section, we assume that F is an algebraically closed field of odd characteristic p.

Remark 4.4. For an abelian p-group G, we have $\prod_{g \in G} g = 1$ since every nontrivial element in G differs from its inverse and their product is the identity element.

Proposition 4.5. *Let* G *be a finite* p*-group of nilpotency class exactly two. Then there exists an element* $y \in ZFG$ *with* $y \notin (G')^+ \cdot FG$ *such that* $y \cdot S^+ = 0$ *holds for all subgroups* $1 \neq S \subseteq G'$.

Proof. Since G' is a nontrivial p-group, there exists a nontrivial group homomorphism $\alpha\colon G' \to F$. We define an element $y := \sum_{g\in G} a_g g \in FG$ by setting $a_g := \alpha(g)$ for $g \in G'$ and $a_g = 0$ otherwise. We have $y \in FG' \subseteq FZ(G) \subseteq ZFG$. Now consider a subgroup $1 \neq S \subseteq G'$. The coefficient of $w \in G$ in the product $y \cdot S^+$ is given by $\sum_{s\in S} a_{ws^{-1}}$. For $w \notin G'$, all summands are zero. For $w \in G'$, we obtain

$$\sum_{s\in S} a_{ws^{-1}} = \sum_{s\in S} \alpha(ws^{-1}) = |S|\cdot\alpha(w) + \sum_{s\in S}\alpha(s^{-1}) = \alpha\left(\prod_{s\in S} s^{-1}\right) = \alpha(1) = 0.$$

In the second and third step, we use that α is a group homomorphism. The fourth equality is due to Remark 4.4. This implies $y \cdot S^+ = 0$ as claimed. □

In this special situation, the condition given in Theorem 3.25 is in fact equivalent to $\operatorname{soc}(ZFG) \trianglelefteq FG$:

Theorem 4.6. *Let G be a finite p-group. Then* $\operatorname{soc}(ZFG)$ *is an ideal in FG if and only if G has nilpotency class at most two.*

Proof. If G is of nilpotency class at most two, we have $G' \subseteq Z(G)$ and hence $\operatorname{soc}(ZFG)$ is an ideal in FG by Theorem 3.25. For the converse implication, we use induction on the nilpotency class of G. Note that $\bar{G} := G/Z(G)$ has nilpotency class $c(G)-1$. First assume $c(G) = 3$. We apply Proposition 4.5 to the group $\bar{G}$ and consider the element $y \in ZF\bar{G}$ constructed therein. Let $\bar{C} \in \overline{\mathrm{Cl}}_{p'}$ be a conjugacy class and let $c \in \bar{C}$. Since $\bar{G}' \subseteq Z(\bar{G})$ holds, the map $\gamma\colon \bar{G} \to \bar{G},\ g \mapsto [g,c]$ is a group homomorphism and hence we have

$$\bar{C} = \left\{gcg^{-1} \colon g \in \bar{G}\right\} = \left\{[g,c]c \colon g \in \bar{G}\right\} = Sc,$$

where $S := \operatorname{Im}\gamma$ is a subgroup of $\bar{G}'$. Note that we have $|S| = |\bar{C}| > 1$. By Proposition 4.5, we have $y\cdot S^+ = 0$ and hence $y\cdot\bar{C}^+ = y\cdot(Sc)^+ = 0$. Since $y \notin (\bar{G}')^+\cdot F\bar{G}$ holds, $\operatorname{soc}(ZFG)$ is not an ideal of FG (see Lemma 4.2). If G is of nilpotency class $c(G) > 3$, we obtain $\operatorname{soc}(ZF\bar{G}) \ntrianglelefteq F\bar{G}$ by induction. Corollary 3.32 then yields $\operatorname{soc}(ZFG) \ntrianglelefteq FG$. □

Remark 4.7. The analogous construction fails for $p = 2$ since the statement of Remark 4.4 does not hold for groups of even order.

4.3. **Characteristic $p = 2$.** Throughout, let F be an algebraically closed field of characteristic two. Unless otherwise stated, we assume that G is a finite 2-group.

Remark 4.8. Let $C = \{f, g\}$ be a conjugacy class of length two of G. An inner automorphism of G either fixes both f and g, or it interchanges the two elements. For $c := gf^{-1} \in G'$, this yields $C_G(f) = C_G(g) \subseteq C_G(c)$. For $h \in G \setminus C_G(f)$, we have $hch^{-1} = hgf^{-1}h^{-1} = fg^{-1} = c^{-1}$. This shows that the subgroup $\langle c\rangle \subseteq G'$ is normal in G.

For every conjugacy class $C := \{f, g\}$ of length two, we set $Y_C := \langle gf^{-1}\rangle$. In the following, we consider the subgroup

$$Y(G) := \langle Y_C : C \in \mathrm{Cl}(G),\ |C| = 2\rangle .$$

Note that $Y(G)$ is characteristic in G. More precisely, we obtain the following:

Lemma 4.9. *We have $Y(G) \subseteq Z(\Phi(G))$. In particular, $Y(G)$ is abelian.*

Proof. Note that $Y(G) \subseteq G' \subseteq \Phi(G)$ holds. Now let $C = \{f, g\}$ be a conjugacy class of length two. Since $C_G(f)$ is a maximal subgroup of G, Remark 4.8 yields $\Phi(G) \subseteq C_G(f) \subseteq C_G(gf^{-1})$ and hence $\Phi(G)$ centralizes Y_C. Thus $Y(G)$ is contained in the center of $\Phi(G)$, so in particular, it is abelian. □

Lemma 4.10. *We have* $\mathrm{soc}(ZFG) \subseteq Y(G)^+ \cdot FG$.

Proof. Let $y = \sum_{g \in G} a_g g \in \mathrm{soc}(ZFG)$. For a conjugacy class $C = \{f, g\}$ of length two, we have $c := gf^{-1} \in Y(G)$ and the condition $y \cdot C^+ = 0$ yields $a_x = a_{xc^{-1}}$ for all $x \in G$. By induction, this implies $a_x = a_{xc_1^{-1}\cdots c_n^{-1}}$ for every $x \in G$ and all elements $c_1, \ldots, c_n$ arising from G-conjugacy classes of length two as above. This shows that y has constant coefficients on the cosets of $Y(G)$, that is, we obtain $y \in Y(G)^+ \cdot FG$. □

With this preliminary result, we obtain the following characterization of the 2-groups G for which $\mathrm{soc}(ZFG)$ is an ideal in FG.

Theorem 4.11. *The socle* $\mathrm{soc}(ZFG)$ *is an ideal in* FG *if and only if* $G' \subseteq Y(G)Z(G)$ *holds.*

Proof. Suppose $G' \not\subseteq Y(G)Z(G)$, so $Y(G)Z(G) \cap G'$ is a proper subgroup of G'. By [6, Theorem III.7.2], there exists a subgroup $N \trianglelefteq G$ with $Y(G)Z(G) \cap G' \subseteq N \subseteq G'$ and $|G' : N| = 2$. We set $M := Y(G)Z(G)N$. Note that $M^+ \in ZFG$ holds since M is a normal subgroup of G. We now show that M^+ annihilates the basis of $J(ZFG)$ given in Theorem 3.14.

For $z \in Z(G) \subseteq M$, we have $(1 + z) \cdot M^+ = 0$. For a G-conjugacy class $C = \{f, g\}$ of length two, we obtain $C^+ \cdot Y(G)^+ = fY(G)^+ + gY(G)^+ = 0$ since $gf^{-1} \in Y(G)$ holds. Hence M^+ annihilates C^+. Every conjugacy class $C \in \mathrm{Cl}(G)$ with $|C| \geq 4$ contains an even number of elements in every coset of N since C is contained in a coset of G' and $|G' : N| = 2$ holds. This implies that C^+ is annihilated by N^+ and hence by M^+. Summarizing, we obtain $M^+ \in \mathrm{soc}(ZFG)$. Moreover, $M \cap G' = N \subsetneq G'$ implies $M^+ \notin (G')^+ \cdot FG$. By Lemma 3.3, this yields $\mathrm{soc}(ZFG) \ntrianglelefteq FG$.

Conversely, assume that $G' \subseteq Y(G)Z(G)$ holds. By Lemmas 4.1 and 4.10, we have

$$\mathrm{soc}(ZFG) \subseteq (Y(G)Z(G))^+ \cdot FG \subseteq (G')^+ \cdot FG$$

and hence $\mathrm{soc}(ZFG)$ is an ideal of FG (see Lemma 3.3). □

This completes the proof of Theorem B.

Remark 4.12. Similarly to the case of odd characteristic, $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds if G has nilpotency class at most two.

However, the next example demonstrates that in contrast to the case of odd characteristic, the nilpotency class of a finite 2-group G for which $\mathrm{soc}(ZFG)$ is an ideal in FG can be arbitrarily large.

Example 4.13.

(i) Let $G = D_{2^n} = \langle r, s \colon r^{2^{n-1}} = s^2 = 1,\ srs = r^{-1} \rangle$ with $n \in \mathbb{N}$ be the dihedral group of order 2^n. For $n \leq 2$, G is abelian and hence $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds. For $n \geq 3$, we have $G' = \langle r^2 \rangle = Y(G)Z(G)$ and hence $\mathrm{soc}(ZFG) \trianglelefteq FG$ follows by Theorem 4.11. The 2-groups of maximal class of a fixed order are isoclinic. Therefore, by Theorem 4.3, $\mathrm{soc}(ZFG)$ is an ideal in FG if G is a semihedral or generalized quaternion 2-group.

(ii) By [3, Theorem 4.12], every 2-group G of order at most 16 satisfies $\mathrm{soc}(ZFG) \trianglelefteq FG$. Up to isomorphism, there exist 51 groups of order 32. Out of those, 7 groups are abelian and 26 groups have nilpotency class precisely two. Additionally, 13 groups satisfy the property $G' \subseteq Y(G)Z(G)$.

(iii) Consider the holomorph $G = \mathbb{Z}/8\mathbb{Z} \rtimes (\mathbb{Z}/8\mathbb{Z})^\times$ of $\mathbb{Z}/8\mathbb{Z}$, which has order 32. It has 11 conjugacy classes and we have $|Z(G)| = 2$. Since $G/Z(G) \cong D_8 \times C_2$ has precisely 10 conjugacy classes, the images of the non-central conjugacy classes of G in $G/Z(G)$ are pairwise distinct. For every such conjugacy class C, we therefore have $Z(G)C \subseteq C$ and hence $\nu_{Z(G)}(C^+) = 0$. This proves $J(ZFG)^2 = 0$, so $J(ZFG) = \operatorname{soc}(ZFG)$ follows. In particular, we obtain $\dim \operatorname{soc}(ZFG) = \dim J(ZFG) = 10$. Due to $|G'| = 4$, the space $(G')^+ \cdot FG$ is eight-dimensional, so it does not contain $\operatorname{soc}(ZFG)$. By Lemma 3.3, $\operatorname{soc}(ZFG)$ is not an ideal in FG.

We conclude this part with a generalization of Theorem 4.14 to arbitrary finite groups, which is a stronger variant of Theorem 3.25:

Theorem 4.14. *Let G be an arbitrary finite group which satisfies $G' \subseteq Y(O_2(G))Z(O_2(G))$. Then $\operatorname{soc}(ZFG)$ is an ideal of FG.*

Proof. The given condition implies $G' \subseteq O_2(G)$, so by Theorem 3.4, we have $G = P \rtimes H$ with $P := O_2(G) \in \operatorname{Syl}_2(G)$ and an abelian $2'$-group H. Note that G' is abelian as $Y(P)$ is abelian (see Lemma 4.9). By Remark 3.6, we have

$$P = C_P(H)G' = C_P(H)Y(P)Z(P). \tag{4.1}$$

Since G' is abelian, $C_P(H)'$ is normal in P. We consider the group $\bar{P} := P/C_P(H)'$ and denote the image of $S \subseteq P$ in $\bar{P}$ by $\bar{S}$. Then we have

$$\overline{Y(P)} \subseteq \bar{P}' = \overline{(C_P(H)Y(P))'} = [\overline{C_P(H)}, \overline{Y(P)}] \subseteq [\bar{P}, \overline{Y(P)}].$$

This implies $\overline{Y(P)} = 1$, so $Y(P) \subseteq C_P(H)'$ follows.

By (4.1), we then have $[P, H] = [Z(P), H]$ and hence $P = C_P(H)[Z(P), H]$ follows. Since $C_P(H)$ centralizes $W := H[Z(P), H] \subseteq HZ(P)$ and $C_P(H) \cap [Z(P), H] = 1$ follows by [5, Theorem 5.3.6], we obtain $G = C_P(H) \times W$. It is then easily verified that $Y(P) = Y(C_P(H))$ holds. With Dedekind's identity, we obtain

$$C_P(H)' \subseteq G' \cap C_P(H) \subseteq Y(C_P(H))Z(P) \cap C_P(H) \subseteq Y(C_P(H)) \cdot Z(C_P(H)).$$

By Theorem 4.11, $\operatorname{soc}(ZFC_P(H))$ is an ideal of $FC_P(H)$. Since $\operatorname{soc}(ZFW) \trianglelefteq FW$ follows by Theorem 3.25, $\operatorname{soc}(ZFG)$ is an ideal in FG by Theorem 3.33. □

This completes the proof of Theorem C.

5. Decomposition of G into a central product

Let F be an algebraically closed field of characteristic $p > 0$. We consider an arbitrary finite group G for which $\operatorname{soc}(ZFG)$ is an ideal in FG. By Theorem 3.4, we may write $G = P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H. In this section, we prove Theorem D. Combined with the results on p-groups from the last section, it reduces our investigation to the case that G' is a Sylow p-subgroup of G.

Theorem 5.1 (Theorem D)**.** *We have $G = C_P(H) * O^p(G)$. Moreover, $\operatorname{soc}(ZFC_P(H))$ and $\operatorname{soc}(ZFO^p(G))$ are ideals in $FC_P(H)$ and $FO^p(G)$, respectively. The socle of ZFG is explicitly given by*

$$\operatorname{soc}(ZFG) = (Z(P)G')^+ \cdot FG.$$

Proof. By Proposition 3.20, we have $G = C_P(H) * O^p(G)$. Theorem 3.33 then implies that $\mathrm{soc}(ZFC_P(H))$ and $\mathrm{soc}(ZFO^p(G))$ are ideals in $FC_P(H)$ and $FO^p(G)$, respectively. It therefore remains to determine the structure of $\mathrm{soc}(ZFG)$. By the above decomposition, we have $Z(C_P(H)) \subseteq Z(G)$. By Corollary 3.19, we obtain $\mathrm{soc}(ZFG) \subseteq Z(C_P(H))^+ \cdot FG$. Together with Lemma 3.3, this implies

$$\mathrm{soc}(ZFG) \subseteq \left(Z(C_P(H))G'\right)^+ \cdot FG \subseteq \left(Z(P)G'\right)^+ \cdot FG.$$

In the last step, we used $Z(P) = Z(C_P(H))Z([G', H]) \subseteq Z(C_P(H))G'$. On the other hand, we have $(Z(P)G')^+ \cdot FG \subseteq \mathrm{soc}(ZFG)$ by Lemma 3.16, which completes the proof of Theorem D. □

This result on the structure of $\mathrm{soc}(ZFG)$ generalizes the corresponding statement in Lemma 4.1. Note that, by Theorem D, the hypothesis that $\mathrm{soc}(ZFG)$ is an ideal in FG implies $\dim \mathrm{soc}(ZFG) = |G : G'Z(G)|$. In particular, the dimension of $\mathrm{soc}(ZFG)$ is a divisor of $|G|$. We also observe that in this situation, $\mathrm{soc}(ZFG)$ is a principal ideal of FG generated by a central element. Furthermore, we obtain the following reduction:

Remark 5.2. Since the structure of the p-group $C_P(H)$ is determined by Theorem B, it suffices to investigate the group $O^p(G)$. Inductively, we may assume $O^p(G) = G$. By Remark 3.6, this yields $P = G' = [G', H]$. In particular, $C_{G'}(H) \subseteq G'' \subseteq Z(G')$ follows (see [5, Theorem 5.2.3]), which implies $C_{G'}(H) \subseteq Z(G)$. If additionally $O_{p'}(G) = 1$ holds, we obtain $C_{G'}(H) = Z(G)$.

Moreover, we state the following consequence of Theorem D:

Theorem 5.3. *We have* $\mathrm{soc}(ZFP) \trianglelefteq FP$. *In particular, the group P is metabelian and its nilpotency class is at most two if p is odd.*

Proof. By Theorem D, we have $P = C_P(H) * [P, H]$ and $\mathrm{soc}(ZFC_P(H))$ is an ideal in $FC_P(H)$. Since $[P, H] \subseteq G'$ has nilpotency class at most two (see Proposition 3.20), we obtain $\mathrm{soc}(ZF[P, H]) \trianglelefteq F[P, H]$ by Theorem B. By Theorem 3.33, this yields $\mathrm{soc}(ZFP) \trianglelefteq FP$. In particular, it follows that P is metabelian and that the nilpotency class of P is at most two if p is odd (see Theorem B). □

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